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Local null and insensitive controllability for the respective models: complete Ladyzhenskaya-Boussinesq and Ladyzhenskaya-Smagorinsky

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Niterói

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Tese submetida ao Programa de Pós-Graduação em Matemática da Universidade Federal Fluminense como requisito parcial para a obtenção do grau de Doutor em Matemática.

Orientador: Prof. Juan Bautista Límaco Ferrel

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por

João Carlos Fernandes Barreira

apresentada ao Programa de Pós-Graduação em Matemática como requesito parcial para a obtenção do grau de

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Para os devidos fins, declaro **estar ciente** do conteúdo desta **versão corrigida** elaborada em atenção às sugestões dos membros da banca examinadora na sessão de defesa do trabalho, manifestando-me **favoravelmente** ao seu encaminhamento e publicação no **Repositório Institucional da UFF**.

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Nome do orientador(a)

I dedicate it to my mother Marisa
and my grandmother Nedina

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RESUMO

Nesta tese os objetivos principais são apresentar três diferentes pesquisas que foram desenvolvidas durante o curso de doutorado do Programa de Pós-Graduação em Matemática da Universidade Federal Fluminense (UFF), sob orientação do professor Juan Bautista Límaco Ferrel. O principal campo de estudo desta tese é a teoria de controle e a boa colocação de equações diferenciais parciais, onde apresentamos três pesquisas sobre os seguintes temas: controle insensível; controlabilidade nula local; e existência e unicidade de solução forte.

Em resumo, os principais objetivos são estabelecer:

i) A existência de controles insensitivos para o sistema de Ladyzhenskaya-Smagorinsky,

$$\begin{cases} y_t - \nabla \cdot ((\nu_0 + \nu_1 \|\nabla y\|_{L^2}^2) Dy) + (y \cdot \nabla)\, y + \nabla p = f + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in} \quad \Omega, \end{cases}$$

onde $Dy := \dfrac{1}{2}(\nabla y + \nabla^T y)$.

ii) A controlabilidade nula local dos modelos Ladyzhenskaya-Boussinesq N-dimensionais completos

$$\begin{cases} y_t - \nabla \cdot (\nu(\nabla y) Dy) + (y \cdot \nabla) y + \nabla P = v\chi_\omega + \nu_0 \theta e_N & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla \cdot (\nu(\nabla y)\nabla\theta) + y \cdot \nabla\theta = v_0\chi_\omega + \nu(\nabla y) Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases}$$

onde

$$\nu(\nabla y) := \nu_0 + \nu_1 \int_\Omega |\nabla y|^2 dx$$

e

$$\begin{cases} y_t - \nabla \cdot (\bar{\nu}(\nabla y) Dy) + (y \cdot \nabla) y + \nabla P = v\chi_\omega + \nu_0 \theta e_N & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla \cdot (\bar{\nu}(\nabla\theta)\nabla\theta) + y \cdot \nabla\theta = v_0\chi_\omega + \bar{\nu}(\nabla y) Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases}$$

onde $\bar{\nu}(\nabla\varsigma) := \nu_0 + \nu_1 \|\nabla\varsigma\|_{L^p}^2$, for $3 < p \leq 6$. Em ambos os sistemas

$$e_N = \begin{cases} (0,1) \text{ se } N = 2, \\ (0,0,1) \text{ se } N = 3, \end{cases}$$

e

$$Dy : \nabla y = \sum_{i,j=1}^{N} \frac{1}{2} \left(\frac{\partial y_j}{\partial x_i} + \frac{\partial y_i}{\partial x_j} \right) \frac{\partial y_i}{\partial x_j}.$$

iii) A existência de solução forte para a equação de Navier-Stokes em domínio não cilíndrico

$$\begin{cases} u' - \nu\Delta u + (u\cdot\nabla)u = f - \nabla p & \text{in} \quad \widehat{Q}, \\ \nabla\cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega_0, \end{cases}$$

por meio do método penalizante.

Palavras-chave: equações diferenciais parciais, equações para fluidos viscosos incompressíveis, teoria de controle, controlabilidade nula, controle insensibilizante, Ladyzhenskaya-Smagorinsky, Ladyzhenskaya-Boussinesq, domínio não-cilíndrico, sistema de Navier-Stokes, soluções fortes, método penalizante.

ABSTRACT

In this thesis the main objectives are to present three different researches that were developed during the doctoral course of the Graduate Program in Mathematics at the Universidade Federal Fluminense (UFF), under the guidance of professor Juan Bautista Límaco Ferrel. The main field of study of this thesis is control theory and the well-posedness of partial differential equations, where we present three research studies on the following topics: insensitive control; local null controllability; and existence and uniqueness of strong solution.

In summary, the main objectives are to establish:

i) The existence of insensitive controls for the system of the Ladyzhenskaya-Smagorinsky

$$\begin{cases} y_t - \nabla\cdot((\nu_0+\nu_1\|\nabla y\|^2_{L^2})Dy) + (y\cdot\nabla)\,y + \nabla p = f + v\chi_\omega & \text{in} \quad Q,\\ \nabla\cdot y = 0 & \text{in} \quad Q,\\ y = 0 & \text{on} \quad \Sigma,\\ y(0) = y^0 + \tau\widehat{y}^0 & \text{in} \quad \Omega, \end{cases}$$

where $Dy := \dfrac{1}{2}(\nabla y + \nabla^T y)$.

ii) The local null controllability of complete N-dimensional Ladyzhenskaya-Boussinesq models

$$\begin{cases} y_t - \nabla\cdot(\nu(\nabla y)Dy) + (y\cdot\nabla)y + \nabla P = v\chi_\omega + \nu_0\theta e_N & \text{in} \quad Q,\\ \nabla\cdot y = 0 & \text{in} \quad Q,\\ \theta_t - \nabla\cdot(\nu(\nabla y)\nabla\theta) + y\cdot\nabla\theta = v_0\chi_\omega + \nu(\nabla y)Dy:\nabla y & \text{in} \quad Q,\\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma,\\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases}$$

where

$$\nu(\nabla y) := \nu_0 + \nu_1\int_\Omega |\nabla y|^2 dx$$

and

$$\begin{cases} y_t - \nabla\cdot(\bar\nu(\nabla y)Dy) + (y\cdot\nabla)y + \nabla P = v\chi_\omega + \nu_0\theta e_N & \text{in} \quad Q,\\ \nabla\cdot y = 0 & \text{in} \quad Q,\\ \theta_t - \nabla\cdot(\bar\nu(\nabla\theta)\nabla\theta) + y\cdot\nabla\theta = v_0\chi_\omega + \bar\nu(\nabla y)Dy:\nabla y & \text{in} \quad Q,\\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma,\\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases}$$

where $\bar\nu(\nabla\varsigma) := \nu_0 + \nu_1\|\nabla\varsigma\|^2_{L^p}$, for $3 < p \le 6$. In both systems

$$e_N = \begin{cases} (0,1) \text{ if } N = 2,\\ (0,0,1) \text{ if } N = 3, \end{cases}$$

and

$$Dy:\nabla y = \sum_{i,j=1}^{N} \frac{1}{2}\left(\frac{\partial y_j}{\partial x_i} + \frac{\partial y_i}{\partial x_j}\right)\frac{\partial y_i}{\partial x_j}.$$

iii) The existence of strong solutions for the Navier-Stokes equations in non-cylindrical domain

$$\begin{cases} u' - \nu\Delta u + (u\cdot\nabla)u = f - \nabla p & \text{in} \quad \widehat{Q}, \\ \nabla\cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega_0, \end{cases}$$

by means the penalizing method.

Keywords: partial differential equations, equations for incompressible viscous fluids, control theory, null controllability, insensitizing control, Ladyzhenskaya-Smagorinsky, Ladyzhenskaya-Boussinesq, non-cylindrical domain, Navier-Stokes system, strong solutions, penalty method.

Contents

Introduction

This thesis comprises results within the scope of control theory and the existence and uniqueness of strong solutions for systems governed by partial differential equations (PDEs). Control theory is a consolidated field of study in mathematical literature, engineering and related areas. It provides us with tools to better understand, apply and manipulate mathematical equations described by (PDEs), or Ordinary Differential Equations (ODEs), which model physical, chemical, biological and even economic behaviors.

In general, we can define a controllability problem as follows: Suppose a state system governed by an PDE (or system of PDEs) that evolves in a time interval $[0, T]$, with certain initial and boundary conditions. We can act on the system by means an appropriate control (the right side of the system) that is taken from a set of admissible controls. Then, given a final state, we are interested in finding a control such that the PDE solution corresponds to both the initial state at time $t = 0$ and the final state at time $t = T$.

Of course, depending on the objective the concept of controllability can be more specific. We precisely define certain commonly considered control problems. To establish the ideas, consider U_{ad} the set of admissible controls, $v \in U_{ad}$ the control, H a Banach space where the equation makes sense and denote by $y_v = y_v(t)$ the system solution associated with the control v. Assuming boundary conditions for the equation and that $y_0, y_d \in H$ are the states at $t = 0$ and $t = T$, respectively. We will say that the system is *exactly controllable* at the time T, if it is possible to find a control v, for any $y_0, y_d \in H$, which "drive" the equation from y_0 to y_d so that

$$y_v(T) = y_d.$$

Nonetheless, if we need to relax the previous condition, we can modify it to

$$y_v(T) \text{ is close enough to } y_d$$

i.e.

$$\|y_v(T) - y_d\|_H < \epsilon, \ \forall \ \epsilon > 0.$$

When the previous condition is possible, we will say that the system is *approximately controllable* at the time T. Thus, in the latter case, the objective is to find a control such that the solution of the system in T is sufficiently close to the state y_d in a “small” neighborhood of H. On the other hand, we will say that the system is *null controllable* at time T if for all $y_0 \in H$ there is a control v such that y_v satisfies

$$y_v(T) = 0.$$

Now, assuming $\tilde{y}_0 \in H, \tilde{v} \in U_{ad}$ and $\tilde{y}$ an associated trajectory, well defined in $[0, T]$. We will say that the system is *exactly controllable for trajectories* at time T, if for any $y_0 \in H$ it is possible to obtain

$$y_v(T) = \tilde{y}(T).$$

The "best" among all the existing controls achieving the desired goal is frequently referred to as the *optimal control*.

The search for a certain controllability will depend on the PDE, as well as its initial and boundary conditions. There is a vast literature on this theory, we cite [LM67], [Lio88], [Rus78], [E Z05], and for a perspective from its origins to some of its many possible applications, see, for example, [ZF03].

The concepts exposed above characterize the main ways of obtaining the existence of controls for a given system. However, variations of these definitions are often necessary to achieve the objective of controlling a system. For example, the notion of null controllability may be more restrictive in the following sense: Given $T > 0$, we will say that the system is *locally null controllable* at time T, if there exists $\epsilon > 0$ such that for any $y_0 \in H$ with $\|y_0\|_H \leq \epsilon$, it is possible to find a control v such that the associated solution y_v satisfies

$$y_v(T) = 0.$$

If it is possible to show the existence of a $T > 0$ large enough such that the previous condition to be valid, then we say that the system is *null controllable in large time*.

We also have the concept of insensitive control which, as will be seen later, can be rewritten as a null controllability problem. The formulation of this type of control derives from the concept of sentinel (which comes from the French term *sentinelles* and can be translated as "observers") used in studies of distributed systems with incomplete data (systems described by EDPs) to "observe" the evolution of the system. This was one of the important contributions made by Jacques-Louis Lions throughout the abstract and applied theoretical development of control theory, see [Lio92] and [BF95]. The general idea of insensitive control can be expressed as follows: Let ϕ be a differentiable functional (called *sentinel functional*) defined on the solution set to which y belongs, for a problem with incomplete initial data, let us say $\tau\widehat{y}^0$. We say that control v insensitizes $\phi(y)$ if

$$\left.\frac{\partial \phi(y(x,t;v,\tau))}{\partial \tau}\right|_{\tau=0} = 0 \; \forall \; \widehat{y}^0 \text{ given in a suitable Hilbert or Banach space } Y \text{ with } \|\widehat{y}^0\|_Y = 1, \quad (1)$$

where x represents the spatial variable, t the time variable, τ an unknown small real number and $\frac{\partial \phi(y)}{\partial \tau}|_{\tau=0}$ denotes the derivative of $\phi(y)$ with respect to τ at $\tau = 0$. Hence, when (1) holds the functional ϕ is *locally insensitive* to the perturbation $\tau\widehat{y}^0$.

Among some of the possible applications for insensitizing controls, we have:

• Parabolic river and lake pollution problems where the initial conditions of the pollutants, or even the boundary conditions, may be unknown or only partially known due to difficulties such as inaccessibility in measuring contaminants, the purpose is to find a control (human action) suitable so that depollution can be carried out even with the uncertainty of some data;

• Oceanographic and meteorological problems where there is a wide variety of possibilities regarding the choice of the initial moment. Hence, the previous reasoning is also useful for this case.

In this thesis we address the Navier-Stokes and Ladyzhenskaya-Smagorinsky systems that govern, under very general conditions, the flow of incompressible and viscous fluids, that is, in which the mass and volume of the fluid do not change even under pressure. Furthermore, we will also study a control problem regarding a thermally conducting fluid obtained by combining a model proposed by Ladyzhenskaya with a nonlinear Boussinesq-like equation, the complete Ladyzhenskaya-Boussinesq model. Hence, the thesis is structured into three chapters plus a part dedicated to additional comments and open problems on the three previous chapters that constitute a compilation of the research we developed during the doctorate. The research themes are: insensitive control; local null controllability; and existence and uniqueness of strong solution.

In the chapter first, deals with N-dimensional Ladyzhenskaya-Smagorinsky kind differential turbulence model with partially known initial data. We are interested in the existence of insensitive controls with $N-1$ scalar controls in an arbitrary control domain for the local L^2 - norm of the solution of model, that will be given by means of a functional. In other words, the goal is to find a control function $v = (v_1, .., v_N)$, having one vanishing component (e.g v_N), such that some functional of the state is locally insensitive to the perturbations of these initial data.

More precisely, we will deal with the following model of the Ladyzhenskaya-Smagorinsky kind, which describes a model for the movement of incompressible viscous flows with incomplete data:

$$\begin{cases} y_t - \nabla \cdot ((\nu_0 + \nu_1 \|\nabla y\|^2_{L^2}) Dy) + (y \cdot \nabla)\, y + \nabla p = f + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in} \quad \Omega, \end{cases} \tag{2}$$

where $y = y(x,t)$, $p = p(x,t)$ represent the “average" velocity field and pressure of a turbulent fluid whose particles in Ω are during the time interval $(0,T)$; $v = (v_j)_{1 \le j \le N}$ is a function which must be viewed as a control acting on the system, $f(x,t) = (f_i(x,t))_{1 \le i \le N} \in L^2(Q)^N$ a given force, applied externally; ν_0 and ν_1 are positive constants, where ν_0 represents the kinematic viscosity and $\nu_1 \|\nabla y\|^2_{L^2}$ the turbulent viscosity and Dy stands for the symmetrized gradient of y: $Dy = \frac{1}{2}(\nabla y + \nabla^T y)$. Moreover, $y(0)$ is the time average velocity $t = 0$ partially unknown.

In our context, the application of insensitizing controls is related to the study of fluids over an Ω region contained, for instance, in the ocean where it is difficult to measure the initial velocity and therefore such velocity is only partially known, see [FGO03] for insensitive controls for a linear quasi-geostrophic ocean model. Therefore, we want to find a control located in an accessible region $\omega \subset \Omega$ so that it provides us with information by means the functional about the fluid and this information is not influenced by lack of knowledge of the initial data.

In [HLC18], the local null controllability was studied for system (2) by means $N-1$ scalar controls for an arbitrary control domain. In our case, within the scope of the insensitive control problem, we will need to define weights with behaviors different from those considered in [HLC18], which will allow us to obtain estimates for the solution of a coupled linear system. Through these estimates we will be able to overcome the difficulties imposed by the term $\nabla \cdot ((\nu_0 + \nu_1 \|\nabla y\|^2_{L^2}) Dy)$.

Notice that, the system (2) is a particular case of

$$\begin{cases} y_t - \nabla \cdot \mathbf{T}(y,p) + (y \cdot \nabla) y = f & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ \text{etc.}, \end{cases} \tag{3}$$

where f is an external force field, $\mathbf{T}(y,p) := -pI + (\nu_0 + \nu_1|Dy|^{r-2})Dy$ is the stress tensor with $r > 2$ and

$$|Dy| := \left[\sum_{i,j=1}^{N} \frac{1}{2}\left(\frac{\partial y_j}{\partial x_i} + \frac{\partial y_i}{\partial x_j}\right)^2\right]^{1/2}.$$

The first mathematical studies on this type of equations were introduced by O. Ladyzhenskaya in the 1960s and can be found in [Lad66; Lad67; Lad68; Lad69]. Just as J.-L. Lions considered in his relevant book [Lio69] the case in which Dy is replaced by ∇y, that is, when the tensor stress is of the form $\mathbf{T_1}(y,p) = -pI + (\nu_0 + \nu_1|\nabla y|^{r-2})\nabla y$ and obtained important results of existence, uniqueness and regularity of solutions. For some regularity properties for the solutions of (3), see for instance [Vei07].

When $N = r = 3$ the model (3) is the classical turbulence model approached by Smagorinsky in [Sma63].

For additional investigations within the scope of control theory on variations of the (3) model, we recommend: [Car+22] in which analyzed the null controllability property when the stress tensor is the same as that considered by J.-L. Lions, that is, dependent on the state gradient; E. Fernández - Cara et al [FLM15], where the existence of local null controls was guaranteed for the case in which stress tensor is equal to $-pI + (\nu_0 + \nu_1(\|Dy\|^2_{L^2}))Dy)$ with ν_1 being a continuously differentiable function, that is, $0 \leq \nu_1 \leq C$ and $|\nu_1'| < C$. In this, the authors also provided a numerical approximation and illustrated the behavior of the algorithm with examples. And, finally, Guerrero and Takahashi on [GT21] that considered $\|\text{curl}(y)\|^2_{L^2}$ instead of $|Dy|^{r-2}$ and demonstrated the controllability by trajectories. To obtain this result, the authors needed to prove a Carleman estimate for the adjoint of a linear system equipped with a nonlocal spatial term.

Next, we will cite some articles on insensitive controls present in the literature.

Considering the semilinear heat system with globally Lipschitz nonlinearities of class $\mathcal{C}^1$ and $\omega \cap \mathcal{O} \neq \emptyset$, where $\omega \subset \Omega$ is the control set and $\mathcal{O} \subset \Omega$ is the observation set, Bodart and Fabre [BF95] weakened the definition of insensitizing controls and proved the existence of ε - insensitizing controls, i.e. they proved that, given $\varepsilon > 0$, there is a control v such that $|\frac{\partial \phi(y)}{\partial \tau}|_{\tau=0}| \leq \varepsilon$. For the same problem, [Ter00] extended the case by proving the existence of insensitive controls.

Still in the context of semilinear heat equations, [Ter97] and [BGP04a] proved the existence of insensitive controls in unbounded domains and for superlinear nonlinearities with regular bounded domains, respectively. For the case of the linear heat equation with disjoint regions of control and observation, that is, with $\omega \cap \mathcal{O} = \emptyset$ the authors of [MOT04] gave an example for the existence of ε - insensitizing controls.

Moreover, in [Ter00] it was proved also for the linear heat equation that when the control does not act everywhere in Ω we cannot expect that the insensitivity holds for all initial data. Thus, here we will assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y^0 \equiv 0$ which are classic hypotheses in insensitization problems. Also for the heat equation, [TZ09] performed a study on the possible conditions of the initial data that can be insensitized, for this the authors removed the condition $y^0 = 0$ when $\mathcal{O} \subset \omega$ and when $\mathcal{O} = \Omega$ and they concluded that if this is not the case, negative results occur. Therefore, this is a delicate issue to address.

With regard to insensitizing controls for fluid equations, the first result was obtained in [Pér04], Section 2.3, where the author established the existence of ε - insensitizing controls with one vanishing component, that is, of the form $(v_1, v_2, 0)$ for the three-dimensional Stokes system. Subsequently, also for the Stokes system, [Gue07a] obtained the existence of insensitive controls both for the case in which the sentinel is given by the L^2–norm of the state and the L^2–norm of the curl of the state. For the Navier–Stokes system we reference [CG14], which proves, extending the results of [Gue13], the existence of insensitive controls having one vanishing component. We also indicate [CP23], which insensitized the rotational of the solution using controls with one component fixed at zero.

For other studies of parabolic equations, see, for example, [Gue07b], [TK10], and [Liu12]. The first article to establish the existence of insensitizing controls for the L^2-norm of the gradient of solutions of linear heat equations. The second paper found the existence of ε-insensitizing controls for some parabolic equations when the control region and the observability region do not intersect, and the third proved the existence of insensitive controls in a Hölder space, for a class of quasilinear parabolic equations with homogeneous Dirichlet boundary conditions. In the latter, the author made use of fixed-point techniques. We also mention [ST19], where the authors proved the existence of insensitizing controls for the nonlinear Ginzburg-Landau equation considering a functional that depends on the gradient of the state. We also mention [CCC16], in which a nonlinear parabolic system modeling phase field phenomena is considered. Such a system is formed from two coupled parabolic equations, where the first one describes the temperature of the material and the second one describes a phase field function. In addition, we mention [BGP04b], which presents the existence of insensitizing controls for a semilinear heat equation in a bounded domain of $R^p\,(p \geq 1)$. Such semilinearity involves the state and gradient terms with homogeneous Dirichlet boundary conditions.

For the Boussinesq system without any control used on the temperature equation, [Car17] showed the existence of the insensitizing controls such that the control acting on the fluid equation can be chosen to have one vanishing component. Also about the Boussinesq system, but with controls act on both equations, when the case is three-dimensional [CGG15] demonstrated the existence of insensitive controls with two vanishing components, and for the two-dimensional case, the authors concluded that no control on the velocity equation is required.

Concerning insensitive controls for hyperbolic equations, we cite [Dág06] which provides a study on insensitizing controls for the uni-dimensional wave equation. The author of this work involves two cases: when the control acts in an interior region, and when it acts on the boundary. They conclude that, in both cases, the ϵ-insensitizing controllability holds when the control time is sufficiently large.

We now cite works that have been recently carried out in this area but that involve domain variations. In [ELP22], a quadratic functional involving the solution of the linear heat equation with respect to domain variations was insensitized. Boundary variations of the spatial domain on which the solution of the PDE is defined at each time were considered, and three main issues were investigated: approximate insensitizing, approximate insensitizing combined with an exact insensitizing for a finite-dimensional subspace, and exact insensitizing, which were defined by the authors. In [LPS19], a semi-linear heat equation with Dirichlet boundary conditions and globally Lipschitz nonlinearity was considered, posed on a bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), assumed to be an unknown perturbation of a reference domain.

For existence of insensitizing controls for a fourth-order nonlinear parabolic equation, see [YL22] and also [BV22]. In the second reference, it was addressed the existence of insensitizing controls was considered for a nonlinear coupled system of fourth, and second-order parabolic equations known as the stabilized Kuramoto-Sivashinsky model.

Finally, for a numerical proposal, [BHT19] addressed an insensitizing control problem in the discrete setting of finite-differences. The authors proved the existence of a control that insensitizes the norm of the observed solution of a 1-D semi-discrete parabolic equation and dealt with the problem of computing numerical approximations of insensitizing controls, featuring numerical illustrations, for the heat equation by using the Hilbert Uniqueness Method (HUM).

The second chapter addresses about local null controllability and large time null controllability for the complete Ladyzhenskaya-Boussinesq-type system with distributed controls supported in small sets. We consider the term $(\nu_0 + \nu_1\|\nabla y\|^2_{L^p})Dy : \nabla y$ on the right side of the temperature equation (represented by the variable θ), the which makes the system more realistic, difficult to analyze and control. We treat separately the cases in which $p = 2$ and in which $3 < p \leq 6$. In these equations describing a temperature-coupled differential turbulence model, we find local and nonlocal nonlinearities: the recurring transport terms and a turbulent viscosity that depends on the global in space energy dissipated by the mean flow. More specifically, we will study the null controllability for the nonlinear systems:

$$\begin{cases} y_t - \nabla\cdot(\nu(\nabla y)Dy) + (y\cdot\nabla)y + \nabla P = v\chi_\omega + \nu_0\theta e_N, \ \ \nabla\cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla\cdot(\nu(\nabla y)\nabla\theta) + y\cdot\nabla\theta = v_0\chi_\omega + \nu(\nabla y)Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{4}$$

where

$$\nu(\nabla y) := \nu_0 + \nu_1 \int_\Omega |\nabla y|^2 dx \tag{5}$$

and

$$\begin{cases} y_t - \nabla\cdot(\bar\nu(\nabla y)Dy) + (y\cdot\nabla)y + \nabla P = v\chi_\omega + \nu_0\theta e_N, \ \ \nabla\cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla\cdot(\bar\nu(\nabla\theta)\nabla\theta) + y\cdot\nabla\theta = v_0\chi_\omega + \bar\nu(\nabla y)Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x),\ \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{6}$$

where $\bar\nu(\nabla\varsigma) := \nu_0 + \nu_1\|\nabla\varsigma\|^2_{L^p}$, for $3 < p \leq 6$, $\omega\times(0,T)$ is the control domain and v (force) and v_0 (heat sources) represent the controls acting on the system, and in both systems

$$e_N = \begin{cases} (0,1) \text{ if } N = 2, \\ (0,0,1) \text{ if } N = 3. \end{cases}$$

As we are assuming on the right side of the heat equation the quadratic term $\nu(\nabla y)Dy : \nabla y$ or $\bar\nu(\nabla y)Dy : \nabla y$ which is related to the work done by viscous forces, the systems (4) and (6) can be considered generalizations of the complete Boussinesq model (which corresponds to several conservation laws involving momentum, mass and energy). Moreover, when $\nu_0 = 1$ and $\nu_1 = 0$ in (4), E. Fernández - Cara et al [FLH] proved that such system is locally null controllable. And, when we remove the entire term $\nu(\nabla y)Dy : \nabla y$ from the right side of equation (4), Huaman et al [HLC18] demonstrated that such a system is locally null controllable by means of $N - 1$ scalar controls for an arbitrary control domain.

We will now list some articles present in the literature that provided relevant controllability results for the Boussinesq system.

On the exact local controllability of trajectories, [Gue06] dealt with the Boussinesq system with $N + 1$ distributed scalar controls supported in small sets. In this interesting work, firstly, a Carleman inequality was proved for a linearized version of the Boussinesq system, which leads to its null controllability at any time $T > 0$. And from this, the result of exact controllability of trajectories was obtained. Still this context we mention, [Fer+06] in which the authors proved that through some hypotheses were imposed on the control domain and the trajectories, the Boussinesq system is locally exactly controllable by $N-1$ scalar controls at a time $T>0$ to the trajectories. Moreover, removing the geometric conditions imposed by [Fer+06], [Car12] concludes the exact local controllability to a particular class of trajectories with internal controls having two vanishing components. Also [FI98] and [FI99] proved, respectively, the local exact boundary controllability to the trajectories of the Boussinesq system with $N+1$ scalar controls acting over the whole boundary in a bounded domain of $\mathbb{R}^N$ $(N = 2\,\text{or}\,3)$ with C^∞-boundary and the local exact controllability to the same trajectories with $N+1$ scalar distributed controls, when the torus is the domain.

Considering a generalized Boussinesq equation in a periodic domain, a unit circle in the plane, [Zha98] showed that depending on the location of the control, whether in the entire domain or in a subdomain, and the amplitude of the initial and terminal states it is possible conclude that the system is globally exactly controllable.

The third chapter is dedicated to the existence and uniqueness of strong solutions for the Navier–Stokes equations in non-cylindrical domains. To do this, we make a modification to the penalty method introduced by Lions, J.-L. in 1964, using two penalty terms which have an elliptical relationship between them instead of a single term used by Lions, J.-L., the decay of the solutions is also proven. This method can also be used to obtain regular solutions in other nonlinear equations in non-cylindrical domains.

The use of the penalization method for evolution inequalities was systematically applied in [Lio68a], [Lio68b] and [FS69] introduced studies for nonlinear evolution problems, specifically for the equation of Navier-Stokes, in non-cylindrical open sets.

For other studies on the Navier-Stokes equation in non-cylindrical domains we cite: [Boc77] in which the author shows that under some smoothness conditions it is possible to obtain a Kiselev-Ladyzhenskaya type estimate in a way that takes into account the non-cylindrical nature of the domain, we also refer to [Sal88] which shows that for dimension 3 some regularity and decay results of this equation known in the cylindrical case are transferred in modified form to the non-cylindrical case. And, also for the three-dimensional case, [NS98] who added a boundary condition analogous to the Neumann condition, thus calling his problem the second boundary problem for the Navier-Stokes system and showed the existence of a strong solution for such. In [ÔY78], the authors demonstrate the existence of solutions using subdifferential operator theory. Specifically, the problem is transformed into an abstract equation in an appropriate Hilbert space, which can be considered a perturbed equation resulting from a time-dependent subdifferential operator. The desired solutions are subsequently constructed using the successive approximation method. An alternative method, see [MT82], involves reducing the problem to a cylindrical domain. This reduction is based on the assumption that a diffeomorphism exists, which maps the given time-dependent domain to a cylindrical domain, and further assumes that the Jacobian of the diffeomorphism depends solely on the time variable. In [Lím+05] we can see the Navier-Stokes equation as a particular case of a study that shows the existence of weak solutions of equations that represent non-homogeneous viscous incompressible fluid flows in a non-cylindrical domain in $\mathbb{R}^3$. And finally, [ML97] studies the Navier-Stokes system using singular perturbation method that consists of transforming a parabolic problem into a family of elliptic problems indexed by a parameter $\varepsilon > 0$. The authors solve the problems using elliptic methods to achieve the solution of the original parabolic problem as a limit as ε tends to zero.

A study that used the method of penalizing the Dirichlet problem for the Navier-Stokes-Fourier system was carried out by [Bas+22]. In this, the authors demonstrated a strong convergence of penalized solutions to the solution of the limit problem and presented numerical simulations illustrating the robustness and efficiency of the proposed penalization strategy in solving the system in complex domains.

Other interesting works concerning the application of the penalization method in non-cylindrical domains are, for example, [LCM04] and [CLB08]. In the first, the authors establish existence, uniqueness and regularity of solutions for a mixed problem associated with equations of Benjamin-Bona-Mahony type in a domain non-cylindrical with moving boundary. The technique consists of transforming the non-cylindrical domain into a Q cylinder using a diffeomorphism and applying the Faedo-Galerkin method on Q to the transformed mixed problem. The uniqueness of the solutions is proved using the energy method in the non-cylindrical domain. The second, presents results on the existence of global solutions and an estimate of the decay rate of weak global solutions for energy associated with an initial and boundary value problem for a beam evolution equation (which describes a small vertical flexion fully clamped) with variable coefficients in non-cylindrical domains. Moreover, the penalty method (refer to [AG93]) is employed for the numerical approximation of the Navier-Stokes problem in a non-cylindrical domain.

For other subjects about the Navier-Stokes equation see for example: [Fer+04] and [Fer+06], where in the first the authors deal with the exact local controllability of the Navier-Stokes system with distributed controls supported in small sets. In it, they present a new Carleman inequality for the linearized Navier-Stokes system, obtaining from this the null controllability and consequently a local result related to the exact controllability for the trajectories of the Navier-Stokes system. And the second extends [Fer+04] presenting some new results for the N-dimensional Navier-Stokes system.

CHAPTER 1

Insensitizing controls with N-1 components for the N-Dimensional Ladyzhenskaya-Smagorinsky system

1.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^N (N = 2$ or 3), be an open, connected, bounded non-empty set with boundary $\Gamma = \partial\Omega$ of class $\mathcal{C}^\infty$. We fix $T > 0$ and denote by Q the cylinder $Q = \Omega \times (0,T)$, with side boundary $\Sigma = \Gamma \times (0,T)$. Let us also consider $\omega \subset \Omega$ open (small) non-empty, which is the *control set*, $\chi_\omega \in C_0^\infty(\Omega)$ satisfies $0 < \chi_\omega \leq 1$ in ω, and $\chi_\omega = 0$ outside ω. Let $\mathcal{O} \subset \Omega$ be another open set called the *observation set*. The symbol C will be used to design a generic positive constant.

We denote by $(.,.)$ and $\|.\|$, respectively, the inner product and norm in L^2 in Ω and Q.

Let us recall the definition of some vector spaces in the context of incompressible fluids:

$$\mathrm{V} = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \in \Omega\}$$

and

$$\mathrm{H} = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega, y \cdot \mathrm{n} = 0 \text{ on } \partial\Omega\},$$

where n is the normal vector exterior to $\partial\Omega$, $L^2(\Omega)$ is the space of square integrable functions and $H_0^1(\Omega)$ is the closure of the space of test functions in Ω, $\mathcal{D}(\Omega)$, in $H^1(\Omega)$ the standard Hilbert space.

Consider the following model of Ladyzhenskaya-Smagorinsky type, which describes a model for the movement of incompressible viscous flows (see, [Lad67; Lad68; Lad69] for more details), with incomplete data:

$$\begin{cases} y_t - \nabla \cdot ((\nu_0 + \nu_1 \|\nabla y\|^2) D y) + (y \cdot \nabla)\, y + \nabla p = f + v\chi_\omega & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $y = y(x,t)$, $p = p(x,t)$ represent the "average" velocity field and pressure of a turbulent fluid whose particles in Ω are during the time interval $(0,T)$; $v = (v_j)_{1 \leq j \leq N}$ is a function which must be viewed as a control acting on the system; $f(x,t) = (f_i(x,t))_{1 \leq i \leq N} \in L^2(Q)^N$ a given force, applied externally; ν_0 and ν_1 are positive constants, where ν_0 represents the kinematic viscosity and $\nu_1\|\nabla y\|^2$ the turbulent viscosity and Dy stands for the symmetrized gradient of y: $Dy = \frac{1}{2}(\nabla y + \nabla^T y)$. Moreover, $y(0)$ is the time average velocity $t = 0$ partially unknown in the following sense:

- $y^0 \in \mathrm{V}$ is known;
- $\widehat{y}^0 \in \mathrm{V}$ is unknown with $\|\widehat{y}^0\|_{H_0^1(\Omega)^N} = 1$; and
- τ is a small, unknown real number.

When $\tau = 0$, f and $v\chi_\omega \in L^2(\Omega \times (0,T))^N$, the guarantee of exactly one strong (y,p) solution of (1.1) in the class $y \in L^2(0,T; D(A)) \cap C^0([0,T]; \mathrm{V})$, $y_t \in L^2(0,T; \mathrm{H})$ was given by [FLM15; HLC18], where $D(A) = H^2(\Omega)^N \cap \mathrm{V}$.

Here, we are interested in proving the existence of controls that insensitize some functional that depends on the velocity field y, following the literature [Lio92], the usual sentinel functional is given by the square of the norm in the L^2 local of state variable y, that is,

$$J_\tau(y) = \frac{1}{2} \iint\limits_{\mathcal{O}\times(0,T)} |y|^2 \, dx \, dt. \tag{1.2}$$

Then, the insensitizing control problem is to find a control v such that the influence of the unknown initial data $\tau\widehat{y}^0$ is imperceptible to our functional, i.e.,

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = 0 \; \forall \, \widehat{y}^0 \in H_0^1(\Omega)^N \text{ such that } \|\widehat{y}^0\|_{H_0^1(\Omega)^N} = 1. \tag{1.3}$$

When (1.3) holds, the functional J_τ is locally insensitive to the perturbation $\tau\widehat{y}^0$ and then we say that the control v insensitizes J_τ.

Of course, several options for the functional are possible, such as

$$K_\tau(y) = \tfrac{1}{2}\|\nabla y\|^2_{L^2(\mathcal{O}\times(0,T))^N}, I_\tau(y) = \tfrac{1}{2}\|\nabla \times y\|^2_{L^2(\mathcal{O}\times(0,T))^{N(N-1)/2}}, \tag{1.4}$$

and others. The choice of these functionals determines the degree of complexity of the cascade system, some of which are open problems. This will be discussed in more detail later.

The main goal of this chapter is to obtain the existence of insensitizing controls for (1.1) having one vanishing component, that is, $v_i \equiv 0$ for any given $i \in \{1, \ldots, N\}$. That said, we are interested in solving the following theorem:

Theorem 1.1. *Let $i \in \{1, \ldots, N\}$ and $m \geq 10$ be a real number. Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y^0 \equiv 0$. Then, there exist $\delta > 0$ and $C > 0$, depending on $\omega, \Omega, \mathcal{O}$, and T, such that for any f satisfying $e^{C/t^m} f, e^{C/t^m} f_t \in L^2(Q)^N$, $(e^{C/t^m} f)(0) \in H_0^1(\Omega)^N$ with $\|e^{C/t^m} f\|_{L^2(Q)^N} + \|e^{C/t^m} f_t\|_{L^2(Q)^N} + \|(e^{C/t^m} f)(0)\|_{H_0^1(\Omega)^N} < \delta$, there exists a control $v \in L^2(\omega \times (0,T))^N$ with $v_i \equiv 0$, which insensitizes the functional* (1.2).

This chapter is organized as follows: In Section 1.2, we show that if the solution of the nonlinear cascade system governed by one equation forward in time and one backwards verifies the null controllability in the variable retrograde in time, then the function v insensitizes the sentinel (1.2) in the sense of (1.3). In other words, we reduce the insensitivity problem for (1.1) to a non-standard null controllability problem. In Section 1.3, we established null controllability for the linearization of the cascade system. This is done using already known Carleman estimates. Furthermore, we prove some technical lemmas that will allow us to obtain regularity estimates that will be of great importance for the null controllability of the cascaded nonlinear system. In Section 1.4, we established the existence of insensitive controls for (1.1), which will be done through the null controllability of the cascaded nonlinear system. To prove this null controllability, we will reduce the controllability problem to an abstract equation, defining an operator $\mathcal{A}$ through the equations and initial conditions of the mentioned cascade system. This abstract equation satisfies the conditions of the Liusternik's Inverse Mapping Theorem, and this is confirmed by the proof of three lemmas. This guarantees the veracity of Theorem 1.1. Finally, we added Appendix A.1 which contains the well-posedness of the nonlinear cascade system studied throughout the study.

1.2 Reduction of the insensitizing problems

In order to prove the existence of insensitizing controls (Theorem 1.1), as usual, we need to reduce the insensibilizing problem to a nonstandard null controllability problem of a nonlinear cascade system. As already indicated, the coupling term is linked to the derivative of the functional (1.2) with respect to τ at $\tau = 0$. That being said, we note the following.

Remark 1.1. *First, we will denote by (y_τ, p_τ) the derivatives of the solution (y, p) of* (1.1) *with respect to τ. Therefore, considering the functional* (1.2)*, we have*

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = \int_0^T \int_{\mathcal{O}} w y_\tau \, dx \, dt \tag{1.5}$$

where w is the solution of (1.1) *when $\tau = 0$ and (y_τ, p_τ) satisfies*

$$\begin{cases} \mathcal{L}_w(y_\tau) + \nabla p_\tau = 0, \;\; \nabla \cdot y_\tau = 0 & in \quad Q, \\ y_\tau = 0 & on \quad \Sigma, \\ y_\tau(0) = \widehat{y}^0 & in \quad \Omega, \end{cases}$$

where the operator $\mathcal{L}_w$ is such that

$$\mathcal{L}_w(y_\tau) = y_{\tau,t} - \left(\nu_0 + \nu_1 \|\nabla w\|^2\right) \Delta y_\tau - 2\nu_1 \left(\nabla w, \nabla y_\tau\right) \Delta w + \left(y_\tau \cdot \nabla\right) w + (w \cdot \nabla) y_\tau.$$

If (w, z, p^0, q) solves the system

$$\begin{cases} \mathcal{L}_1(w) + \nabla p^0 = f + v\chi_\omega, \;\; \nabla \cdot w = 0 & in \quad Q, \\ \mathcal{L}_w^*(z) + \nabla q = w\chi_{\mathcal{O}}, \;\; \nabla \cdot z = 0 & in \quad Q, \\ w = 0, \;\; z = 0 & on \quad \Sigma, \\ w(0) = 0, \;\; z(T) = 0 & in \quad \Omega, \end{cases} \tag{1.6}$$

where $\mathcal{L}_1$ is an operator such that $\mathcal{L}_1(w) = w_t - \nabla \cdot \big((\nu_0 + \nu_1 \|\nabla w\|^2) Dw\big) + (w \cdot \nabla)\, w$ and $\mathcal{L}_w^*$ is the adjoint operator of $\mathcal{L}_w$, that is,

$$\mathcal{L}_w^*(z) = -z_t - \nabla \cdot \big((\nu_0 + \nu_1 \|\nabla w\|^2) Dz\big) + 2\nu_1 \left((\Delta w, z)_{L^2} \Delta w\right) + \left(z \cdot \nabla^t\right) w - (w \cdot \nabla)\, z,$$

then it is verified due to (1.5), (1.6)*, and the definition of the adjoint that*

$$\begin{aligned} \left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} &= \int_0^T \int_{\mathcal{O}} w y_\tau \, dx \, dt = \int_0^T \int_{\Omega} \left(\mathcal{L}_w^*(z) + \nabla q\right) y_\tau \, dx \, dt \\ &= \int_0^T \int_{\Omega} z \left(\mathcal{L}_w(y_\tau) + \nabla p_\tau\right) \, dx \, dt + \int_{\Omega} \widehat{y}^0 z(0) \, dx = \int_{\Omega} \widehat{y}^0 z(0) \, dx. \end{aligned}$$

Therefore, so that we have $\frac{\partial J_\tau(y)}{\partial \tau}|_{\tau=0} = 0$*, we need to solve the controllability problem* (1.6) *with* $z(0) = 0$*.*

In summary, when choosing the functional J_τ*, condition* (1.3) *becomes equivalent to solving a non-standard null controllability problem of a nonlinear coupled system governed by a forward in time equation and another backwards.*

The change in the sentinel functional would directly imply the second equation of the cascade system (1.6)*. More specifically, considering the functional* K_τ *(resp.* I_τ*), given in* (1.4)*, we would have the substitution of* $w\chi_{\mathcal{O}}$ *in* (1.6) *by* $\nabla\cdot(\nabla w\chi_{\mathcal{O}})$ *(resp.* $\nabla\times((\nabla\times w)\chi_{\mathcal{O}})$*), that is, the coupling terms depend on the sentinel considered. Certainly, these new terms would offer us greater difficulties in obtaining regularities and Carleman estimates, and are therefore still open problems. For more details on the difficulties imposed by these functionals for the Stokes, parabolic, and Ginzburg–Landau equations, we strongly recommend reading [Gue07a], [Gue07b], and [ST19], respectively.*

The formulation of the result presented in Remark 1.1 is given by the following proposition.

Proposition 1.1. *Let* y *be the solution of* (1.1) *with* $y^0 = 0$*. If the* (w, z) *of the solution of the nonlinear cascade system:*

$$\begin{cases} w_t - \nabla\cdot\big((\nu_0+\nu_1\|\nabla w\|^2)Dw\big) + (w\cdot\nabla)\,w + \nabla p^0 = f + v\chi_\omega & \text{in} \quad Q,\\ \nabla\cdot w = 0 & \text{in} \quad Q,\\ -z_t - \nabla\cdot\big((\nu_0+\nu_1\|\nabla w\|^2)Dz\big) + 2\nu_1\left((\Delta w, z)_{L^2}\Delta w\right) + \left(z\cdot\nabla^t\right)w & \\ -\,(w\cdot\nabla)\,z + \nabla q = w\chi_{\mathcal{O}},\ \ \nabla\cdot z = 0 & \text{in} \quad Q,\\ w = 0,\ \ z = 0 & \text{on} \quad \Sigma,\\ w(0) = 0,\ \ z(T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.7}$$

satisfies $z(0) = 0$ *in* Ω*, then* v *insensitizes the functional* J_τ *(defined by* (1.2)*). We have denoted*

$$\left(\left(z\cdot\nabla^t\right)w\right)_i = \sum_{j=1}^N z_j\partial_i w_j,\ i = 1,\ldots,N,$$

and w *the solution of* (1.1) *when* $\tau = 0$*.*

Proof. To demonstrate this result it is sufficient to prove that v satisfies (1.3). Throughout the proof, we will see that the equation of z corresponds to a formal adjoint of the equation governed by the derivative of y with respect to τ at $\tau = 0$. Let be any $\widehat{y}_0 \in H_0^1(\Omega)^N$ with $\|\widehat{y}_0\|_{H_0^1(\Omega)^N} = 1$ and denote by y the solution of equation (1.1) associated to τ and v. Then,

$$\begin{aligned} \left.\frac{\partial J_\tau(y)}{\partial\tau}\right|_{\tau=0} &= \frac{1}{2}\lim_{\tau\to 0}\int_0^T\int_{\mathcal{O}}\frac{y^2-w^2}{\tau}dx\,dt\\ &= \frac{1}{2}\lim_{\tau\to 0}\int_0^T\int_{\mathcal{O}}(y+w)\frac{y-w}{\tau}dx\,dt, \end{aligned}$$

where w is the solution of (1.1) when $\tau = 0$, that is, (w, p^0) is the solution of

$$\begin{cases} w_t - \nabla\cdot((\nu_0+\nu_1\|\nabla w\|^2)Dw) + (w\cdot\nabla)w + \nabla p^0 = f + v\chi_\omega, & \\ \nabla\cdot w = 0 & \text{in} \quad Q,\\ w = 0 & \text{on} \quad \Sigma,\\ w(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

We want convergence

$$y \longrightarrow w \ \text{ in} \, L^2(Q)^N, \text{ as} \, \tau \longrightarrow 0. \tag{1.9}$$

Write $h = y - w$. Then, h satisfies

$$\begin{cases} h_t - \nabla \cdot ((\nu_0 + \nu_1 \|\nabla y\|^2) Dy - (\nu_0 + \nu_1 \|\nabla w\|^2) Dw) + (y \cdot \nabla) y & \\ -(w \cdot \nabla) w + \nabla p - \nabla p^0 = 0, \ \ \nabla \cdot h = 0 & \text{in} \quad Q, \\ h = 0 & \text{on} \quad \Sigma, \\ h(0) = \tau \widehat{y}^0 & \text{in} \quad \Omega, \end{cases} \tag{1.10}$$

Multiplying both sides of the first equation of (1.10) by h and integrating it in Ω, we obtain that

$$\frac{1}{2} \frac{d}{dt} \int\limits_{\Omega} |h^2(t)| \, dx + \nu_0 \|h\|_{\mathrm{V}}^2 + \int\limits_{\Omega} \nu_1 \|\nabla y\|^2 |\nabla h|^2 dx \leq \frac{\nu_0}{2} \|h\|_{\mathrm{V}}^2 + C \|\Delta w\|_{L^2(\Omega)^N}^2 \|h\|_{\mathrm{H}}^2.$$

Multiplying by 2 and integrating from 0 to t,

$$\|h(t)\|^2 + \int_0^t \nu_0 \|h\|_{\mathrm{V}}^2 \, ds + 2 \int_0^t \int\limits_{\Omega} \nu_1 \|\nabla y\|^2 |\nabla h|^2 dx \, ds \leq \|h(0)\|^2$$

$$+ C \int_0^t \|\Delta w\|^2 \|h\|^2 ds$$

By the Gronwall's Lemma, we have

$$\|h(t)\|_{\mathrm{H}}^2 + \int_0^T \nu_0 \|h\|_{\mathrm{V}}^2 + 2 \int_0^T \int\limits_{\Omega} \nu_1 \|\nabla y\|^2 |\nabla h| dx \, ds \leq \tau^2 \|\widehat{y}^0\|_{H_0^1(\Omega)^N}^2 e^{\int_0^T \|\Delta w\|^2 dt}. \tag{1.11}$$

Consequently, $|h|_{C([0,T];\mathrm{H})} \longrightarrow 0, \text{ as} \, \tau \longrightarrow 0$. This yields (1.9). Furthermore, we also obtain from (1.11) that

$$|h|_{L^2(0,T;\mathrm{V})} \longrightarrow 0, \ \text{ as } \ \tau \longrightarrow 0. \tag{1.12}$$

Next, we will see that

$$\frac{y - w}{\tau} \longrightarrow y_\tau \ \text{ in } \ L^2(Q)^N, \ \text{ as } \ \tau \longrightarrow 0. \tag{1.13}$$

where y_τ is the derivative of the solution y with respect to τ at $\tau = 0$, that is, since we have y solution of (1.1) differentiating y with respect to τ at $\tau = 0$ we obtain (y_τ, p_τ) solution of the system

$$\begin{cases} y_{\tau,t} - \left(\nu_0 + \nu_1 \|\nabla w\|^2\right) \Delta y_\tau - 2\nu_1 \left(\nabla w, \nabla y_\tau\right) \Delta w + \left(y_\tau \cdot \nabla\right) w & \\ +(w \cdot \nabla) y_\tau + \nabla p_\tau = 0, \ \ \nabla \cdot y_\tau = 0 & \text{in} \quad Q, \\ y_\tau = 0 & \text{on} \quad \Sigma, \\ y_\tau(0) = \widehat{y}^0 & \text{in} \quad \Omega. \end{cases} \tag{1.14}$$

To show (1.13), let $h_\tau = \dfrac{y - w}{\tau} - y_\tau$. Then, h_τ satisfies

$$\begin{cases} h_{\tau,t} - \nu_0 \, \Delta h_\tau - \nu_1 \left[\dfrac{\|\nabla y\|^2 \Delta y - \|\nabla w\|^2 \Delta w}{\tau} - \|\nabla w\|^2 \Delta y_\tau - 2 \left(\nabla w, \nabla y_\tau\right) \Delta w \right] & \\ + \dfrac{(y \cdot \nabla) \, y - (w \cdot \nabla) \, w}{\tau} - (y_\tau \cdot \nabla) \, w - (w \cdot \nabla) \, y_\tau + \dfrac{\nabla p - \nabla p^0}{\tau} - \nabla p_\tau = 0 & \text{in} \quad Q, \\ \nabla \cdot h_\tau = 0 & \text{in} \quad Q, \\ h_\tau = 0 & \text{on} \quad \Sigma, \\ h_\tau(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.15}$$

Rephrasing the first equation of the system (1.15),

$$
\begin{aligned}
&h_{\tau,t} - \nu_0\,\Delta h_\tau - \nu_1[\|\nabla w\|^2 \left(\frac{\Delta y - \Delta w}{\tau} - \Delta y_\tau\right) + \left(\frac{\|\nabla y\|^2 - \|\nabla w\|^2}{\tau}\right)\Delta y.\\
&-2\,(\nabla w, \nabla y_\tau)\,\Delta w] + \frac{((y-w)\cdot\nabla)y}{\tau} + (w\cdot\nabla)\left(\frac{y-w}{\tau}\right) - (y_\tau\cdot\nabla)\,w - (w\cdot\nabla)\,y_\tau\\
&+\frac{\nabla p - \nabla p^0}{\tau} - \nabla p_\tau\\
&= h_{\tau,t} - \nu_0\,\Delta h_\tau - \nu_1\left[\|\nabla w\|^2\Delta h_\tau + \left(\frac{\|\nabla y\|^2 - \|\nabla w\|^2}{\tau}\right)\Delta y - 2\,(\nabla w, \nabla y_\tau)\,\Delta w\right]\\
&+(w\cdot\nabla)\,h_\tau + \frac{((y-w)\cdot\nabla)y}{\tau} - (y_\tau\cdot\nabla)\,w + \frac{\nabla p - \nabla p^0}{\tau} - \nabla p_\tau.
\end{aligned}
$$

So, we can rewrite the system (1.15) as follows:

$$
\begin{cases}
h_{\tau,t} - (\nu_0 + \nu_1\|\nabla w\|^2)\Delta h_\tau - \nu_1\left(\dfrac{\|\nabla y\|^2 - \|\nabla w\|^2}{\tau}\right)\Delta y & \\
+\,2\nu_1\,(\nabla w, \nabla y_\tau)\,\Delta w + (w\cdot\nabla)\,h_\tau + \dfrac{((y-w)\cdot\nabla)y}{\tau} - (y_\tau\cdot\nabla)\,w & \\
+\dfrac{\nabla p - \nabla p^0}{\tau} - \nabla p_\tau = 0,\ \ \nabla\cdot h_\tau = 0 & \text{in} \quad Q,\\
h_\tau = 0 & \text{on} \quad \Sigma,\\
h_\tau(0) = 0 & \text{in} \quad \Omega.
\end{cases}
\tag{1.16}
$$

Multiplying (1.16) by h_τ and integrating it in Ω we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|h_\tau\|^2 + \nu_0\|\nabla h_\tau\|^2 + \int_\Omega \nu_1\|\nabla w\|^2|\nabla h_\tau|^2\,dx\\
&= \nu_1\int_\Omega \left[\left(\frac{\|\nabla y\|^2 - \|\nabla w\|^2}{\tau}\right)\Delta y\,h_\tau - 2\,(\nabla w, \nabla y_\tau)\,\Delta w\,h_\tau\right]dx.
\end{aligned}
$$

Notice that,

$$
\begin{aligned}
&\bullet\ \nu_1\int_\Omega \left[\left(\frac{\|\nabla y\|^2 - \|\nabla w\|^2}{\tau}\right)\Delta y\,h_\tau - 2\,(\nabla w, \nabla y_\tau)\,\Delta w\,h_\tau\right]dx\\
&= \nu_1\int_\Omega \left[\frac{(\nabla y, \nabla y) - (\nabla y, \nabla w) + (\nabla y, \nabla w) - (\nabla w, \nabla w)}{\tau}\right]\Delta y\,dx\\
&\quad -\nu_1\int_\Omega 2(\nabla w, \nabla y_\tau)\Delta w\,h_\tau\,dx\\
&= \nu_1\int_\Omega \left[\frac{(\nabla y + \nabla w, \nabla y - \nabla w)}{\tau}\,\Delta y - 2(\nabla w, \nabla y_\tau)\Delta w\right]\,h_\tau\,dx\\
&= \nu_1\int_\Omega (\nabla y, \nabla h_\tau)\Delta y\,h_\tau\,dx + \nu_1\int_\Omega(\nabla w, \nabla h_\tau)\Delta y\,h_\tau\,dx\\
&\quad +\nu_1\int_\Omega(\nabla h, \nabla y_\tau)\Delta y\,h_\tau\,dx + \ 2\nu_1\int_\Omega(\nabla w, \nabla y_\tau)\Delta h\,h_\tau\,dx\\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}
\tag{1.17}
$$

By the Hölder's and Young's inequality, for any $\epsilon > 0$

$$K_1 \le \nu_1 \|\nabla y\| \, \|\nabla h_\tau\| \int_\Omega |\Delta y \, h_\tau| \, dx \le \nu_1 \|\nabla y\| \, \|\nabla h_\tau\| \, \|\Delta y\| \, \|h_\tau\|$$

$$\le \frac{\nu_0}{\epsilon} \|\nabla h_\tau\|^2 + C_\epsilon \|h_\tau\|^2;$$

$$K_2 \le \nu_1 \|\nabla w\| \, \|\nabla h_\tau\| \, \|\Delta y\| \, \|h_\tau\| \le \frac{\nu_0}{\epsilon} \|\nabla h_\tau\|^2 + C_\epsilon \|h_\tau\|^2;$$

$$K_3 \le \nu_1 \|\nabla h\| \, \|\nabla y_\tau\| \, \|\Delta y\| \, \|h_\tau\| \le \frac{\nu_0}{\epsilon} \|\nabla y_\tau\|^2 \, \|\nabla h\|^2 + C_\epsilon \|h_\tau\|^2;$$

Also,

$$K_4 = 2\nu_1 \int_\Omega (\nabla w, \nabla y_\tau) \Delta h \, h_\tau \, dx = -2\nu_1 \int_\Omega (\nabla w, \nabla y_\tau) \, \nabla h \, \nabla h_\tau \, dx$$

$$\le 2\nu_1 \|\nabla w\| \, \|\nabla y_\tau\| \, \|\nabla h\| \, \|\nabla h_\tau\|$$

$$\le \frac{\nu_0}{\epsilon} \|\nabla h_\tau\|^2 + C_\epsilon \|\nabla y_\tau\|^2 \|\nabla h\|^2.$$

Therefore, from (1.17) we have

$$\nu_1 \int_\Omega \left(\frac{\|\nabla y\|^2 - \|\nabla w\|^2}{\tau} \right) \Delta y \, h_\tau - 2 \, (\nabla w, \nabla y_\tau) \, \Delta w \, h_\tau \, dx$$

$$\le 3 \frac{\nu_0}{\epsilon} \|\nabla h_\tau\|^2 + \left(\frac{\nu_0}{\epsilon} + C_\epsilon \right) \|\nabla y_\tau\|^2 \, \|\nabla h\|^2 + C_\epsilon \|h_\tau\|^2.$$

Thus,

$$\frac{1}{2} \frac{d}{dt} \|h_\tau\|^2 + \nu_0 \|\nabla h_\tau\|^2 + \int_\Omega \nu_1 \|\nabla w\|^2 |\nabla h_\tau|^2 \, dx$$

$$\le 3 \frac{\nu_0}{\epsilon} \|\nabla h_\tau\|^2 + \left(\frac{\nu_0}{\epsilon} + C_\epsilon \right) \|\nabla y_\tau\|^2 \, \|\nabla h\|^2 + C_\epsilon \|h_\tau\|^2.$$

Taking, $\epsilon = 6$,

$$\frac{1}{2} \frac{d}{dt} \|h_\tau\|^2 + \frac{\nu_0}{2} \|\nabla h_\tau\|^2 + \int_\Omega \nu_1 \|\nabla w\|^2 |\nabla h_\tau|^2 \, dx \le \left(\frac{\nu_0}{6} + C \right) \|\nabla y_\tau\|^2 \, \|\nabla h\|^2$$

$$+ C \|h_\tau\|^2.$$

So, multiplying the previous inequality by 2 and integrating from 0 to t,

$$\|h_\tau(t)\|^2 + \int_0^t \nu_0 \|\nabla h_\tau\|^2 \, ds + 2\nu_1 \int_0^t \int_\Omega \nu_1 \|\nabla w\|^2 |\nabla h_\tau|^2 \, dx \, ds$$

$$\le 2 \left(\frac{\nu_0}{6} + C \right) \int_0^t \|\nabla y_\tau\|^2 \, \|\nabla h\|^2 \, ds + C \int_0^t \|h_\tau\|^2 \, ds.$$

Since $y_\tau \in C(0, T; \mathrm{V})$ (see Appendix A.1), we can make use of Gronwall's Lemma and (1.12) to deduce that

$$\|h_\tau(t)\|^2 + \int_0^t \nu_0 \|\nabla h_\tau\|^2 \, ds + 2\nu_1 \int_0^t \int_\Omega \nu_1 \|\nabla w\|^2 |\nabla h_\tau|^2 \, dx \, ds$$

$$\le C \left(\int_0^T \|h\|_{\mathrm{V}}^2 \right) e^{\int_0^T C} \longrightarrow 0, \text{ as } \tau \longrightarrow 0.$$

and consequently we have the convergence

$$h_\tau \longrightarrow 0 \text{ in } C(0, T; \mathrm{H}), \text{ as } \tau \longrightarrow 0.$$

Then, (1.13) holds.

Combining (1.9) and (1.13), we find that

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = \frac{1}{2}\lim_{\tau\to 0}\int_0^T \int_{\mathcal{O}} (y+w)\frac{y-w}{\tau}\,dx\,dt = \int_0^T\int_{\mathcal{O}} w y_\tau\,dx\,dt. \tag{1.18}$$

On the other hand, multiplying the first equation of (1.14) by z, integrating it in Q and using integration by parts, one obtains that

$$\iint_Q \big[-z_t - \nabla\cdot\big((\nu_0+\nu_1\|\nabla w\|^2)Dz\big) + 2\nu_1\left((\Delta w, z)\,\Delta w\right) + (z\cdot\nabla^t)w - (w\cdot\nabla)z$$
$$+\nabla q\big]\,y_\tau\,dx\,dt = \,(y_\tau(0), z(0)) - (y_\tau(T), z(T))\,.$$

Hence, since every term in brackets is equal to $w\chi_{\mathcal{O}}$ and $z(T)=0$ we obtain

$$\int_0^T\int_{\mathcal{O}} w\,y_\tau\,dx\,dt = \int_\Omega \widehat{y}^0 z(0)\,dx \;\;\forall\;\widehat{y}^0\in H_0^1(\Omega)^N \text{ with } \|\widehat{y}^0\|_{H_0^1(\Omega)^N}=1. \tag{1.19}$$

By (1.18) and (1.19), it follows that

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = 0 \;\text{ if and only if }\; z(0)=0 \,\text{in}\,\Omega.$$

Therefore, v insensitizes the functional J_τ, and the proof of Proposition 1.1 is complete. □

Remark 1.2. *Notice that it is natural to think of the turbulence model* (1.7)*, since* (1.19) *is obtained from the solution by transposition of* (1.14)*. Indeed, a solution by transposition of* (1.14) *is a unique function* $y_\tau\in L^2(Q)^N$ *satisfying*

$$\iint_Q y_\tau\,\mathrm{h}\,dx\,dt = \int_\Omega \widehat{y}^0 z(0)\,dx,\;\forall\,\mathrm{h}\in L^2(Q)^N,$$

where, for each $\mathrm{h}\in L^2(Q)^N$*, the associated solution* (z,q) *satisfies the corresponding adjoint system*

$$\begin{cases} \mathcal{L}_w^*(z)+\nabla q = \mathrm{h}, \;\; \nabla\cdot z = 0 & in \quad Q,\\ z=0 & on \quad \Sigma,\\ z(T)=0 & in \quad \Omega,\end{cases}$$

in which w *is the solution of* (1.8) *and*

$$\mathcal{L}_w^*(z) = -z_t - \nabla\cdot\big((\nu_0+\nu_1\|\nabla w\|^2)Dz\big) + 2\nu_1\left((\Delta w, z)_{L^2}\,\Delta w\right) + \left(z\cdot\nabla^t\right)w - (w\cdot\nabla)\,z.$$

Therefore, considering in particular $\mathrm{h}=w\chi_{\mathcal{O}}$ *we acquire the system* (1.7).

1.3 Preliminary results

Carleman Estimates

We will list here some global estimates of Carleman. To establish these inequalities, let us introduce some weight functions. Let ω_0 be a non-empty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega\cap\mathcal{O}$, and $\eta\in\mathcal{C}^2(\overline{\Omega})$ such that

$$|\nabla\eta|>0 \,\text{in}\, \overline{\Omega\setminus\omega_0}, \quad \eta>0\,\text{in}\,\Omega, \;\text{ and }\; \eta\equiv 0 \;\text{ on}\,\partial\Omega.$$

The existence of such a functions η is given in [FI96]. Also, let $\ell \in \mathcal{C}^\infty([0,T])$ be a positive function in $(0,T)$ satisfying

$$\begin{cases} \ell(t) = t, \quad \forall t \in [0, T/4], \qquad\qquad \ell(t) = T - t, \quad \forall t \in [3T/4, T], \\ \ell(t) \le \ell(T/2), \quad \forall t \in [0,T]. \end{cases}$$

Then, for all $\lambda \ge 1$ and $m \ge 10$, we consider the following weight functions:

$$\begin{aligned} \alpha(x,t) &= \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell(t)^m}, \qquad & \xi(x,t) &= \frac{e^{\lambda\eta(x)}}{\ell(t)^m} \\ \alpha^*(t) &= \max_{x\in\overline{\Omega}} \alpha(x,t), & \xi^*(t) &= \min_{x\in\overline{\Omega}} \xi(x,t), \\ \hat{\alpha}(t) &= \min_{x\in\overline{\Omega}} \alpha(x,t), & \hat{\xi}(t) &= \max_{x\in\overline{\Omega}} \xi(x,t). \end{aligned} \tag{1.20}$$

Let us to introduce a Carleman estimate for the Stokes coupled system:

$$\begin{cases} -\varphi_t - \nu_0\Delta\varphi + \nabla\pi = g^0 + \psi\chi_{\mathcal{O}}, \;\; \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \psi_t - \nu_0\Delta\psi + \nabla\kappa = g^1, \;\; \nabla\cdot\psi = 0 & \text{in} \quad Q, \\ \varphi = 0, \;\; \psi = 0 & \text{on} \quad \Sigma, \\ \varphi(T) = 0, \;\; \psi(0) = \psi^0 & \text{in} \quad \Omega, \end{cases} \tag{1.21}$$

where $g^0 \in L^2(Q)^N$, $g^1 \in L^2(0,T;\mathrm{V})$, and $\psi^0 \in \mathrm{H}$. The following proposition gives us the desired inequality.

Proposition 1.2. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists a constant λ_0, such that for any $\lambda \ge \lambda_0$, there exists a constant $C > 0$ depending only on λ, Ω, ω, and ℓ such that for any $i \in \{1,\dots,N\}$, any $g^0 \in L^2(Q)^N$, any $g^1 \in L^2(0,T;\mathrm{V})$, and any $\psi^0 \in \mathrm{H}$, the solution (φ,ψ) of* (1.21) *satisfies*

$$\begin{aligned} &s^4 \iint\limits_Q e^{-7s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint\limits_Q e^{-4s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt \\ &\le C\left(s^9 \iint\limits_Q e^{-3s\alpha - s\alpha^*}\xi^9|g^0|^2 dx\,dt + \iint\limits_Q e^{-s\alpha^*}(|g^1|^2 + |\nabla g^1|^2) dx\,dt \right. \\ &\qquad \left. + s^{13} \sum_{j=1, j\neq i}^{N} \iint\limits_{\omega\times(0,T)} e^{-3s\alpha - s\alpha^*}\xi^{13}|\varphi_j|^2 dx\,dt \right), \end{aligned} \tag{1.22}$$

for every $s \ge C$.

In order to obtain Proposition 1.2, the authors in [CG14] divide the proof into three parts. First, they estimate a general Carleman inequality for the Stokes system with local terms only in $\Delta(.)_j$, $j \neq i$. In the second part, they deduce a Calerman estimate for the ψ in (1.21). And finally, in the third part, estimate a Carleman for φ to then obtain (1.22).

Null controllability of the linear system

Here we will comment on the already known null controllability of the linear system

$$\begin{cases} \mathcal{L}(w) + \nabla p^0 = f^0 + v\chi_\omega, \;\; \nabla\cdot w = 0 & \text{in} \quad Q, \\ \mathcal{L}^*(z) + \nabla q = f^1 + w\chi_{\mathcal{O}}, \;\; \nabla\cdot z = 0 & \text{in} \quad Q, \\ w = 0, \;\; z = 0 & \text{on} \quad \Sigma, \\ w(0) = 0, \;\; z(T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.23}$$

with f^0 and f^1 in appropriate weighted function spaces, $\mathcal{L}(w) = w_t - \nu_0 \Delta w$, and $\mathcal{L}^*(z) = -z_t - \nu_0 \Delta z$, which is the adjoint operator of $\mathcal{L}$. Therefore, we want to find a control v with $v_i \equiv 0$, for some given $i \in \{1, \dots, N\}$, such that the associated solution of (1.23) satisfies $z(0) = 0$.

For that purpose, we need a Calerman inequality with weight functions not vanishing in $t = T$. We introduce the following weight functions:

$$
\begin{aligned}
&\beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\tilde{\ell}(t)^m}, && \gamma(x,t) = \frac{e^{\lambda\eta(x)}}{\tilde{\ell}(t)^m} \\
&\beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t), && \gamma^*(t) = \min_{x\in\overline{\Omega}} \gamma(x,t), \\
&\widehat{\beta}(t) = \min_{x\in\overline{\Omega}} \beta(x,t), && \widehat{\gamma}(t) = \max_{x\in\overline{\Omega}} \gamma(x,t),
\end{aligned}
$$

where

$$
\tilde{\ell}(t) = \begin{cases} \ell(t), & 0 \le t \le T/2, \\ \|\ell\|_\infty, & T/2 < t \le T. \end{cases}
$$

So, we have the following lemma.

Lemma 1.1. *Let $i \in \{1, \dots, N\}$ and let s and λ be like in Proposition 1.2. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution (φ, ψ) of* (1.21) *satisfies*

$$
\begin{aligned}
&\iint_Q e^{-7s\beta^*} (\gamma^*)^4 |\varphi|^2 dx\, dt + \iint_Q e^{-4s\beta^*} (\gamma^*)^5 |\psi|^2 dx\, dt \\
&\le C \left(\iint_Q e^{-3s\hat{\beta} - s\beta^*} (\widehat{\gamma})^9 |g^0|^2 + \iint_Q e^{-s\beta^*} (|g^1|^2 + |\nabla g^1|^2)\, dx\, dt \right. \\
&\qquad \left. + \sum_{j=1, j\neq i}^{N} \iint_{\omega\times(0,T)} e^{-3s\widehat{\beta} - s\beta^*} (\widehat{\gamma})^{13} |\varphi_j|^2 dx\, dt \right).
\end{aligned} \tag{1.24}
$$

For proof of the previous Lemma, see Lemma 4.1 in [CG14].

Now, we introduce an appropriate weighted functional space that allows us to obtain a null controllability result for system (1.23). Consider the following space, for $N = 2$ or 3 and $i \in \{1, \cdots, N\}$:

$$
\begin{aligned}
E_N^i = \{(w, p^0, z, q, v) :\ & e^{3/2s\widehat{\beta} + 1/2s\beta^*} (\widehat{\gamma})^{-9/2} w \in L^2(Q)^N,\ e^{1/2s\beta^*} z \in L^2(0,T; H^{-1}(\Omega)^N), \\
& e^{3/2s\widehat{\beta} + 1/2s\beta^*} (\widehat{\gamma})^{-13/2} v\chi_\omega \in L^2(Q)^N,\ v_i \equiv 0,\ z(T) = 0, \\
& e^{7/4s\beta^*} w \in L^2(0,T; H^2(\Omega)^N) \cap L^\infty(0,T; \mathrm{V}), \\
& e^{1/2s\beta^*} (\gamma^*)^{-2-2/m} z \in L^2(0,T; H^2(\Omega)^N) \cap L^\infty(0,T; \mathrm{V}), \\
& p^0, q \in L^2(0,T; H^1(\Omega)), \textstyle\int_\Omega p^0 dx = \int_\Omega q dx = 0 \text{ a.e., for } f^0 = \mathcal{L}(w) + \nabla p^0 - v\chi_\omega, \\
& f^1 = \mathcal{L}^*(z) + \nabla q - w\chi_{\mathcal{O}},\ e^{7/2s\beta^*} (\gamma^*)^{-2} f^0 \in L^2(Q)^N,\ e^{2s\beta^*} (\gamma^*)^{-5/2} f^1 \in L^2(Q)^N\},
\end{aligned}
$$

which is a Banach space with the norm.

$$
\begin{aligned}
&\|(w,p^0,z,q,v)\|_{E_N^i} := \Big(\|e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-9/2}w\|^2_{L^2(Q)^N}\\
&+\|e^{1/2s\beta^*}z\|^2_{L^2(0,T;H^{-1}(\Omega)^N)}+\|e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-13/2}v\chi_\omega\|^2_{L^2(Q)^N}\\
&+\|e^{7/4s\beta^*}w\|^2_{L^2(0,T;H^2(\Omega)^N)}+\|e^{7/4s\beta^*}w\|^2_{L^\infty(0,T;\mathrm{V})}\\
&+\|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\|^2_{L^2(0,T;H^2(\Omega)^N)}+\|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\|^2_{L^\infty(0,T;\mathrm{V})}\\
&+\|e^{7/2s\beta^*}(\gamma^*)^{-2}f^0\|^2_{L^2(Q)^N}+\|e^{2s\beta^*}(\gamma^*)^{-5/2}f^1\|^2_{L^2(Q)^N}\Big)^{1/2}.
\end{aligned}
$$

Therefore, the following result is obtained.

Proposition 1.3. *Assume the hypothesis of Lemma 1.1 and*

$$
e^{7/2s\beta^*}(\gamma^*)^{-2}f^0\in L^2(Q)^N \text{ and } e^{2s\beta^*}(\gamma^*)^{-5/2}f^1\in L^2(Q)^N.
$$

Then, we can find a control v such that the associated solution (w,p^0,z,q) to (1.23) *satisfies $(w,p^0,z,q,v)\in E_N^i$. In particular, $v_i\equiv 0$ and $z(0)=0$.*

The proof of Proposition 1.3 can be found in Proposition 4.3 of [CG14].

Additional estimates for the States Solutions

This subsection will be devoted to the proof of technical lemmas that will contain weighted energy estimates that will be needed, in Section 1.4. More precisely, we will show in this subsection that not only the state-control (w,p^0,z,q,v) founded for equation (1.23) in Proposition 1.3 belong to weighted L^2 spaces, but also w_t,∇w, Δw, ∇w_t, Δw_t, w_{tt}, v_t, and Δv belong to such spaces. Furthermore, throughout this subsection we will consider $e^{7/2s\beta^*}(\gamma^*)^{-2}f^0\in L^2(Q)^N$ and $e^{2s\beta^*}(\gamma^*)^{-5/2}f^1\in L^2(Q)^N$.

In order to simplify the notation, we fix λ and s and we set

$$
\begin{cases}
\rho_0=e^{7/2s\beta^*}(\gamma^*)^{-2},\ \widehat{\rho_0}=e^{2s\beta^*}(\gamma^*)^{-5/2},\\
\rho_1=e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-9/2},\ \rho_2=e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-13/2}.
\end{cases}
$$

This guarantees that $\rho_2\le C\rho_1\le C\rho_0$. That said, we have the following lemma.

Lemma 1.2. *Let us define $\rho_3=e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-15/2}$ and $\rho_4=e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-17/2}$. We have*

$$
\sup_{[0,T]}\left(\int_\Omega \rho_3^2|w|^2dx\right)+\iint_Q \rho_3^2|\nabla w|^2dx\,dt\le C\mathcal{K}_0(f^0,f^1), \tag{1.25}
$$

and

$$
\iint_Q \rho_4^2(|w_t|^2+|\Delta w|^2)dx\,dt+\sup_{[0,T]}\left(\int_\Omega \rho_4^2|\nabla w|^2dx\right)\le C\mathcal{K}_0(f^0,f^1), \tag{1.26}
$$

where

$$
\mathcal{K}_0(f^0,f^1):=\iint_Q \rho_0^2|f^0|^2dx\,dt+\iint_Q \widehat{\rho_0}^2|f^1|^2\,dx\,dt.
$$

Proof. Let $A : D(A) \to \mathrm{H}$ be the Stokes operator, and consider for each $n \geq 1$, $v_n(t,.)$, $f_n^0(t,.)$ and $y_n^0(.)$ as, respectively, the projections of $v(t,.)$, $f^0(t,.)$, and $y^0(.)$ on the first n eigenfunctions, which we will denote by λ_n, and we will talk about in more detail in the Appendix A.1. We define y_n as the solution corresponding to the approximate finite-dimensional Stokes system. For simplicity, during the proof of this lemma we will omit the subscript n, and we confirm that the constants C that will appear next are independent of n.

Multiplying $\rho_3^2 w$ as a test function on the first line of the system (1.23), doing some integrations by parts, it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho_3^2|w|^2\,dx\right) + \nu_0\int_\Omega \rho_3^2|\nabla w|^2\,dx &= \int_\omega \rho_3^2 v\, w\,dx + \int_\Omega \rho_3^2 f^0\, w\,dx \\ &\quad + \frac{1}{2}\int_\Omega \frac{d}{dt}(\rho_3^2)|w|^2\,dx. \end{aligned} \tag{1.27}$$

Since, $\rho_3 \leq C\rho_2 \leq C\rho_1 \leq C\rho_0$, then

$$\int_\omega \rho_3^2 v\, w\,dx \leq C\left(\int_\omega \rho_2^2|v|^2\,dx + \int_\Omega \rho_1^2|w|^2\,dx\right), \tag{1.28}$$

and

$$\int_\Omega \rho_3^2 f^0\, w\,dx \leq C\left(\int_\Omega \rho_0^2|f^0|^2\,dx + \int_\Omega \rho_1^2|w|^2\,dx\right). \tag{1.29}$$

Also, of $|\frac{d}{dt}(\rho_3^2)| \leq C\rho_1^2$,

$$\int_\Omega \frac{d}{dt}(\rho_3^2)|w|^2\,dx \leq C\int_\Omega \rho_1^2|w|^2\,dx. \tag{1.30}$$

Applying (1.28), (1.29) and (1.30) in (1.27), integrating in time together with an inequality that can be obtained in the proof of Proposition 1.3 gives us that

$$\iint_Q \rho_1^2|w|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_2^2|v|^2\,dx\,dt < C\mathcal{K}_0(f^0,f^1),$$

we can conclude (1.25).

Now, multiplying $\rho_4^2(w_t - \nu_0 A w)$ as a test function on the first line of the system (1.23). Hence, we get

$$\begin{aligned} \int_\Omega \rho_4^2\left(|w_t|^2 + \nu_0|\Delta w|^2\right)\,dx + \nu_0\frac{d}{dt}\left(\int_\Omega \rho_4^2|\nabla w|^2\,dx\right) &= \int_\omega \rho_4^2 v\,(w_t - \nu_0 A w)\,dx \\ &\quad + \int_\Omega \rho_4^2 f^0\,(w_t - \nu_0 A w)\,dx + \nu_0\int_\Omega \frac{d}{dt}(\rho_4^2)|\nabla w|^2\,dx. \end{aligned} \tag{1.31}$$

For any $\epsilon > 0$, using the fact that $\rho_4 \leq C\rho_2$, $\rho_4 \leq C\rho_0$ and $|\frac{d}{dt}(\rho_4^2)| \leq C\rho_3^2$ follows that

$$\int_\omega \rho_4^2 v\,(w_t - \nu_0 A w)\,dx \leq C\left[\frac{1}{\epsilon}\int_\omega \rho_2^2|v|^2\,dx + \epsilon\int_\Omega \rho_4^2(|w_t|^2 + |\Delta w|^2)\,dx\right], \tag{1.32}$$

$$\int_\Omega \rho_4^2 f^0\,(w_t - \nu_0 Aw)\,dx \le C\left[\frac{1}{\epsilon}\int_\Omega \rho_0^2|f^0|^2\,dx + \epsilon\int_\Omega \rho_4^2(|w_t|^2 + |\Delta w|^2)\,dx\right], \tag{1.33}$$

$$\int_\Omega \frac{d}{dt}(\rho_4^2)|\nabla w|^2\,dx \le C\int_\Omega \rho_3^2|\nabla w|^2\,dx. \tag{1.34}$$

Thus, for ϵ sufficiently small the terms that have w in (1.32) and (1.33) are absorbed by the left side of (1.31). Moreover, from (1.34) and (1.25), integrating in time the third term on the right hand side of (1.31) is bounded by $C\mathcal{K}_0(f^0, f^1)$. Hence, we can conclude (1.26) in Galerkin approximates with w_n instead of the actual solution w. So, by default arguments, when $n \to \infty$, we get that (1.26) is valid for w. □

The next result will provide us with a regularity for the control v. This regularity will be fundamental to obtain more estimates with other weight functions, which we will define later, so that we can show the existence of insensitive controls for equation (1.1).

Lemma 1.3. *$\rho_5 = e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-19/2}$, then*

$$\rho_5 v \in L^2(0,T;[H^2(\omega)\cap H_0^1(\omega)]^N)\cap C^0([0,T];V),\ (\rho_5 v)_t \in L^2(\omega\times(0,T))^N,$$

with the estimate

$$\int_0^T\int_\omega [|(\rho_5 v)_t|^2 + |\rho_5\Delta v|^2]dx\,dt + \sup_{[0,T]}\|\rho_5 v\|_V^2 \le C\mathcal{K}_0(f^0, f^1).$$

Proof. Note that, considering only the Stokes system

$$\begin{cases} \mathcal{L}(w) + \nabla p^0 = f^0 + v\chi_\omega, \ \ \nabla\cdot w = 0 & \text{in} \quad Q,\\ w = 0 & \text{on} \quad \Sigma,\\ w(0) = 0 & \text{in} \quad \Omega,\end{cases} \tag{1.35}$$

and its consequent adjoint system

$$\begin{cases} \mathcal{L}^*(\varphi) + \nabla\pi = g^0, \ \ \nabla\cdot\varphi = 0 & \text{in} \quad Q,\\ \varphi = 0 & \text{on} \quad \Sigma,\\ \varphi(T) = 0 & \text{in} \quad \Omega.\end{cases} \tag{1.36}$$

Following the ideas of Proposition 1.3 together with those contained in [Car+22] (see, Lemma 2.4), we can define

$$\begin{cases} w = \rho_1^{-2}(\mathcal{L}^*\varphi + \nabla\pi), & \text{in} \quad Q\\ v_j = -\rho_2^{-2}\varphi_j\chi_\omega, \ j\neq i, \ v_i \equiv 0 & \text{in} \quad Q,\\ \overline{w} = \rho_2^{-2}\varphi, & \text{in} \quad Q.\end{cases} \tag{1.37}$$

Thus, note that,

$$\begin{aligned} \mathcal{L}^*(\rho_5\overline{w}) &= \mathcal{L}^*(e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-19/2}e^{-3s\widehat{\beta}-s\beta^*}(\widehat{\gamma})^{13}\varphi)\\ &= \mathcal{L}^*(e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\varphi)\\ &= -\frac{d}{dt}(e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2})\varphi + e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-11/2}w\\ &\quad -e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\nabla\pi. \end{aligned} \tag{1.38}$$

In order to apply a regularity result to Stokes systems, we will study each term of this inequality. Since,

$$\begin{aligned}\left|\frac{d}{dt}\left(e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\right)\right| &\leq C\, e^{(-3/2s\widehat{\beta}-1/2s\beta^*)}(-\tilde{\ell}(t)^{-m})'\,\tilde{\ell}(t)^{-7/2m} \\ &\quad + C\, e^{(-3/2s\widehat{\beta}-1/2s\beta^*)}\tfrac{7}{2}\, m\, \tilde{\ell}(t)^{-7/2m-1} \\ &\leq C\, e^{(-3/2s\widehat{\beta}-1/2s\beta^*)}\, \tilde{\ell}(t)^{-9/2m-1}\end{aligned} \tag{1.39}$$

and known that $e^{(-3/2s\widehat{\beta}-1/2s\beta^*)}\,\tilde{l}(t)^{-9/2m-1} \longrightarrow 0$, as $t \to 0^+$ and is bounded as $t \to T^-$, we can deduce

$$\left|\frac{d}{dt}\left(e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\right)\right| \leq C,$$

and as consequence we obtain

$$-\frac{d}{dt}\left(e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\right)\varphi \in L^2(Q)^N.$$

Moreover,

$$e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-11/2} \leq C\, e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-9/2} = C\,\rho_1$$

and therefore $e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-11/2}w \in L^2(Q)^N$.

The weight of the last term is limited, i.e.

$$|e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}| \leq C.$$

Therefore, defining $u = \rho_5\overline{w} = e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-19/2}\,\overline{w}$ and $\overline{\pi} = e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\pi$. By (1.38), we have $(u, \overline{\pi})$ as the solution of the Stokes system

$$\begin{cases} \mathcal{L}^*(u) + \nabla\overline{\pi} = F,\ \nabla\cdot u = 0 & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(T) = 0, & \text{in} \quad \Omega. \end{cases}$$

where

$$F := -\frac{d}{dt}\left(e^{-3/2s\widehat{\beta}-1/2s\beta^*}(\widehat{\gamma})^{7/2}\right)\varphi + e^{3/2s\widehat{\beta}+1/2s\beta^*}(\widehat{\gamma})^{-11/2}w \in L^2(Q)^N.$$

Hence, from the standard regularity for solutions of Stokes systems, we can infer from the definitions of u and v the stated regularity for $\rho_5 v = -u\chi_\omega$. □

By the two previous lemmas, the subsequent lemma is feasible.

Lemma 1.4. *Let us set $\rho_6 = e^{7/4s\beta^*}(\widehat{\gamma})^{-21/2}$. Supposing $\rho_5 f_t^0 \in L^2(Q)^N$, we have the following estimates:*

$$\sup_{[0,T]}\left(\int_\Omega \rho_5^2|w_t|^2dx\right) + \iint_Q \rho_5^2|\nabla w_t|^2dx\,dt \leq C\mathcal{K}_1(f^0, f^1). \tag{1.40}$$

Moreover, if $(\rho_6 f^0)(0) \in H_0^1(\Omega)^N$,

$$\iint_Q \rho_6^2(|w_{tt}|^2 + |\Delta w_t|^2)dx\,dt + \sup_{[0,T]}\left[\int_\Omega \rho_6^2(|\nabla w_t|^2 + |\Delta w|^2)dx\right] \leq C\mathcal{K}_2(f^0, f^1), \tag{1.41}$$

where

$$\mathcal{K}_1(f^0, f^1) := \mathcal{K}_0(f^0, f^1) + \iint_Q \rho_5^2|f_t^0|^2dx\,dt$$

and

$$\mathcal{K}_2(f^0, f^1) := \mathcal{K}_1(f^0, f^1) + \|(\rho_6 f^0)(0)\|^2_{H_0^1(\Omega)^N}.$$

Proof. We establish the current estimates by proceeding in the same structure as the proof of Lemma 1.2. Let us differentiate the first line of the system (1.23) with respect to time, and use $\rho_5^2 w_t$ as a test function:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int\limits_\Omega \rho_5^2|w_t|^2\,dx\right)+\nu_0\int\limits_\Omega \rho_5^2|\nabla w_t|^2\,dx &= \int\limits_\omega \rho_5^2\,v_t\,w_t\,dx+\int\limits_\omega \rho_5^2\,f_t^0\,w_t\,dx\\ &\quad+\frac{1}{2}\int\limits_\Omega \frac{d}{dt}(\rho_5^2)|w_t|^2\,dx.\end{aligned}\tag{1.42}$$

Since,

$$\left|\frac{d}{dt}(\rho_5^2)\right|\le C\rho_4^2,\quad \rho_5\le C\rho_4\le C\rho_2,$$

then

$$\int\limits_\omega \rho_5^2 v_t\,w_t\,dx\le C\left[\int\limits_\omega\left(|\rho_2 v|^2+|(\rho_5 v)_t|^2\right)\,dx+\int\limits_\Omega \rho_4^2|w_t|^2dx\right],\tag{1.43}$$

$$\int\limits_\Omega \rho_5^2 f_t^0\,w_t\,dx\le C\left[\int\limits_\Omega \rho_5^2|f_t^0|^2\,dx+\int\limits_\Omega \rho_4^2|w_t|^2\,dx\right]\tag{1.44}$$

and

$$\int\limits_\Omega \frac{d}{dt}(\rho_5^2)|w_t|^2\,dx\le C\int\limits_\Omega \rho_4^2|w_t|^2\,dx.\tag{1.45}$$

From Lemma 1.2 and Lemma 1.3,

$$\iint\limits_Q \rho_4^2|w_t|^2\,dx\,dt+\int_0^T\int\limits_\omega |(\rho_5 v)_t|^2\,dx\,dt\le \mathcal{K}_0(f^0,f^1).$$

Furthermore, note that, as $|\frac{d}{dt}(\rho_5)|\le C\rho_0$, then

$$\frac{d}{dt}(\rho_5 f^0)=\frac{d}{dt}(\rho_5)f^0+\rho_5 f_t^0\in L^2(Q)^N$$

and consequently $\rho_5 f^0\in C^0([0,T];L^2(\Omega)^N)$ (see, Chapter 5, Section 9 in [Eva10]). Therefore, by first line of the system (1.23) and Lemma 1.3 it is simple to get the estimate

$$\begin{aligned}\rho_5^2(0)\|w_t(0)\|^2_{L^2(\Omega)^N} &\le C\rho_5^2(0)\left(\|f^0(0)\|^2_{L^2(\Omega)^N}+\|v(0)\|^2_{L^2(\omega)^N}\right)\\ &\le C\left(\sup_{[0,T]}\|\rho_5 f^0\|^2_{L^2(Q)^N}+\sup_{[0,T]}\|\rho_5 v\|^2_{\mathrm{V}}\right)\\ &\le C\left(\|\rho_5 f^0\|^2_{L^2(Q)^N}+\|\frac{d}{dt}(\rho_5 f^0)\|^2_{L^2(Q)^N}+\sup_{[0,T]}\|\rho_5 v\|^2_{\mathrm{V}}\right)\\ &\le C\,\mathcal{K}_1(f^0,f^1).\end{aligned}\tag{1.46}$$

Thus, using (1.43), (1.44) and (1.45) in (1.42), and integrating in time, it follows by (1.46) that

$$\begin{aligned}\sup_{[0,T]}\left(\int\limits_\Omega \rho_5^2|w_t|^2\,dx\right)+\iint\limits_Q \rho_5^2|\nabla w_t|^2\,dx\,dt &\le C\Big(\rho_5^2(0)\|w_t(0)\|^2_{L^2(\Omega)^N}\\ &\qquad+\mathcal{K}_0(f^0,f^1)+\iint\limits_Q \rho_5^2|f_t^0|^2dx\,dt\Big)\\ &\le C\mathcal{K}_1(f^0,f^1),\end{aligned}$$

proving (1.40).

Next, for $A : D(A) \to \mathrm{H}$, the Stokes operator defined in Lemma 1.2, we use $\rho_6^2(w_{tt} - \nu_0 A w_t)$ as a test function in $w_{tt} - \nu_0 \Delta w_t + \nabla p_t^0 = f_t^0 - v_t \chi_\omega$, i.e., differentiating the first line of system (1.23) with respect to time. Then, we deduce

$$\begin{aligned}\int_\Omega \rho_6^2(|w_{tt}|^2 + \nu_0|\Delta w_t|^2)\,dx + \nu_0 \frac{d}{dt}\left(\int_\Omega \rho_6^2|\nabla w_t|^2\,dx\right) &= \int_\omega \rho_6^2\, v_t(w_{tt} - \nu_0 A w)\,dx \\ &+ \int_\Omega \rho_6^2\, f_t^0\,(w_{tt} - \nu_0 A w_t)\,dx + \nu_0 \int_\Omega \frac{d}{dt}(\rho_6^2)|\nabla w_t|^2\,dx.\end{aligned} \tag{1.47}$$

Given that $|\frac{d}{dt}(\rho_6^2)| \leq C\rho_5^2$, we get

$$\int_\Omega \frac{d}{dt}(\rho_6^2)|\nabla w_t|^2\,dx \leq C \int_\Omega \rho_5^2|\nabla w_t|^2\,dx. \tag{1.48}$$

And from $\rho_6 \leq C\rho_5$, for any $\epsilon > 0$, we have

$$\int_\omega \rho_6^2\, v_t\,(w_{tt} - \nu_0 A w_t)\,dx \leq C\left[\frac{1}{\epsilon}\int_\omega \rho_5^2|v_t|^2\,dx + \epsilon\int_\Omega \rho_6^2(|w_{tt}|^2 + |\Delta w_t|^2)\,dx\right]; \tag{1.49}$$

and

$$\int_\Omega \rho_6^2\, f_t^0\,(w_{tt} - \nu_0 A w_t)\,dx \leq C\left[\frac{1}{\epsilon}\int_\Omega \rho_5^2|f_t^0|^2\,dx + \epsilon\int_\Omega \rho_6^2(|w_{tt}|^2 + |\Delta w_t|^2)\,dx\right]. \tag{1.50}$$

We fix ϵ small enough, the second term on the right side of (1.49) and (1.50) are absorbed by the left side of (1.47). So, using (1.48) in (1.47), we infer

$$\begin{aligned}\int_\Omega \rho_6^2(|w_{tt}|^2 + |\Delta w_t|^2)\,dx + \frac{d}{dt}\left(\int_\Omega \rho_6^2|\nabla w_t|^2\,dx\right) \leq C\Bigg[&\int_\omega \rho_5^2|v_t|^2\,dx \\ &+ \int_\Omega \rho_5^2|f_t^0|^2\,dx + \int_\Omega \rho_5^2|\nabla w_t|^2\,dx\Bigg].\end{aligned} \tag{1.51}$$

Integrating from 0 to t and using Lemma 1.3 and (1.40), we get

$$\begin{aligned}\iint_Q \rho_6^2(|w_{tt}|^2 + |\Delta w_t|^2)\,dx\,dt + \sup_{[0,T]}\left(\int_\Omega \rho_6^2|\nabla w_t|^2\,dx\right) \leq C\Big[&\mathcal{K}_0(f^0, f^1) \\ &+ \mathcal{K}_2(f^0, f^1) + \iint_Q \rho_5^2|f_t^0|^2 dx\,dt + \mathcal{K}_1(f^0, f^1)\Big].\end{aligned} \tag{1.52}$$

By the same previous reasoning,

$$\begin{aligned}\rho_6^2(0)\|\nabla w_t(0)\|^2_{L^2(\Omega)^N} &\leq C\rho_6^2(0)\left(\|\nabla v(0)\|^2_{L^2(\Omega)^N} + \|\nabla f^0(0)\|^2_{L^2(\Omega)^N}\right) \\ &\leq C\left(\sup_{[0,T]}\|\rho_5 v\|^2_{\mathrm{V}} + \rho_6^2(0)\|f^0(0)\|^2_{H_0^1(\Omega)^N}\right) \\ &\leq C\mathcal{K}_2(f^0, f^1).\end{aligned}$$

Consequently,

$$\iint\limits_Q \rho_6^2(|w_{tt}|^2 + |\Delta w_t|^2)\,dx\,dt + \sup_{[0,T]}\left(\int\limits_\Omega \rho_6^2|\nabla w_t|^2\,dx\right) \le C\mathcal{K}_2(f^0,f^1). \tag{1.53}$$

Finally, we use $-\rho_6^2\Delta w_t$ in the first line of the system (1.23), and we obtain

$$\begin{aligned}
\int\limits_\Omega \rho_6^2|\nabla w_t|^2\,dx + \frac{\nu_0}{2}\frac{d}{dt}\left(\int\limits_\Omega \rho_6^2|\Delta w|^2\,dx\right) &= -\int\limits_\omega \rho_6^2\,v\,\Delta w_t\,dx - \int\limits_\Omega \rho_6^2 f^0\,\Delta w_t\,dx \\
&\quad + \frac{\nu_0}{2}\int\limits_\Omega \frac{d}{dt}(\rho_6^2)|\Delta w|^2\,dx \\
&\le C\left[\int\limits_\omega \rho_2^2|v|^2\,dx + \int\limits_\Omega \rho_0^2|f^0|^2\,dx + \int\limits_\Omega \rho_6^2|\Delta w_t|^2\,dx + \int\limits_\Omega \rho_4^2|\Delta w|^2\,dx\right],
\end{aligned} \tag{1.54}$$

in which it was used, $\rho_6 \le C\rho_2$, $\rho_6 \le C\rho_0$, and $|\frac{d}{dt}(\rho_6^2)| \le C\rho_5^2 \le C\rho_4^2$. By Lemma 1.2 and (1.53), we achieved that

$$\sup_{[0,T]}\left(\int\limits_\Omega \rho_6^2|\Delta w|^2\,dx\right) \le C\mathcal{K}_2(f^0,f^1). \tag{1.55}$$

From the estimates (1.53) and (1.55) we get (1.41). In particular, $\rho_6 w \in L^\infty(0,T;H^2(\Omega)^N)$. $\square$

1.4 Insensitizing controls for Equation (1.1)

In this section, we will prove the existence of insensitizing controls for (1.1), which will be a consequence of the null local controllability of the cascade system (1.7). Notice that, in this definition, $\nabla\cdot((\nu_0+\nu_1\|\nabla w\|^2)Dw)$ can be rewritten, using $\nabla\cdot w = 0$, in the form $(\nu_0+\nu_1\|\nabla w\|^2)\Delta w$. This is,

$$\begin{aligned}
\nabla\cdot\big((\nu_0+\nu_1\|\nabla w\|^2)Dw\big) &= \nabla\cdot\big((\nu_0+\nu_1\|\nabla w\|^2)(\nabla w + \nabla^T w)\big) \\
&= (\nu_0+\nu_1\|\nabla w\|^2)\nabla\cdot(\nabla w) + (\nu_0+\nu_1\|\nabla w\|^2)\nabla\cdot(\nabla^T w) \\
&= (\nu_0+\nu_1\|\nabla w\|^2)\Delta w + (\nu_0+\nu_1\|\nabla w\|^2)\nabla(\nabla\cdot w) \\
&= (\nu_0+\nu_1\|\nabla w\|^2)\Delta w.
\end{aligned}$$

Thus, cascade system (1.7) can be rewritten as follows

$$\begin{cases}
\mathcal{L}(w) - \nu_1\|\nabla w\|^2\Delta w + (w\cdot\nabla)\,w + \nabla p^0 = f + v\chi_\omega,\ \nabla\cdot w = 0 & \text{in} \quad Q, \\
\mathcal{L}^*(z) - \nu_1\|\nabla w\|^2\Delta z + 2\nu_1\left((\Delta w, z)_{L^2}\,\Delta w\right) + \left(z\cdot\nabla^t\right)w - (w\cdot\nabla)\,z & \\
+\nabla q = w\chi_{\mathcal{O}},\ \nabla\cdot z = 0 & \text{in} \quad Q, \\
w = 0,\ z = 0 & \text{on} \quad \Sigma, \\
w(0) = 0,\ z(T) = 0 & \text{in} \quad \Omega.
\end{cases} \tag{1.56}$$

Therefore, we want to find a control v, with $v_i \equiv 0$, such that $z(0) = 0$. For this, we introduce the space functional, for $N = 2$ or 3 and $i \in \{1,\cdots,N\}$, given by

$$\begin{aligned}
\overline{E}_N^i \ := \ &\big\{(w,p^0,z,q,v) : (w,p^0,z,q,v) \in E_N^i, (\rho_5 v)_t\,,\ \rho_5\Delta v \in L^2(\omega\times(0,T))^N, \\
&(\rho_6 f^0)(0) \in H_0^1(\Omega)^N,\ \rho_5 f_t^0 \in L^2(Q)^N\big\},
\end{aligned}$$

which is a Banach space with the norm:

$$
\begin{aligned}
\|(w,p^0,z,q,v)\|^2_{\overline{E}^i_N} &:= \|(w,p^0,z,q,v)\|^2_{E^i_N} + \|(\rho_6 f^0)(0)\|^2_{H^1_0(\Omega)^N} \\
&+\|\rho_5 f^0_t\|^2_{L^2(Q)^N} + \|(\rho_5 v)_t\|^2_{L^2(\omega\times(0,T))^N} + \|\rho_5\Delta v\|^2_{L^2(\omega\times(0,T))^N}.
\end{aligned}
$$

Furthermore, in view of Lemmas 1.2 and 1.4, one also has

$$
\|\rho_3 w\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\rho_4 w\|^2_{L^\infty(0,T;V)\cap L^2(0,T;D(A))} \le C\|(w,p^0,z,q,v)\|^2_{\overline{E}^i_N},
$$
$$
\|\rho_6 w\|^2_{L^\infty(0,T;D(A))} \le C\|(w,p^0,z,q,v)\|^2_{\overline{E}^i_N}.
$$

Remark 1.3. *In particular, an element (w,p^0,z,q,v) of $\overline{E}^i_N$ satisfies $z(0)=0$ and $v_i\equiv 0$. Moreover, of Proposition 1.3 and Lemma 1.4, we have that*

$$
\begin{aligned}
\rho_0(w\cdot\nabla)w &\in L^2(Q)^N,\\
\rho_0\|\nabla w\|^2_{L^2(\Omega)^N}\Delta w &\in L^2(Q)^N,\\
\widehat{\rho_0}(z\cdot\nabla^t)w &\in L^2(Q)^N,\\
\widehat{\rho_0}(w\cdot\nabla)z &\in L^2(Q)^N,\\
\widehat{\rho_0}\|\nabla w\|^2_{L^2(\Omega)^N}\Delta z &\in L^2(Q)^N,\\
\widehat{\rho_0}(\Delta w,z)_{L^2(\Omega)^N}\Delta w &\in L^2(Q)^N.
\end{aligned}
$$

We are interested in apply the Mapping Inverse Theorem in infinite dimensional spaces, which can be found in [ATF87] (Chapter 2, Section 2.3.4), and is given below:

Theorem 1.2 (Liusternik's Inverse Mapping Theorem)**.** *Let B_1 and B_2 Banach spaces and let $\mathcal{A}: B_1\to B_2$ satisfy $\mathcal{A}\in\mathcal{C}^1(B_1;B_2)$. Assume that $b_1\in B_1$, $\mathcal{A}(b_1)=b_2$ and that $\mathcal{A}'(b_1): B_1\to B_2$ is surjective. Then, there exists $\delta>0$ such that, for every $b'\in B_2$ satisfying $\|b'-b_2\|<\delta$, there exists a solution of the equation*

$$
\mathcal{A}(b)=b',\ b\in B_1.
$$

The Setup. Let us set

$$
\begin{aligned}
B_1 &= \overline{E}^i_N\\
Z_N &= \{f\in L^2(Q)^N : \rho_0 f,\ \rho_5 f_t\in L^2(Q)^N,\ (\rho_6 f)(0)\in H^1_0(\Omega)^N\},
\end{aligned}
\tag{1.57}
$$

with

$$
\|f\|^2_{Z_N} = \|\rho_0 f\|^2_{L^2(Q)^N} + \|\rho_5 f_t\|^2_{L^2(Q)^N} + \|(\rho_6 f)(0)\|^2_{H^1_0(\Omega)^N}. \tag{1.58}
$$

Then, we define

$$
B_2 = Z_N\times L^2(\widehat{\rho_0}^2;Q)^N,
$$

where the natural product topology is also a Banach space.

Finally, we define the mapping $\mathcal{A}: B_1\longrightarrow B_2$ by

$$
\begin{aligned}
\mathcal{A}(w,p^0,z,q,v) &= \big(\mathcal{L}(w)-\nu_1\|\nabla w\|^2\Delta w + (w\cdot\nabla)w + \nabla p^0 - v\chi_\omega,\\
&\mathcal{L}^*(z) - \nu_1\|\nabla w\|^2\Delta z + 2\nu_1\left((\Delta w,z)_{L^2}\Delta w\right) + \left(z\cdot\nabla^t\right)w\\
&-(w\cdot\nabla)z + \nabla q - w\chi_{\mathcal{O}}\big).
\end{aligned}
\tag{1.59}
$$

In order that Theorem 2.4 can be applied in this setting, we will prove three lemmas.

Lemma 1.5. *The mapping $\mathcal{A} : B_1 \longrightarrow B_2$ is well-defined, and is continuous around the origin.*

Proof. We want to show that $\mathcal{A}(w,p^0,z,q,v) \in B_2$, for every $(w,p^0,z,q,v) \in B_1$.

First, let us denote by $\mathcal{A}_i$ the components of $\mathcal{A}$ for $i = 1,2$ so that

$$\mathcal{A}(w,p^0,z,q,v) := \big(\mathcal{A}_1(w,p^0,z,q,v), \mathcal{A}_2(w,p^0,z,q,v)\big)$$

where

$$\begin{cases} \mathcal{A}_1(w,p^0,z,q,v) := \mathcal{L}(w) - \nu_1\|\nabla w\|^2\Delta w + (w\cdot\nabla)\,w + \nabla p^0 - v\chi_\omega; \\ \mathcal{A}_2(w,p^0,z,q,v) := \mathcal{L}^*(z) - \nu_1\|\nabla w\|^2\Delta z + 2\nu_1\left((\Delta w, z)_{L^2}\Delta w\right) \\ +\left(z\cdot\nabla^t\right)w - (w\cdot\nabla)\,z + \nabla q - w\chi_{\mathcal{O}}, \end{cases} \tag{1.60}$$

for every $(w,p^0,z,q,v) \in B_1$. In this way, we have $\mathcal{A}_1(w,p^0,z,q,v) \in Z_N$ and $\mathcal{A}_2(w,p^0,z,q,v) \in L^2(\widehat{\rho_0}^2;Q)^N$ for every $(w,p^0,z,q,v) \in B_1$. Indeed, analyzing $\mathcal{A}_1$ and $\mathcal{A}_2$ separately, one has:

Analysis of $\mathcal{A}_1$.

Let $(w,p^0,z,q,v) \in B_1$ and write $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1(w,p^0,z,q,v) = a_1(w,p^0,z,q,v) + a_2(w,p^0,z,q,v) + a_3(w,p^0,z,q,v),$$

where

$$\begin{cases} a_1(w,p^0,z,q,v) := \mathcal{L}(w) + \nabla p^0 - v\chi_\omega = f^0; \\ a_2(w,p^0,z,q,v) := -\nu_1\|\nabla w\|^2\Delta w; \\ a_3(w,p^0,z,q,v) := (w\cdot\nabla)\,w. \end{cases}$$

We will show that for each $j = \{1,2,3\}$,

$$\|a_j(w,p^0,z,q,v)\|_{Z_N} \le C\|(w,p^0,z,q,v)\|_{B_1}$$

and consequently we will have $\mathcal{A}_1(w,p^0,z,q,v) \in Z_N$. In effect, by the definition of the Z_N, we have

$$\|a_1(w,p^0,z,q,v)\|^2_{Z_N} \le \|(w,p^0,z,q,v)\|^2_{B_1}. \tag{1.61}$$

For a_2, note that:

$$\begin{aligned} &\|a_2(w,p^0,z,q,v)\|^2_{Z_N} = \iint_Q \big(\rho_0^2| - \nu_1\|\nabla w\|^2\Delta w|^2 \\ &+\rho_5^2|(-\nu_1\|\nabla w\|^2\Delta w)_t|^2\big)\,dx\,dt + \|(-\nu_1\rho_6\|\nabla w\|^2\Delta w)(0)\|^2_{H_0^1(\Omega)^N} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{1.62}$$

Let us show that each term of a_2 is bounded. In fact,

$$\begin{aligned} I_1 &= \|e^{7/2s\beta^*}(\gamma^*)^{-2}\|\nabla w\|^2\Delta w\|^2_{L^2(Q)^N} \le C\|e^{7/2s\beta^*}\|\nabla w\|^2\Delta w\|^2_{L^2(Q)^N} \\ &= C\int_0^T |e^{7/4s\beta^*}\|\nabla w\|\|\nabla w\||^2 \int_\Omega |e^{7/4s\beta^*}\Delta w|^2 dx\,dt \\ &\le C\|w\|^2_{L^\infty(0,T;\mathrm{V})}\|e^{7/4s\beta^*}w\|^2_{L^\infty(0,T;\mathrm{V})}\|e^{7/4s\beta^*}w\|^2_{L^2(0,T;H^2(\Omega)^N)} \\ &\le C\|(w,p^0,z,q,v)\|^6_{B_1}, \end{aligned} \tag{1.63}$$

since $(\gamma^*)^{-2}$ is bounded and $e^{7/4s\beta^*}w \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;\mathrm{V})$.

Also, since

$$
\begin{aligned}
\left(-\nu_1\|\nabla w\|^2\Delta w\right)_t &= (-\nu_1(\nabla w,\nabla w)_t\,\Delta w)-\nu_1(\nabla w,\nabla w)\Delta w_t\\
&= -2\nu_1(\nabla w,\nabla w_t)\Delta w-\nu_1(\nabla w,\nabla w)\Delta w_t
\end{aligned}
$$

we get,

$$
\begin{aligned}
I_2 &= \iint\limits_Q \rho_5^2|(-\nu_1\|\nabla w\|^2\Delta w)_t|^2 d\,dt\\
&= \iint\limits_Q \rho_5^2|-2\nu_1(\nabla w,\nabla w_t)\Delta w-\nu_1(\nabla w,\nabla w)\Delta w_t|^2 dx\,dt\\
&\leq 2\nu_1\iint\limits_Q \rho_5^2|\,\|\nabla w\|\|\nabla w_t\|\Delta w\,|^2+\nu_1\iint\limits_Q \rho_5^2|\,\|\nabla w\|^2\Delta w_t\,|^2 dx\,dt\\
&= 2\nu_1\overline{M_1}+\nu_1\overline{M_2}.
\end{aligned}
\tag{1.64}
$$

So we need to check that $\overline{M_1}$ and $\overline{M_2}$ are bounded. Well, by Lemma 1.4 and using the fact that $e^{3s\widehat{\beta}-5/2s\beta^*}(\widehat{\gamma})^{-19}\leq C\rho_6^2$, we arrive at

$$
\begin{aligned}
\overline{M_1} &\leq C\iint\limits_Q \rho_5^2|\,\|\nabla w\|\|\nabla w_t\|\Delta w\,|^2 dx\,dt = C\iint\limits_Q \rho_5^2\|\nabla w\|^2\|\nabla w_t\|^2|\Delta w|^2 dx\,dt\\
&= C\int\limits_0^T \rho_5^2\|\nabla w\|^2\|\nabla w_t\|^2\left(\int\limits_\Omega |\Delta w|^2 dx\right)dt\\
&\leq C\,\|w\|^2_{L^\infty(0,T;\mathrm{V})}\int_0^T \rho_5^2\|\nabla w_t\|^2\left(\int\limits_\Omega |\Delta w|^2 dx\right)dt\\
&\leq C\,\|w\|^2_{L^\infty(0,T;\mathrm{V})}\sup_{[0,T]}\left(\int\limits_\Omega \rho_6^2|\Delta w|^2 dx\right)\left(\int_0^T\int\limits_\Omega \rho_5^2|\nabla w_t|^2\rho_6^{-2}dx\,dt\right)\\
&\leq C\,\|w\|^2_{L^\infty(0,T;\mathrm{V})}\|\rho_6\Delta w\|^2_{L^\infty(0,T;L^2(\Omega)^N)}\|\rho_5\nabla w_t\|^2_{L^2(0,T;L^2(\Omega)^N)}\\
&\leq C\,\|(w,p^0,z,q,v)\|^6_{B_1}.
\end{aligned}
\tag{1.65}
$$

And,

$$
\begin{aligned}
\overline{M_2} &\leq C\iint\limits_Q \rho_5^2\|\nabla w\|^2|\Delta w_t|^2 dx\,dt\\
&= C\iint\limits_Q e^{7/2s\beta^*}e^{-7/2s\beta^*}\rho_5^2\|\nabla w\|^2|\Delta w_t|^2 dx\,dt\\
&= C\iint\limits_Q e^{7/2s\beta^*}\|\nabla w\|^2\,e^{3s\widehat{\beta}-5/2s\beta^*}(\widehat{\gamma})^{-19}|\Delta w_t|^2 dx\,dt\\
&\leq C\iint\limits_Q e^{7/2s\beta^*}\|\nabla w\|^2\rho_6^2|\Delta w_t|^2 dx\,dt\\
&\leq C\sup_{[0,T]}\left(\int_\Omega e^{7/2s\beta^*}|\nabla w|^2 dx\right)\left(\iint\limits_Q \rho_6^2|\Delta w_t|^2 dx\,dt\right)\\
&\leq C\|e^{7/4s\beta^*}w\|^2_{L^\infty(0,T;\mathrm{V})}\|\rho_6\Delta w_t\|^2_{L^2(0,T;L^2(\Omega)^N)}\\
&\leq C\|(w,p^0,z,q,v)\|^4_{B_1}
\end{aligned}
\tag{1.66}
$$

By (1.65) and (1.66) in (1.64), we have

$$I_2 \leq C\|(w,p^0,z,q,v)\|_{B_1}^4 + C\|(w,p^0,z,q,v)\|_{B_1}^6. \tag{1.67}$$

Furthermore, given that $w(0) = 0$, we get

$$I_3 = \|(-\nu_1\rho_6\|\nabla w\|^2\Delta w)(0)\|_{H_0^1(\Omega)^N}^2 = 0. \tag{1.68}$$

Then, from (1.63), (1.67), and (1.68) in (1.62), we have

$$\|a_2(w,p^0,z,q,v)\|_{Z_N}^2 \leq C\|(w,p^0,z,q,v)\|_{B_1}^4 \left(1 + \|(w,p^0,z,q,v)\|_{B_1}^2\right). \tag{1.69}$$

Now, we need to show that the same occurs for $a_3(w,p^0,z,q,v)$. Observe that,

$$\begin{aligned} \|a_3(w,p^0,z,q,v)\|_{Z_N}^2 &= \iint\limits_Q \left(\rho_0^2|(w\cdot\nabla)w|^2 + \rho_5^2|\left[(w\cdot\nabla)w\right]_t|^2\right) dx\,dt \\ &\quad + \|\left(\rho_6(w\cdot\nabla)w\right)(0)\|_{H_0^1(\Omega)^N}^2 = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3. \end{aligned} \tag{1.70}$$

Since $\tilde{I}_3 = \|\left(\rho_6(w\cdot\nabla)w\right)(0)\|_{H_0^1(\Omega)^N}^2 = 0$, we just check the other terms. By the definition of B_1,

$$e^{7/2s\beta^*}w \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;\mathrm{V})$$

and we then used the continuous immersions $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, of $N \leq 3$, and one has

$$e^{7/4s\beta^*}w \in L^2(0,T;L^\infty(\Omega)^N) \text{ and } \nabla(e^{7/4s\beta^*}w) \in L^\infty(0,T;L^2(\Omega)^{N\times N}).$$

Consequently,

$$\begin{aligned} \tilde{I}_1 &= \|e^{7/2s\beta^*}(\gamma^*)^{-2}(w\cdot\nabla)w\|_{L^2(Q)^N}^2 \\ &\leq C\|e^{7/2s\beta^*}(w\cdot\nabla)w\|_{L^2(Q)^N}^2 \\ &= C\|(e^{7/4s\beta^*}w\cdot\nabla)e^{7/4s\beta^*}w\|_{L^2(Q)^N}^2 \\ &= C\int_0^T\int\limits_\Omega |e^{7/4s\beta^*}w|^2\,|\nabla(e^{7/4s\beta^*}w)|^2 dx\,dt \\ &\leq C\int_0^T \sup_\Omega |e^{7/4s\beta^*}w|^2 \left(\int\limits_\Omega |\nabla(e^{7/4s\beta^*}w)|^2 dx\right) dt \\ &\leq C\|e^{7/4s\beta^*}w\|_{L^2(0,T;L^\infty(\Omega)^N)}^2\|e^{7/4s\beta^*}w\|_{L^\infty(0,T;\mathrm{V})}^2 \\ &\leq C\|(w,p^0,z,q,v)\|_{B_1}^4, \end{aligned} \tag{1.71}$$

since $(\gamma^*)^{-2}$ bounded. Analogously, from Lemma 1.2 and Lemma 1.4,

$$
\begin{aligned}
\tilde{I}_2 &= \iint_Q \rho_5^2 |(w_t\cdot\nabla)w + (w\cdot\nabla)w_t|^2 dx\,dt \\
&\le 2\iint\limits_Q \rho_5^2|(w_t\cdot\nabla)w|^2 dx\,dt + 2\iint\limits_Q \rho_5^2|(w\cdot\nabla)w_t|^2 dx\,dt \\
&\le 2\iint\limits_Q \rho_5^2\rho_3^{-2}\rho_3^2|w_t|^2|\nabla w|^2 dx\,dt + 2\iint\limits_Q \rho_5^2 e^{7/2s\beta^*}e^{-7/2s\beta^*}|w|^2|\nabla w_t|^2 dx\,dt \\
&\le C\left(\sup_{[0,T]}\int\limits_\Omega \rho_5^2|w_t|^2\right)\int_0^T\int\limits_\Omega \rho_3^2|\nabla w|^2 dx\,dt \\
&\quad + C\left(\sup_{[0,T]}\int\limits_\Omega e^{7/2s\beta^*}|w|^2 dx\right)\int_0^T\int\limits_\Omega \rho_5^2|\nabla w_t|^2 dx\,dt \\
&\le C\,\|\rho_5 w_t\|^2_{L^\infty(0,T;L^2(\Omega)^N)}\|\rho_3\nabla w\|^2_{L^2(0,T;L^2(\Omega)^N)} \\
&\quad + C\,\|e^{7/4s\beta^*}w\|^2_{L^\infty(0,T;\mathrm{V})}\|\rho_5\nabla w_t\|^2_{L^2(0,T;L^2(\Omega)^N)} \\
&\le C\|(w,p^0,z,q,v)\|^4_{B_1}.
\end{aligned}
\tag{1.72}
$$

Thus, from (1.71) and (1.72) in (1.70), we get

$$
\|a_3(w,p^0,z,q,v)\|^2_{Z_N} \le C\|(w,p^0,z,q,v)\|^4_{B_1}. \tag{1.73}
$$

Therefore, by (1.61), (1.69), and (1.73),

$$
\begin{aligned}
\|\mathcal{A}_1(w,p^0,z,q,v)\|^2_{Z_N} \;\le\; & C\left(1+\|(w,p^0,z,q,v)\|^2_{B_1}+\|(w,p^0,z,q,v)\|^4_{B_1}\right) \\
& \|(w,p^0,z,q,v)\|^2_{B_1}
\end{aligned}
\tag{1.74}
$$

and consequently $\mathcal{A}_1(w,p^0,z,q,v)\in Z_N$, for any $(w,p^0,z,q,v)\in B_1$.

Analysis of $\mathcal{A}_2$.

Following in a similar way to what was done for $\mathcal{A}_1$, let $(w,p^0,z,q,v)\in B_1$ and decompose $\mathcal{A}_2$ as follows:

$$
\mathcal{A}_2(w,p^0,z,q,v) = \overline{a_1}(w,p^0,z,q,v) + \overline{a_2}(w,p^0,z,q,v),
$$

where

$$
\begin{cases}
\overline{a_1}(w,p^0,z,q,v) = \mathcal{L}^*(z) + \nabla q - w\chi_{\mathcal{O}} = f^1; \\
\overline{a_2}(w,p^0,z,q,v) = -\nu_1\|\nabla w\|^2\Delta z + 2\nu_1\left((\Delta w,z)_{L^2}\Delta w\right) + \left(z\cdot\nabla^t\right)w \\
\qquad\qquad - (w\cdot\nabla)\,z = \bar{I}_1+\bar{I}_2+\bar{I}_3+\bar{I}_4.
\end{cases}
$$

We will show that, for each $l=\{1,2\}$,

$$
\|\overline{a_l}(w,p^0,z,q,v)\|_{L^2(\widehat{\rho_0}^2;Q)^N} \le \|(w,p^0,z,q,v)\|_{B_1}
$$

and consequently, we get $\mathcal{A}_2(w,p^0,z,q,v)\in L^2(\widehat{\rho_0}^2;Q)^N$. First, it is clear that

$$
\|\overline{a_1}(w,p^0,z,q,v)\|^2_{L^2(\widehat{\rho_0};Q)^N} = \int_0^T\int\limits_\Omega |\widehat{\rho_0}f^1|^2 dx\,dt \le \|(w,p^0,z,q,v)\|^2_{B_1}. \tag{1.75}
$$

So, we need to show that each $\bar{I}_i$, $i = \{1,2,3,4\}$ is bounded to conclude that $\overline{a_2}$ is bounded. Note that, $\bar{I}_4$ satisfies

$$\|\bar{I}_4\|^2_{L^2(\widehat{\rho_0}^2;Q)^N} \leq \|(w,p^0,z,q,v)\|^4_{B_1}. \tag{1.76}$$

Indeed, consider the weight function $e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}$. Since $9/4 > 2$ and $-2-2/m > -5/2$, then $\widehat{\rho_0} = e^{2s\beta^*}(\gamma^*)^{-5/2} \leq e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}$. Thus,

$$\begin{aligned}
\|\bar{I}_4\|_{L^2(\widehat{\rho_0}^2;Q)^N} &= \|\widehat{\rho_0}(w\cdot\nabla)z\|_{L^2(Q)^N} \leq \|e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}(w\cdot\nabla)z\|_{L^2(Q)^N}\\
&= \|\left(e^{7/4s\beta^*}w\cdot\nabla\right)e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\|_{L^2(Q)^N}\\
&\leq \|e^{7/4s\beta^*}w\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{1/2s\beta^*(\gamma^*)-2-2/m}z\|_{L^\infty(0,T;\mathrm{V})}\\
&\leq \|(w,p^0,z,q,v)\|^2_{B_1},
\end{aligned}$$

since $e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T,\mathrm{V})$, proving (1.76).

The reasoning used with the $e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}$ weight will be useful for us to study the other terms $\bar{I}_i$, with $i = \{1,2,3\}$. Effectively, for $\bar{I}_3$, notice that

$$\begin{aligned}
\|\bar{I}_3\|^2_{L^2(\widehat{\rho_0}^2;Q)^N} &= \|\widehat{\rho_0}(z\cdot\nabla^t)w\|^2_{L^2(Q)^N} \leq \|e^{9/4s\beta^*}(\gamma)^{-2-2/m}(z\cdot\nabla^t)w\|_{L^2(Q)^N}\\
&= \|\left(e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\cdot\nabla^t\right)e^{7/4s\beta^*}w\|_{L^2(Q)^N}\\
&\leq \|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{7/4s\beta^*}w\|_{L^\infty(0,T;\mathrm{V})}\\
&\leq \|(w,p^0,z,q,v)\|^4_{B_1}.
\end{aligned} \tag{1.77}$$

Also, by the same arguments,

$$\begin{aligned}
&\|\bar{I}_1\|^2_{L^2(\widehat{\rho_0};Q)^N} \leq \|e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}\|\nabla w\|^2_{L^2}\Delta z\|_{L^2(Q)^N}\\
&\leq \|w\|_{L^\infty(0,T;\mathrm{V})}\|e^{7/4s\beta^*}w\|_{L^\infty(0,T;\mathrm{V})}\|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\|_{L^2(0,T;H^2(\Omega)^N)}\\
&\leq C\|(w,p^0,z,q,v)\|^6_{B_1}
\end{aligned} \tag{1.78}$$

Finally, by the previous regularities plus the regularity obtained in Lemma 1.4, $\rho_6 w \in L^\infty(0,T;H^2(\Omega)^N)$, we have

$$\begin{aligned}
&\|\bar{I}_2\|^2_{L^2(\widehat{\rho_0}^2;Q)^N} = \|\widehat{\rho_0}(2\nu_1(\Delta w,z)\Delta w)\|^2_{L^2(Q)^N}\\
&\leq \|e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}(\Delta w,z)\Delta w\|^2_{L^2(Q)^N}\\
&\leq \|e^{9/4s\beta^*}(\gamma^*)^{-2-2/m}\|z\|_{L^2}\|\Delta w\|_{L^2}\Delta w\|^2_{L^2(Q)^N}\\
&= \int_0^T\int_\Omega |e^{7/4s\beta^*}e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}\|z\|_{L^2}\|\Delta w\|_{L^2}\Delta w|^2 dx\,dt\\
&= \int_0^T e^{7/2s\beta^*}e^{s\beta^*}(\gamma^*)^{-4-4/m}\|z\|^2_{L^2}\|\Delta w\|^2_{L^2}\int_\Omega |\Delta w|^2 dx\,dt\\
&= \int_0^T e^{7/2s\beta^*}e^{s\beta^*}(\gamma^*)^{-4-4/m}\|z\|^2_{L^2}\|\Delta w\|^2_{L^2}\|\Delta w\|^2_{L^2}dt\\
&= \int_0^T e^{s\beta^*}(\gamma^*)^{-4-4/m}\|z\|^2_{L^2}\rho_6^2\|\Delta w\|^2_{L^2}\rho_6^2\|\Delta w\|^2_{L^2}e^{-7/2s\beta^*}(\widehat{\gamma})^{42}\\
&\leq C\left(\|e^{1/2s\beta^*}(\gamma^*)^{-2-2/m}z\|^2_{L^2(0,T;L^2(\Omega)^N)}\|\rho_6 w\|^2_{L^\infty(0,T;H^2(\Omega)^N)}\right.\\
&\left.\|\rho_6 w\|^2_{L^\infty(0,T;H^2(\Omega)^N)}\right)\\
&\leq C\,\|(w,p^0,z,q,v)\|^6_{B_1}.
\end{aligned} \tag{1.79}$$

From $(1.76)-(1.79)$, we get

$$\begin{aligned}&\|\overline{a_2}(w,p^0,z,q,v)\|^2_{L^2(\widehat{\rho_0}^2;Q)^N}\\&\leq C(\|(w,p^0,z,q,v)\|^2_{B_1}+\ \|(w,p^0,z,q,v)\|^4_{B_1})\|(w,p^0,z,q,v)\|^2_{B_1}.\end{aligned}\tag{1.80}$$

Consequently, from (1.75) and (1.80),

$$\begin{aligned}&\|\mathcal{A}_2(w,p^0,z,q,v)\|^2_{L^2(\widehat{\rho_0}^2;Q)^N}\\&\leq C\left(1+\|(w,p^0,z,q,v)\|^2_{B_1}+\ \|(w,p^0,z,q,v)\|^4_{B_1}\right)\|(w,p^0,z,q,v)\|^2_{B_1}\end{aligned}\tag{1.81}$$

and $\mathcal{A}_2(w,p^0,z,q,v)\in L^2(\widehat{\rho_0}^2;Q)^N$, for any $(w,p^0,z,q,v)\in B_1$.

Therefore, from (1.74) and (1.81), we have that $\mathcal{A}(w,p^0,z,q,v)\in B_2$, for every $(w,p^0,z,q,v)\in B_1$, with

$$\begin{aligned}&\|\mathcal{A}(w,p^0,z,q,v)\|^2_{B_2}\\&\leq\ C\left(1+\|(w,p^0,z,q,v)\|^2_{B_1}+\|(w,p^0,z,q,v)\|^4_{B_1}\right)\|(w,p^0,z,q,v)\|^2_{B_1}.\end{aligned}$$

and this concludes that $\mathcal{A}:B_1\longrightarrow B_2$ is well defined.

Using similar arguments it, is easy to check that $\mathcal{A}$ is continuous around the origin. □

Lemma 1.6. *The mapping $\mathcal{A}:B_1\longrightarrow B_2$ is continuously differentiable.*

Proof. We will the proof for $N=3$. The proof for the case $N=2$ is similar.

Let us first prove that $\mathcal{A}$ is Gâteaux-differentiable for all $(w,p^0,z,q,v)\in\overline{E}^i_3$ and let us compute the *G-derivative* $\mathcal{A}'(w,p^0,z,q,v)$.

Let us fix $(w,p^0,z,q,v)\in\overline{E}^i_3$ and let us take $(w',p^{0\prime},z',q',v')\in\overline{E}^i_3$ and $\sigma>0$. By the decomposition made at the beginning of the Lemma 1.5, $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2)$, we have:

$$\begin{aligned}&\frac{1}{\sigma}[\mathcal{A}_1\Big((w,p^0,z,q,v)+\sigma(w',p^{0\prime},z',q',v')\Big)-\mathcal{A}_1(w,p^0,q,z,v)]\\&=w'_t-(\nu_0+\nu_1\|\nabla(w+\sigma w')\|^2)\Delta w'-\frac{\nu_1}{\sigma}\left(\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2\right)\Delta w\\&\ +\nabla p^{0\prime}-v'\chi_\omega+(w'\cdot\nabla)w+(w\cdot\nabla)w'+\sigma(w'\cdot\nabla)w'\end{aligned}$$

and

$$\frac{1}{\sigma}[\mathcal{A}_2\Big((w,p^0,z,q,v)+\sigma(w',p^{0\prime},z',q',v')\Big)-\mathcal{A}_2(w,p^0,z,q,v)]=\tilde{\mathcal{A}}_{2\sigma}+\widehat{\mathcal{A}}_{2\sigma},$$

where

$$\begin{aligned}\tilde{\mathcal{A}}_{2\sigma}&=-z'_t-(\nu_0+\nu_1\|\nabla(w+\sigma w')\|^2)\Delta z'-\frac{\nu_1}{\sigma}(\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2)\Delta z\\&\ +\nabla q'-w'\chi_{\mathcal{O}}-(w\cdot\nabla)z'-(w'\cdot\nabla)z-\sigma(w'\cdot\nabla)z'\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathcal{A}}_{2\sigma}&=(z\cdot\nabla^t)w'+(z'\cdot\nabla^t)w+\sigma(z'\cdot\nabla^t)w'+2\nu_1\left[(\Delta w,z')\Delta w\right.\\&\ +(\Delta w,z)\Delta w'+\ \sigma(\Delta w,z')\Delta w'+(\Delta w',z)\Delta w+\sigma(\Delta w',z')\Delta w\\&\ \left.+\ \sigma(\Delta w',z)\Delta w'+\sigma^2(\Delta w',z')\Delta w'\right].\end{aligned}$$

Let us introduce the linear mapping

$$D\mathcal{A}:\overline{E}^i_3\longrightarrow Z_3\times L^2\left(\widehat{\rho_0};Q\right)^3,$$

with $D\mathcal{A}(w,p^0,z,q,v)=D\mathcal{A}=(D\mathcal{A}_1,D\mathcal{A}_2)$, where:

$$\begin{array}{rcl} D\mathcal{A}_1(w',p^{0'},z',q',v') & = & w'_t-(\nu_0+\nu_1\|\nabla w\|^2)\Delta w'-2\nu_1\,(\nabla w,\nabla w')\,\Delta w \\ & & +\nabla p^{0'}-v'\chi_\omega+(w'\cdot\nabla)w+(w\cdot\nabla)w' \end{array} \tag{1.82}$$

and

$$\begin{array}{l} D\mathcal{A}_2(w',p^{0'},z',q',v')=-z'_t-(\nu_0+\nu_1\|\nabla w\|^2)\Delta z'-2\nu_1(\nabla w,\nabla w')\Delta z \\ +\,2\nu_1[(\Delta w,z')\Delta w+(\Delta w,z)\Delta w'+(\Delta w',z)\Delta w]+(z\cdot\nabla^t)w'+\,(z'\cdot\nabla^t)w \\ -(w'\cdot\nabla)z+(w\cdot\nabla)z'+\nabla q'-w'\chi_{\mathcal{O}}, \end{array} \tag{1.83}$$

for all $(w',p^{0'},z',q',v')\in\overline{E}^i_3$.

From the definition of the spaces $\overline{E}^i_3$, $Z_3\times L^2(\widehat{\rho_0}^2;Q)^3$ and (1.82)-(1.83), it becomes that $D\mathcal{A}\in\mathcal{L}(\overline{E}^i_3,Z_3\times L^2(\widehat{\rho_0}^2;Q)^3)$. Moreover,

$$\begin{array}{l} \dfrac{1}{\sigma}[\mathcal{A}_1\left((w,p^0,z,q,v)+\sigma(w',p^{0'},z',q',v')\right)-\mathcal{A}_1(w,p^0,z,q,v)] \\ \longrightarrow D\mathcal{A}_1(w',p^{0'},z',q',v')\text{ strong in }Z_3,\ \text{as}\,\sigma\longrightarrow 0, \end{array} \tag{1.84}$$

and

$$\begin{array}{l} \dfrac{1}{\sigma}[\mathcal{A}_2\left((w,p^0,z,q,v)+\sigma(w',p^{0'},z',q',v')\right)-\mathcal{A}_2(w,p^0,z,q,v)] \\ \longrightarrow D\mathcal{A}_2(w',p^{0'},z',q',v')\text{ strong in }L^2(\widehat{\rho_0}^2;Q)^3,\ \text{as}\,\sigma\longrightarrow 0. \end{array} \tag{1.85}$$

Let us demonstrate that (1.84) is true. Indeed,

$$\begin{array}{l} \|\dfrac{1}{\sigma}[\mathcal{A}_1\left((w,p^0,z,q,v)+\sigma(w',p^{0'},z',q',v')\right)-\mathcal{A}_1(w,p^0,q,z,v)] \\ \quad -D\mathcal{A}_1(w',p^{0'},z',q',v')\|_{Z_3} \\ \le\|\nu_1\left(\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2\right)\Delta w'\|_{Z_3}+\|\sigma(w'\cdot\nabla)w'\|_{Z_3} \\ \quad +\|\dfrac{\nu_1}{\sigma}\left(\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2\right)\Delta w-2\nu_1(\nabla w,\nabla w')\Delta w\|_{Z_3} \\ =L_1+L_2+L_3. \end{array}$$

We will see that for all $i\in\{1,2,3\}$, $L_i\longrightarrow 0$, as $\sigma\longrightarrow 0$. In effect, using Lebesgue's Theorem,

$$\begin{array}{l} L_1^2=\nu_1^2\displaystyle\iint_Q\rho_0^2|\,\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2\,|^2|\Delta w'|^2dx\,dt \\ +\nu_1^2\displaystyle\iint_Q\rho_5^2|\,[(\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2)\Delta w']_t\,|^2dx\,dt \\ +\|\nu_1\rho_6[(\|\nabla(w+\sigma w')\|^2-\|\nabla w\|^2)\Delta w'](0)\|^2_{H^1_0(\Omega)^3}\longrightarrow 0, \end{array}$$

and

$$\begin{array}{rcl} L_2^2 & = & \|\sigma(w'\cdot\nabla)w'\|^2_{Z_3} \\ & = & \displaystyle\iint_Q[\rho_0^2|\sigma(w'\cdot\nabla)w'|^2+\rho_5^2|\sigma(w'_t\cdot\nabla)w'+\sigma(w'\cdot\nabla)w'_t|^2]dx\,dt \\ & & +\|\rho_6\sigma(w'\cdot\nabla)w'(0)\|^2_{H^1_0(\Omega)^N}\longrightarrow 0, \end{array}$$

as $\sigma\longrightarrow 0$, since the integrals are bounded as we prove in (1.69) and (1.73), respectively.

Now, writing L_3, one has

$$L_3^2 = \iint\limits_Q \rho_0^2 |\frac{\nu_1}{\sigma}(\|\nabla(w+\sigma w')\|^2 - \|\nabla w\|^2)\Delta w - 2\nu_1(\nabla w, \nabla w')\Delta w|^2 dx\, dt$$
$$+ \iint\limits_Q \rho_5^2 |\,[\frac{\nu_1}{\sigma}(\|\nabla(w+\sigma w')\|^2 - \|\nabla w\|^2)\Delta w - 2\nu_1(\nabla w, \nabla w')\Delta w]_t\,|^2 dx\, dt$$
$$+ \|\rho_6 \frac{\nu_1}{\sigma}[(\|\nabla(w+\sigma w')\|^2 - \|\nabla w\|^2)\Delta w - 2\nu_1(\nabla w, \nabla w')\Delta w](0)\|^2_{H_0^1(\Omega)^3}$$
$$= N_1 + N_2 + N_3.$$

Observe each N_j, $j = \{1,2,3\}$:

- $N_1 \longrightarrow 0$, as $\sigma \longrightarrow 0$, since

$$\lim_{\sigma\to 0} \frac{1}{\sigma}(\|\nabla(w+\sigma w')\|^2 - \|\nabla w\|^2)\Delta w = 2(\nabla w, \nabla w')\Delta w.$$

Before we analyze N_2, let's rewrite it. To do this, we start by looking at the first term of the sum on the right-hand side:

$$[(\|\nabla(w+\sigma w')\|^2 - \|\nabla w\|^2)\Delta w]_t$$
$$= [(\nabla w + \sigma\nabla w', \nabla w + \sigma \nabla w')\Delta w - (\nabla w, \nabla w)\Delta w]_t$$
$$= 2(\nabla w + \sigma\nabla w', \nabla w_t + \sigma\nabla w'_t)\Delta w + (\nabla w + \sigma \nabla w', \nabla w + \sigma\nabla w')\Delta w_t$$
$$-2(\nabla w, \nabla w_t)\Delta w - (\nabla w, \nabla w)\Delta w_t$$
$$= 2(\nabla w, \sigma\nabla w'_t)\Delta w + \left[2\sigma(\nabla w', \nabla w_t) + 2\sigma^2(\nabla w', \nabla w'_t)\right]\Delta w + \left[\sigma(\nabla w, \nabla w')\right.$$
$$\left. +\sigma(\nabla w', \nabla w) + \sigma^2(\nabla w', \nabla w')\right]\Delta w_t$$

and the second term of the sum,

$$-2\nu_1[(\nabla w, \nabla w')\Delta w]_t \quad = \quad [-2\nu_1(\nabla w_t, \nabla w') - 2\nu_1(\nabla w, \nabla w'_t)]\Delta w$$
$$-2\nu_1(\nabla w, \nabla w')\Delta w_t.$$

Therefore, simplifying some terms, we obtain

- $N_2 = \iint\limits_Q \rho_5^2 \left|2\sigma\nu_1(\nabla w', \nabla w'_t)\Delta w + \nu_1\sigma(\nabla w', \nabla w')\Delta w_t\right|^2 dx \longrightarrow 0$

as $\sigma \longrightarrow 0$, since the integrals are bounded as we prove in (1.67). Moreover, $N_3 = 0$ given that $w(0) = 0$. Therefore,

$$L_3^2 \longrightarrow 0, \text{ as } \sigma \longrightarrow 0.$$

This yields (1.84) and that $\mathcal{A}_1$ is Gâteaux-differentiable.

For the proof of (1.85), take in (1.83)

$$D\mathcal{A}_2(w', p^{0'}, z', q', v') = D\tilde{\mathcal{A}}_2(w', p^{0'}, z', q', v') + D\widehat{\mathcal{A}}_2(w', p^{0'}, z', q', v'),$$

where

$$D\tilde{\mathcal{A}}_2(w', p^{0'}, z', q', v') \quad = \quad -z'_t - (\nu_0 + \nu_1\|\nabla w\|^2)\Delta z' - 2\nu_1(\nabla w, \nabla w')\Delta z$$
$$+\nabla q' - w'\chi_{\mathcal{O}} - (w'\cdot\nabla)z + (w\cdot\nabla)z',$$

and

$$D\widehat{\mathcal{A}}_2(w', p^{0'}, z', q', v') = (z \cdot \nabla^t)w' + (z' \cdot \nabla^t)w + 2\nu_1[(\Delta w, z')\Delta w + (\Delta w, z)\Delta w' + (\Delta w', z)\Delta w].$$

Arguing similarly to $\mathcal{A}_1$, we have

$$\tilde{A}_{2\sigma} \longrightarrow D\tilde{A}_2(w', p^{0'}, z', q', v') \text{ strong in } L^2(\widehat{\rho_0}^2; Q)^3, \tag{1.86}$$

as $\sigma \longrightarrow 0$. And, since

$$\begin{aligned}
&\|\widehat{A}_{2\sigma} - D\widehat{\mathcal{A}}_2(w', p^{0'}, z', q', v')\|_{L^2(\widehat{\rho_0}^2;Q)^3} \\
&\leq \|2\nu_1\sigma[(\Delta w, z')\Delta w' + (\Delta w', z')\Delta w + (\Delta w', z)\Delta w' + \sigma(\Delta w', z')\Delta w']\|_{L^2(\widehat{\rho_0}^2;Q)^3} \\
&\quad + \|\sigma(z' \cdot \nabla^t)w'\|_{L^2(\widehat{\rho_0}^2;Q)^3},
\end{aligned}$$

we can conclude that

$$\widehat{A}_{2\sigma} \longrightarrow D\widehat{\mathcal{A}}_2(w', p^{0'}, z', q', v') \text{ strong in } L^2(\widehat{\rho_0}^2; Q)^3, \tag{1.87}$$

as $\sigma \longrightarrow 0$.

Thus, from (1.86) and (1.87), (1.85) holds and $\mathcal{A}_2$ is Gâteaux-differentiable.

Therefore, $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ is Gâteaux-differentiable at any $(w, p^0, z, q, v) \in \overline{E}_3^i$, with a *G-derivative* $\mathcal{A}'(w, p^0, z, q, v) = D\mathcal{A}$.

Now, in view of the classical results, we will prove that $\mathcal{A}$ is not only Gâteaux-differentiable, but also Fréchet-differentiable. Hence, we will have

$$\mathcal{A} \in C^1\left(\overline{E}_3^i, Z_3 \times L^2(\widehat{\rho_0}^2; Q)^3\right) \text{ with } \mathcal{A}'(w, p^0, z, q, v) = D\mathcal{A}(w, p^0, z, q, v),$$

i.e.

$$\mathcal{A}'(w, p^0, z, q, v)(w', p^{0'}, z', q', v') = D\mathcal{A}(w, p^0, z, q, v)(w', p^{0'}, z', q', v'),$$

for all $(w', p^{0'}, z', q', v') \in \overline{E}_3^i$, where

$$\begin{aligned}
&D\mathcal{A}(w, p^0, z, q, v)(w', p^{0'}, z', q', v') = \big(w'_t - (\nu_0 + \nu_1\|\nabla w\|^2)\Delta w' \\
&-2\nu_1\,(\nabla w, \nabla w')\,\Delta w + (w' \cdot \nabla)w + (w \cdot \nabla)w' + \nabla p^{0'} - v'\chi_\omega, -z'_t \\
&-(\nu_0 + \nu_1\|\nabla w\|^2)\Delta z' - 2\nu_1(\nabla w, \nabla w')\Delta z + 2\nu_1[(\Delta w, z')\Delta w + (\Delta w, z)\Delta w' \\
&+(\Delta w', z)\Delta w] - (w' \cdot \nabla)z + (w \cdot \nabla)z' + (z \cdot \nabla^t)w' + (z' \cdot \nabla^t)w + \nabla q' - w'\chi_{\mathcal{O}}\big).
\end{aligned}$$

For this purpose, just prove that for

$$(w_n, p_n^0, z_n, q_n, v_n) \longrightarrow (w, p^0, z, q, v) \text{ in } \overline{E}_3^i$$

there is $\epsilon_n(w, p^0, z, q, v)$ such that

$$\begin{aligned}
&\|\left(D\mathcal{A}(w_n, p_n^0, z_n, q_n, v_n) - D\mathcal{A}(w, p^0, z, q, v)\right)(w', p^{0'}, z', q', v')\|^2_{Z_3 \times L^2\left(\widehat{\rho_0}^2;Q\right)^3} \\
&\qquad \leq \epsilon_n\|(w', p^{0'}, z', q', v')\|^2_{\overline{E}_3^i},
\end{aligned} \tag{1.88}$$

for all $(w', p^{0'}, z', q', v') \in \overline{E}_3^i$ and $\lim_{n\to\infty} \epsilon_n = 0$.

Let us prove (1.88). Note that, for $D\mathcal{A}_1$, from (1.82),

$$
\begin{aligned}
&D\mathcal{A}_1(w,p^0,z,q,v)(w',p^{0'},z',q',v') = w'_t - (\nu_0+\nu_1\|\nabla w\|^2)\Delta w' \\
&\quad -2\nu_1\,(\nabla w,\nabla w')\,\Delta w + (w'\cdot\nabla)w + (w\cdot\nabla)w' + \nabla p^{0'} - v'\chi_\omega; \\
&D\mathcal{A}_1(w_n,p^0_n,z_n,q_n,v_n)(w',p^{0'},z',q',v') = w'_t - (\nu_0+\nu_1\|\nabla w_n\|^2)\Delta w' \\
&\quad -2\nu_1\,(\nabla w_n,\nabla w')\,\Delta w_n + (w'\cdot\nabla)w_n + (w_n\cdot\nabla)w' + \nabla p^{0'} - v'\chi_\omega,
\end{aligned}
$$

for all $(w',p^{0'},z',q',v')\in\overline{E}^i_3$. Then, we have

$$
\begin{aligned}
&\big(D\mathcal{A}_1(w_n,p^0_n,z_n,q_n,v_n) - D\mathcal{A}_1(w,p^0,z,q,v)\big)\,(w',p^{0'},z',q',v') \\
&= \nu_1(\|\nabla w\|^2 - \|\nabla w_n\|^2)\Delta w' - 2\nu_1(\nabla w_n,\nabla w')\Delta w_n + 2\nu_1(\nabla w,\nabla w')\Delta w \\
&+(w'\cdot\nabla)w_n - (w'\cdot\nabla)w + (w_n\cdot\nabla)w' - (w\cdot\nabla)w',
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\,\big(D\mathcal{A}_1(w_n,p^0_n,z_n,q_n,v_n) - D\mathcal{A}_1(w,p^0,z,q,v)\big)\,(w',p^{0'},z',q',v')\|^2_{Z_3} \\
&\le 3\|(\nu_1\|\nabla w\|^2 - \nu_1\|\nabla w_n\|^2)\Delta w'\|^2_{Z_3} \\
&\quad + 3\| - 2\nu_1(\nabla w_n,\nabla w')\Delta w_n + 2\nu_1(\nabla w,\nabla w')\Delta w\|^2_{Z_3} \\
&\quad + 3\|(w'\cdot\nabla)(w_n-w) + (w_n-w)\cdot\nabla w'\|^2_{Z_3} \\
&= 3D_{1,n} + 12D_{2,n} + 3D_{3,n}.
\end{aligned}
\tag{1.89}
$$

We need to analyze these terms. Using what has already been done for (1.62), we obtain

$$
\begin{aligned}
D_{1,n} &\le C\left(\|\|\nabla(w-w_n)\|\|\nabla w\|\Delta w'\|^2_{Z_3} + \|\|\nabla(w-w_n)\|\|\nabla w_n\|\Delta w'\|^2_{Z_3}\right) \\
&\le \epsilon_{1,n}\|(w',p^{0'},z',q',v')\|^2_{\overline{E}^i_3}.
\end{aligned}
\tag{1.90}
$$

where

$$
\begin{aligned}
\epsilon_{1,n} = C\|(w_n,p^0_n,z_n,q_n,v_n) - (w,p^0,z,q,v)\|^2_{\overline{E}^i_3}\Bigg(&\|(w,p^0,z,q,v)\|^2_{\overline{E}^i_3} \\
&+\|(w_n,p^0_n,z_n,q_n,v_n)\|^2_{\overline{E}^i_3}\Bigg).
\end{aligned}
$$

For $D_{2,n}$ let us first see the following:

$$
\begin{aligned}
&\|(\nabla w_n,\nabla w')\Delta w_n\|^2_{Z_3} = \iint_Q \big(\rho_0^2|(\nabla w_n,\nabla w')\Delta w_n|^2 \\
&+\rho_5^2|[(\nabla w_n,\nabla w')\Delta w_n]_t|^2\big)\,dx\,dt + \|[\rho_6(\nabla w_n,\nabla w')\Delta w_n](0)\|^2_{H^1_0(\Omega)^3} \\
&= X_1 + X_2 + X_3.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
X_1 &= \iint_Q \rho_0^2|(\nabla w_n,\nabla w')\Delta w_n|^2 dx\,dt \\
&\le C\|w'\|^2_{L^\infty(0,T;\mathrm{V})}\|e^{7/4s\beta^*}w_n\|^2_{L^\infty(0,T;\mathrm{V})}\|e^{7/4s\beta^*}w_n\|^2_{L^2(0,T;H^2(\Omega)^3)} < +\infty.
\end{aligned}
\tag{1.91}
$$

From Lemma 1.4,

$$
\begin{aligned}
X_2 &= \iint\limits_Q \rho_5^2 |[(\nabla w_n, \nabla w')\Delta w_n]_t|^2 dx\, dt \\
&\leq 3\iint\limits_Q \rho_5^2 |(\nabla w_{n,t}, \nabla w')\Delta w_n|^2 dx\, dt + 3\iint\limits_Q \rho_5^2 |(\nabla w_n, \nabla w'_t)\Delta w_n|^2 dx\, dt \\
&\quad + 3\iint\limits_Q \rho_5^2 |(\nabla w_n, \nabla w')\Delta w_{n,t}|^2 dx\, dt \\
&\leq C\Big(\|w'\|^2_{L^\infty(0,T;\mathrm{V})} \|\rho_6 \Delta w_n\|^2_{L^\infty(0,T;L^2(\Omega)^N)} \|\rho_5 \nabla w_{n,t}\|^2_{L^2(0,T;L^2(\Omega)^N)} \\
&\quad + \|w_n\|^2_{L^\infty(0,T;\mathrm{V})} \|\rho_6 \Delta w_n\|^2_{L^\infty(0,T;L^2(\Omega)^N)} \|\rho_5 \nabla w'_t\|^2_{L^2(0,T;L^2(\Omega)^N)} \\
&\quad + \|w'\|^2_{L^\infty(0,T;\mathrm{V})} \iint\limits_Q e^{7/2s\beta^*} \|\nabla w_n\|^2 e^{3s\widehat{\beta}-5/2s\beta^*} (\widehat{\gamma})^{-19} |\Delta w_{n,t}|^2 dx\, dt \Big) \\
&\leq C\Big(\|w'\|^2_{L^\infty(0,T;\mathrm{V})} \|\rho_6 \Delta w_n\|^2_{L^\infty(0,T;L^2(\Omega)^N)} \|\rho_5 \nabla w_{n,t}\|^2_{L^2(0,T;L^2(\Omega)^N)} \\
&\quad + \|w_n\|^2_{L^\infty(0,T;\mathrm{V})} \|\rho_6 \Delta w_n\|^2_{L^\infty(0,T;L^2(\Omega)^N)} \|\rho_5 \nabla w'_t\|^2_{L^2(0,T;L^2(\Omega)^N)} \\
&\quad + \|w'\|^2_{L^\infty(0,T;\mathrm{V})} \|e^{7/2s\beta^*} w_n\|^2_{L^\infty(0,T;\mathrm{V})} \|\rho_6 \Delta w_{n,t}\|^2_{L^2(0,T;L^2(\Omega)^N)} \Big) < +\infty,
\end{aligned}
\tag{1.92}
$$

And, $X_3 = 0$. Thus, adding and subtracting $\nu_1(\nabla w_n, \nabla w')\Delta w$ in $D_{2,n}$ follows, from the calculations made for (1.91) and (1.92), such that

$$
\begin{aligned}
&D_{2,n} = \| -\nu_1(\nabla w_n, \nabla w')\Delta(w_n - w) - \nu_1(\nabla(w_n - w), \nabla w')\Delta w\|^2_{Z_3} \\
&\leq C\left(\| -\nu_1(\nabla w_n, \nabla w')\Delta(w_n - w)\|^2_{Z_3} + \| -\nu_1(\nabla(w_n - w), \nabla w')\Delta w\|^2_{Z_3} \right) \\
&\leq \epsilon_{2,n} \|(w', p^{0\prime}, z', q', v')\|^2_{\overline{E}_3^i},
\end{aligned}
\tag{1.93}
$$

where

$$
\begin{aligned}
\epsilon_{2,n} = C\, \|(w_n, p_n^0, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}_3^i} (\|(w_n, p_n^0, z_n, q_n, v_n)\|^2_{\overline{E}_3^i} \\
+ \|(w, p^0, z, q, v)\|^2_{\overline{E}_3^i}).
\end{aligned}
$$

And similarly, we obtain

$$
\begin{aligned}
D_{3,n} &= \|(w' \cdot \nabla)(w_n - w) + (w_n - w)\nabla w'\|^2_{Z_3} \\
&\leq \epsilon_{3,n} \|(w', p^{0\prime}, z', q', v')\|^2_{\overline{E}_3^i},
\end{aligned}
\tag{1.94}
$$

where $\epsilon_{3,n} = C\|(w_n, p_n^0, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}_3^i}$. From (1.90), (1.93), and (1.94) in (1.89), we have

$$
\begin{aligned}
\| \left(D\mathcal{A}_1(w_n, p_n^0, z_n, q_n, v_n) - D\mathcal{A}_1(w, p^0, z, q, v) \right) (w', p^{0\prime}, z', q', v')\|^2_{Z_3} \\
\leq \epsilon_{j,n} \|(w', p^{0\prime}, z', q', v')\|^2_{\overline{E}_3^i},
\end{aligned}
\tag{1.95}
$$

with $\lim\limits_{n\to\infty} \epsilon_{j,n} = 0$, for all $j \in \{1, 2, 3\}$.

In the same way, we will study $D\mathcal{A}_2$. Remembering that, for (1.83)

$$D\mathcal{A}_2(w,p^0,z,q,v)(w',p^{0'},z',q',v') = -z'_t - (\nu_0+\nu_1\|\nabla w\|^2)\Delta z'$$
$$-2\nu_1(\nabla w,\nabla w')\Delta z + 2\nu_1[(\Delta w,z')\Delta w + (\Delta w,z)\Delta w' + (\Delta w',z)\Delta w]$$
$$+(z\cdot\nabla^t)w' + (z'\cdot\nabla^t)w - (w'\cdot\nabla)z + (w\cdot\nabla)z' + \nabla q' - w'\chi_{\mathcal{O}};$$
$$D\mathcal{A}_2(w_n,p^0_n,z_n,q_n,v_n)(w',p^{0'},z',q',v') = -z'_t - (\nu_0+\nu_1\|\nabla w_n\|^2)\Delta z'$$
$$-2\nu_1(\nabla w_n,\nabla w')\Delta z_n + 2\nu_1[(\Delta w_n,z')\Delta w_n + (\Delta w_n,z_n)\Delta w' + (\Delta w',z_n)\Delta w_n]$$
$$+(z_n\cdot\nabla^t)w' + (z'\cdot\nabla^t)w_n - (w'\cdot\nabla)z_n + (w_n\cdot\nabla)z' + \nabla q' - w'\chi_{\mathcal{O}},$$

for all $(w',p^{0'},z',q',v')\in\overline{E}^i_3$. Then, we get

$$\big(D\mathcal{A}_2(w_n,{p^0}_n,z_n,q_n,v_n) - D\mathcal{A}_2(w,p^0,z,q,v)\big)(w',p^{0'},z',q',v')$$
$$= \nu_1(\|\nabla w\|^2 - \|\nabla w_n\|^2)\Delta z' - 2\nu_1(\nabla w_n,\nabla w')\Delta z_n + 2\nu_1(\nabla w,\nabla w')\Delta z$$
$$-(w'\cdot\nabla)z_n + (w_n\cdot\nabla)z' + (w'\cdot\nabla)z - (w\cdot\nabla)z' + (z_n\cdot\nabla^t)w' + (z'\cdot\nabla^t)w_n$$
$$-(z\cdot\nabla^t)w' - (z'\cdot\nabla^t)w + 2\nu_1[(\Delta w_n,z')\Delta w_n + (\Delta w_n,z_n)\Delta w'$$
$$+(\Delta w',z_n)\Delta w_n] - 2\nu_1[(\Delta w,z')\Delta w + (\Delta w,z)\Delta w' + (\Delta w',z)\Delta w].$$

After some simple calculations

$$\begin{aligned}
&\|\big(D\mathcal{A}_2(w_n,{p^0}_n,z_n,q_n,v_n) - D\mathcal{A}_2(w,p^0,z,q,v)\big)(w',p^{0'},z',q',v')\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&\le C\Big[\|(\nu_1\|\nabla w\|^2 - \nu_1\|\nabla w_n\|^2)\Delta z'\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&\quad+\|-2\nu_1(\nabla w_n,\nabla w')\Delta z_n + 2\nu_1(\nabla w,\nabla w')\Delta z\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&\quad+\|(w'\cdot\nabla)(z_n-z) + ((w_n-w)\cdot\nabla)z'\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&\quad+\|(z'\cdot\nabla^t)(w_n-w) + ((z_n-z)\cdot\nabla^t)w'\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&\quad+\|2\nu_1(\Delta(w_n-w),z')\Delta w_n + 2\nu_1(\Delta w,z')\Delta(w_n-w)\\
&\quad+2\nu_1(\Delta w_n,z_n-z)\Delta w' + 2\nu_1(\Delta(w_n-w),z)\Delta w'\\
&\quad+2\nu_1(\Delta w',z_n-z)\Delta w_n + 2\nu_1(\Delta w',z)\Delta(w_n-w)\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\Big]\\
&= C\left[\overline{D}_{1,n} + \overline{D}_{2,n} + \overline{D}_{3,n} + \overline{D}_{4,n} + \overline{D}_{5,n}\right].
\end{aligned} \tag{1.96}$$

Analyze the previous terms, as was done before for (1.78):

$$\begin{aligned}
\overline{D}_{1,n} &= \|\nu_1(\|\nabla w\|^2 - \|\nabla w_n\|^2)\Delta z'\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&= \|\nu_1(\|\nabla w\| - \|\nabla w_n\|)(\|\nabla w\| + \|\nabla w_n\|)\Delta z'\|^2_{L^2((\widehat{\rho_0}^2;Q)^3)}\\
&\le C\Big(\|\nu_1\|\nabla(w-w_n)\|\,\|\nabla w\|\,\Delta z'\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\\
&\qquad+\|\nu_1\|\nabla(w-w_n)\|\,\|\nabla w_n\|\,\Delta z'\|^2_{L^2(\widehat{\rho_0}^2;Q)^3}\Big)\\
&\le \bar\epsilon_{1,n}\|(w',p^{0'},z',q',v')\|^2_{\overline{E}^i_3},
\end{aligned} \tag{1.97}$$

where

$$\bar\epsilon_{1,n} = C\,\|(w_n,{p^0}_n,z_n,q_n,v_n) - (w,p^0,z,q,v)\|^2_{\overline{E}^i_3}\Big(\|(w,p^0,z,q,v)\|^2_{\overline{E}^i_3}$$
$$+\|(w_n,{p^0}_n,z_n,q_n,v_n)\|^2_{\overline{E}^i_3}\Big).$$

Similar to the reasoning used for (1.93), we arrive at

$$\overline{D}_{2,n} \leq \bar{\epsilon}_{2,n} \|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3} \tag{1.98}$$

where

$$\bar{\epsilon}_{2,n} = C\|(w_n, {p^0}_n, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}^i_3} \|(w_n, {p^0}_n, z_n, q_n, v_n)\|^2_{\overline{E}^i_3}.$$

Also

$$\overline{D}_{3,n} \leq \bar{\epsilon}_{3,n} \|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3} \tag{1.99}$$

where

$$\bar{\epsilon}_{3,n} = \|(w_n, {p^0}_n, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}^i_3}.$$

Just like was done for (1.77),

$$\overline{D}_{4,n} \leq \bar{\epsilon}_{4,n} \|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3} \tag{1.100}$$

where

$$\bar{\epsilon}_{4,n} = C\,\|w_n, {p^0}_n, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}^i_3}.$$

Finally, note that some terms of $\overline{D}_{5,n}$ are bounded by

$$C\,\|(w_n, {p^0}_n, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}^i_3} \Big(\|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3} \\ \|(w_n, {p^0}_n, z_n, q_n, v_n)\|^2_{\overline{E}^i_3} \Big)$$

or by

$$C\,\|(w_n, {p^0}_n, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}^i_3} \Big(\|(w, p^0, z, q, v)\|^2_{\overline{E}^i_3} \\ \|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3} \Big).$$

Thus,

$$\overline{D}_{5,n} \leq \bar{\epsilon}_{5,n} \|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3} \tag{1.101}$$

where

$$\bar{\epsilon}_{5,n} = C\,\|(w_n, {p^0}_n, z_n, q_n, v_n) - (w, p^0, z, q, v)\|^2_{\overline{E}^i_3} \Big(\|(w_n, {p^0}_n, z_n, q_n, v_n)\|^2_{\overline{E}^i_3} \\ + \|(w, p^0, z, q, v)\|^2_{\overline{E}^i_3} \Big).$$

From (1.97)-(1.101) in (1.96), we have

$$\begin{aligned} &\| \left(D\mathcal{A}_2(w_n, {p^0}_n, z_n, q_n, v_n) - D\mathcal{A}_2(w, p^0, z, q, v) \right) (w', p^{0'}, z', q', v') \|^2_{L^2(\widehat{\rho_0}^2; Q)^3} \\ &\leq \bar{\epsilon}_{j,n} \|(w', p^{0'}, z', q', v')\|^2_{\overline{E}^i_3}, \end{aligned} \tag{1.102}$$

with $\lim_{n\to\infty} \bar{\epsilon}_{j,n} = 0$, for all $j \in \{1, 2, 3, 4, 5\}$. Therefore, from (1.95) and (1.102), (1.88) is holds and $\mathcal{A} \in C^1 \left(\overline{E}^i_3, Z_3 \times L^2(\widehat{\rho_0}^2; Q)^3 \right)$ with $\mathcal{A}'(w, p^0, z, q, v) = D\mathcal{A}(w, p^0, z, q, v)$. □

Lemma 1.7. *Let $\mathcal{A}$ be the mapping defined by* (1.59)*. Then $\mathcal{A}'(0, 0, 0, 0, 0)$ is onto.*

Proof. Let $(f^0, f^1) \in B_2$, from Proposition 1.3 we know that there exists (w, p^0, z, q, v) satisfying

$$\begin{cases} \mathcal{L}(w) + \nabla p^0 = f^0 + v\chi_\omega, \;\; \nabla \cdot w = 0 & \text{in} \quad Q, \\ \mathcal{L}^*(z) + \nabla q = f^1 + w\chi_{\mathcal{O}}, \;\; \nabla \cdot z = 0 & \text{in} \quad Q, \\ w = 0, \;\; z = 0 & \text{on} \quad \Sigma, \\ w(0) = 0, \;\; z(T) = 0 & \text{in} \quad \Omega, \end{cases}$$

remembering that $\mathcal{L}(w) = w_t - \nu_0 \Delta w$ and $\mathcal{L}^*(z) = -z_t - \nu_0 \Delta z$, where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. By the estimates proved in the Lemmas 1.2–1.4, the membership $(w, p^0, z, q, v) \in B_1$ is valid. Moreover,

$$\begin{aligned} \mathcal{A}'(0,0,0,0,0)(w, p^0, z, q, v) &= (\mathcal{L}(w) + \nabla p^0 - v\chi_\omega, \, \mathcal{L}^*(z) + \nabla q - w\chi_{\mathcal{O}}) \\ &= (f^0, f^1). \end{aligned}$$

Hence, $\mathcal{A}'(0,0,0,0,0)$ is onto. $\square$

The Proof of Theorem 1.1 According to the previous Lemmas 1.5–1.7, it is legitimate to apply Theorem 1.2. Then, in Theorem 1.2, consider $b_1 = (0,0,0,0,0)$ and $b_2 = (0,0)$. In particular, this gives the existence of a positive number $\delta > 0$ such that, if $\|e^{C/t^m} f\|_{L^2(Q)^N} + \|e^{C/t^m} f_t\|_{L^2(Q)^N} + \|(e^{C/t^m} f)(0)\|_{H_0^1(\Omega)^N} < \delta$, for some $C > 0$, then we can find a control v, with $v_i \equiv 0$, such that the associated solution (w, p^0, z, q) to cascade system (1.56) satisfies $z(0) = 0$.

By Proposition 1.1, the proof of Theorem 1.1 is completed.

CHAPTER **2**

Local null controllability of the complete N-Dimensional Ladyzhenskaya-Boussinesq model

2.1 Problem Formulation

Here we are interested in studying a system that models viscous flows, where viscosity is in function of the velocity gradient, in which thermal effects are taken into account. We will consider $\Omega \subset \mathbb{R}^N$ ($N = 2$ or $N = 3$) be a non-empty bounded connected open set, with regular boundary $\partial\Omega$ and let $T > 0$ be given. We will us denote by Q the cylinder $\Omega \times (0, T)$ with side boundary $\Sigma = \partial\Omega \times (0, T)$.

Let $\omega \subset \Omega$ be a (small) non-empty open set. We denote by $(.,.)$ and $\|.\|$ respectively the L^2 scalar product and norm in Ω. We will use C to denote a generic positive constant. Thus, we will study the local null controllability for the nonlinear systems:

$$\begin{cases} y_t - \nabla\cdot(\nu(\nabla y)Dy) + (y\cdot\nabla)y + \nabla P = v\chi_\omega + \nu_0\theta e_N, \;\; \nabla\cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla\cdot(\nu(\nabla y)\nabla\theta) + y\cdot\nabla\theta = v_0\chi_\omega + \nu(\nabla y)Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \;\; \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \;\; \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where

$$\nu(\nabla y) := \nu_0 + \nu_1 \int_\Omega |\nabla y|^2 dx \tag{2.2}$$

and

$$\begin{cases} y_t - \nabla\cdot(\bar\nu(\nabla y)Dy) + (y\cdot\nabla)y + \nabla P = v\chi_\omega + \nu_0\theta e_N, \;\; \nabla\cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla\cdot(\bar\nu(\nabla\theta)\nabla\theta) + y\cdot\nabla\theta = v_0\chi_\omega + \bar\nu(\nabla y)Dy : \nabla y & \text{in} \quad Q, \\ y(x,t) = 0, \;\; \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \;\; \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.3}$$

where $\bar\nu(\nabla\varsigma) := \nu_0 + \nu_1\|\nabla\varsigma\|^2_{L^p}$, for $3 < p \leq 6$, and in both systems

$$e_N = \begin{cases} (0,1) \text{ if } N = 2, \\ (0,0,1) \text{ if } N = 3. \end{cases}$$

In (2.1) and (2.3), $y = y(x,t)$ stands the "averaged" velocity field, $\theta = \theta(x,t)$ and $P = P(x,t)$ represent, respectively, temperature and pressure of a fluid whose particles are in Ω during the time interval $(0,T)$; ν_0 and ν_1 are positive constants representing the kinematic viscosity and turbulent viscosity, respectively. (y^0, θ^0) are the initial states, that is to say, the states at time $t = 0$; $\chi_\omega \in C_0^\infty(\Omega)$ such that $0 < \chi_\omega \leq 1$ in ω and $\chi_\omega = 0$ outside ω; Dy stands for the symmetrized gradient of y: $Dy = \frac{1}{2}(\nabla y + \nabla^T y)$ and

$$Dy : \nabla y := \sum_{i,j=1}^{N} \frac{1}{2}\left(\frac{\partial y_j}{\partial x_i} + \frac{\partial y_i}{\partial x_j}\right)\frac{\partial y_i}{\partial x_j}. \tag{2.4}$$

Furthermore, $\omega \times (0,T)$ is the control domain and v (force) and v_0 (heat sources) represent the controls acting on the system.

The proof of local null controllability is based on well-known arguments: Carleman estimates and Liusternik's Inverse Mapping Theorem. However, some difficulties arise due to the nonlinear terms added to both the velocity equation and the temperature equation. Furthermore, we will prove a result, only for the case $p = 2$, of null controllability in large time. The proof consists of evolving the system in question without its controls and demonstrating that the system's solutions have an asymptotic behavior so that we can then use the first result of local null controllability.

The following vector spaces, frequently used in the context of incompressible fluids, which will be used throughout the text:

$$H := \{u \in L^2(\Omega)^N : \nabla \cdot u = 0 \,\text{in}\, \Omega, u \cdot \eta = 0 \,\text{on}\, \partial\Omega\}$$

and

$$V^p := \{u \in W_0^{1,p}(\Omega)^N : \nabla \cdot u = 0 \,\text{in}\, \Omega\},$$

where η is the normal vector exterior to $\partial\Omega$ and $W_0^{1,p}(\Omega)$ is the closure of the space of test functions in Ω, $\mathcal{D}(\Omega)$, in $W^{1,p}(\Omega)$(the standard Sobolev space). In particular, when $p = 2$ we will denote $V = V^p$.

When $(y^0, \theta^0) \in V \times H_0^1(\Omega)$, [HLC18] proved that (2.1) is locally null controllable by means $N - 1$ scalar controls for an arbitrary control domain.

For $N = 2$, $y^0 \in V$, $\theta^0 \in W_0^{1,3/2}(\Omega)$, and any $v \in L^2(\omega \times (0,T))^N$, $v_0 \in L^2(\omega \times (0,T))$ sufficiently small in their respective spaces, (2.1) possesses exactly a strong solution (y, p, θ) with

$$\begin{cases} y \in L^2(0,T; H^2(\Omega)^N \cap V) \cap C^0([0,T]; V), \; y_t \in L^2(0,T; H) \\ \theta \in L^2(0,T; W^{2,3/2}(\Omega)), \; \theta_t \in L^2(0,T; L^{3/2}(\Omega)). \end{cases} \tag{2.5}$$

And, for $N = 3$ this is true if y^0, θ^0, v and v_0 are sufficiently small in their respective spaces, that is, there exists $R > 0$ such that

$$\|v\|^2_{L^2(\omega\times(0,T))^N} + \|v_0\|^2_{L^2(\omega\times(0,T))} + \|y^0\|_V + \|\theta^0\|_{W_0^{1,3/2}(\Omega)} < R.$$

The proof of these statements can be seen later in the Appendix B.1 and will be used opportunely to achieve a result of null controllability in a long time, as stated in Theorem 2.3.

Definition 2.1. *Let any non-empty open set $\omega \subset \Omega$. It will be said that* (2.1) *(resp.* (2.3)*) is locally null-controllable at time $T > 0$ if there exists $\delta > 0$ such that, for every $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$ (resp. $(y^0, \theta^0) \in V^p \times W_0^{1,p}(\Omega)$) with*

$$\|(y^0, \theta^0)\|_{V \times W_0^{1,3/2}(\Omega)} < \delta \; (\textit{resp. } \|(y^0, \theta^0)\|_{V^p \times W_0^{1,p}(\Omega)} < \delta),$$

there exists controls $v \in L^2(\omega \times (0,T))^N$, $v_0 \in L^2(\omega \times (0,T))$ and associated solutions (y,p,θ) satisfying

$$y(x,T) = 0 \text{ and } \theta(x,T) = 0 \text{ in } \Omega. \tag{2.6}$$

Thus, the main results are given by the following:

Theorem 2.1. *The nonlinear system* (2.1) *is locally null-controllable at any $T > 0$.*

Theorem 2.2. *The nonlinear system* (2.3) *is locally null-controllable at any $T > 0$.*

In order to prove Theorems 2.1 and 2.2, we will first see a result of null controllability for the linear system associated with (2.1) and (2.3)

$$\begin{cases} \mathcal{L}_1 y + \nabla P = v\chi_\omega + \nu_0 \theta e_N + F_1, \;\; \nabla \cdot y = 0 & \text{in} \quad Q, \\ \mathcal{L}_2 \theta = v_0 \chi_\omega + F_2 & \text{in} \quad Q, \\ y(x,t) = 0, \;\; \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x), \;\; \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.7}$$

where, $\mathcal{L}_1 y := y_t - \nu_0 \Delta y$ and $\mathcal{L}_2 \theta := \theta_t - \nu_0 \Delta \theta$.

Once the null controllability of (2.7) has been proven, we will define a Banach space that will contain a remodeling of the null controllability problem. In other words, we rewrite the null controllability property of (2.1) and (2.3)), separately, as abstract equations (see (2.40) and (2.56)) in well chosen spaces of "admissible" state-controls; see (2.36) and (2.39) for (2.1) and (2.54) and (2.55) for (2.3). In particular, through the definitions applied to the equations and "admissible" spaces, it is possible to show that such applications are well defined and C^1 and, also its derivatives analyzed at zero are surjective. This will allow us to achieve the null controllability of the systems in question.

Furthermore, when $N = 2$ we also show that for certain conditions in the initial data it is possible to obtain a result of null controllability in a large time for the solutions of the system (2.1). To do this, we will show that such solutions have asymptotic behavior when $t \to \infty$. Therefore, we have the following theorem:

Theorem 2.3. *[Large time null-controllability] For $N = 2$, let $(y^0, \theta^0) \in V \times H_0^1(\Omega)$ and $r > 0$ a positive constant given by Theorem B.1 (see Appendix B.1) such that $\|(y^0, \theta^0)\|_{V \times H_0^1(\Omega)} < r$, then there exists a sufficiently large time $T > 0$ such that the nonlinear system* (2.1) *is null-controllable at T.*

This chapter is organized as follows: In Section 2.2 we will talk about some already known results for parabolic problems and Stokes systems and also Carleman estimates, which will be extremely important for the null controllability of the system (2.7). In Section 2.3, based on [Gue06], we will obtain the null controllability of (2.7) and prove estimates, in Banach spaces with weights, for the solutions of the system linear (2.7) as well as for the controls v and v_0 that will be useful (in Section 2.4) to achieve the null controllability of the systems (2.1) and (2.3). In Section 2.4 we establish the null controllability for the systems (2.1) and (2.3) which will be done, as previously described, through Liusternik's Inverse Mapping Theorem. And, in Section 2.5 the proof of the Theorem 2.3 which will be carried out through a lemma that guarantees that under certain conditions imposed on the initial data the solution of the system (2.1) without controls v and v_0 have asymptotic behavior as $t \to \infty$. Finally, still on this chapter, we added an Appendix B.1 that contains results of existence and uniqueness of solution for the system (2.1) and the proof of the Lemma stated in Section 2.5.

2.2 Some previous results

Our goal in the present section is to present well-posedness results for parabolic problems and Stokes systems, as well as Carleman estimates for the adjoint system of (2.7) which is given by

$$\begin{cases} \mathcal{L}_1^*\varphi + \nabla\pi = G_1, \ \ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \mathcal{L}_2^*\psi = \varphi e_N + G_2 & \text{in} \quad Q, \\ \varphi(x,t) = 0, \ \ \psi(x,t) = 0 & \text{on} \quad \Sigma, \\ \varphi(x,T) = \varphi^T(x), \ \ \psi(x,T) = \psi^T(x) & \text{in} \quad \Omega, \end{cases} \tag{2.8}$$

where $\mathcal{L}_1^*\varphi := -\varphi_t - \nu_0\Delta\varphi$, $\mathcal{L}_2^*\psi := -\psi_t - \nu_0\Delta\psi$, $\varphi^T \in H$, $\psi^T \in L^2(\Omega)$, $G_1 \in L^2(Q)^N$ and $G_2 \in L^2(Q)$.

Well-posedness results

The results of this subsection will be applied when we study the null controllability of system (2.7) (Section 2.3), since once we have the appropriate regularity for θ^0 and y^0 the results described here can be applied to equation formed by $(2.7)_1$ and $(2.7)_2$.

The first lemma we mention here is applied to parabolic equations in $L^p - L^q$ spaces and its verification can be based on [DHP07]:

Lemma 2.1. *Let $1 < r, s < \infty$ and suppose that $\phi^0 \in W^{1,s}(\Omega)$ and $h \in L^r(0,T;L^s(\Omega))$. Then the problem*

$$\begin{cases} \phi_t - \Delta\phi = h & \textit{in} \quad Q, \\ \phi = 0 & \textit{on} \quad \Sigma, \\ \phi(0) = \phi^0 & \textit{in} \quad \Omega \end{cases}$$

admits a unique solution

$$\phi \in W^{1,r}(0,T;L^s(\Omega)) \cap L^r(0,T;W^{2,s}(\Omega)),$$

Furthermore, there exist a constant $C > 0$ such that

$$\|\phi_t\|_{L^r(0,T;L^s(\Omega))} + \|\Delta\phi\|_{L^r(0,T;L^s(\Omega))} \le C(\|\phi^0\|_{W^{1,s}(\Omega)} + \|h\|_{L^r(0,T;L^s(\Omega))}). \tag{2.9}$$

Remark 2.1. *Remembering that the Sobolev space $W^{1,r}(0,T;X)$, where X denote a real Banach space with norm $\|.\|_X$, consists of all functions $u \in L^r(0,T;X)$ such that $u' = u_t$ exists in the weak sense and belongs to $L^r(0,T;X)$. Furthermore,*

$$\|u\|_{W^{1,r}(0,T;X)} := \begin{cases} \left[\int_0^T (\|u(t)\|_X^r + \|u'(t)\|_X^r)dt\right]^{1/r} & (1 \le r < \infty) \\ \operatorname{ess\,sup}_{[0,T]}(\|u(t)\|_X + \|u'(t)\|_X) & (r = \infty). \end{cases}$$

More details about this space can be found at [Eva10].

The second result is valid for Stokes systems with homogeneous Dirichlet boundary conditions and can be found in [Tem97]:

Lemma 2.2. *For every $T > 0$, $u^0 \in V$ and $f \in L^2(Q)^N$, there exists a unique solution $(u, q) \in \big(L^2(0,T;H^2(\Omega)^N \cap V) \cap L^\infty(0,T;V) \times L^2(0,T;H^1(\Omega)\big)$ to the Stokes system*

$$\begin{cases} u_t - \Delta u + \nabla q = f, \;\; \nabla \cdot u = 0 & \textit{in} \quad Q, \\ u = 0 & \textit{on} \quad \Sigma, \\ u(0) = u^0 & \textit{in} \quad \Omega. \end{cases}$$

The next result concerns the regularity of solutions of the Stokes system in $L^p - L^q$ spaces, it was proven in [GS91], and complemented by [Gue06] where the author comments on the application of Helmotz's decomposition to have it in the following form:

Lemma 2.3. *Let $1 < p_1, p_2 < \infty$ and suppose that $u^0 \in W^{1,p_2}(\Omega)^N$ and $f \in L^{p_1}(0,T;L^{p_2}(\Omega))$. Then, the weak solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ of system*

$$\begin{cases} u_t - \Delta u + \nabla q = f, \;\; \nabla \cdot u = 0 & \textit{in} \quad Q, \\ u = 0 & \textit{on} \quad \Sigma, \\ u(0) = u^0 & \textit{in} \quad \Omega \end{cases}$$

actually verifies, together with a pressure q, that

$$(u, \nabla q) \in \big(L^{p_1}(0,T;W^{2,p_2}(\Omega)^N) \cap W^{1,p_1}(0,T;L^{p_2}(\Omega)^N)\big) \times L^{p_1}(0,T;L^{p_2}(\Omega)^N).$$

Moreover, there exists a positive constant C just depending on Ω such that

$$\begin{aligned} &\|u\|_{L^{p_1}(0,T;W^{2,p_2}(\Omega)^N) \cap W^{1,p_1}(0,T;L^{p_2}(\Omega)^N)} + \|\nabla q\|_{L^{p_1}(0,T;L^{p_2}(\Omega)^N)} \\ &\leq C(\|f\|_{L^{p_1}(0,T;L^{p_2}(\Omega)^N)} + \|u^0\|_{W^{1,p_2}(\Omega)^N}). \end{aligned}$$

Carleman estimates

We dedicate this subsection to Carleman estimate, which will be fundamental to achieving the null controllability of (2.7).

Let's introduce a new non-empty open set $\omega_0 \Subset \omega$. Due to Fursikov and Imanuvilov [FI96] we have the following result:

Lemma 2.4. *There exists a function $\eta^0 \in C^2(\overline{\Omega})$ satisfying*

$$\begin{cases} \eta^0(x) > 0, & \forall x \in \Omega, \\ \eta^0(x) = 0, & \forall x \in \partial\Omega, \\ |\nabla \eta^0(x)| > 0, & \forall x \in \overline{\Omega} \setminus \omega_0. \end{cases}$$

Let us introduce the function $\ell \in C^\infty([0,T])$ such that

$$\ell(t) = \begin{cases} \dfrac{T^2}{4}, & 0 \leq t \leq T/2, \\ t(T-t), & T/2 < t \leq T. \end{cases}$$

Thus, for all $\lambda > 0$ and $m > 4$, we consider the following weight functions:

$$\alpha(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4}, \;\; \xi(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)^4},$$

$$\alpha^*(t) = \max_{x \in \overline{\Omega}} \alpha(x,t), \;\; \xi^*(t) = \min_{x \in \overline{\Omega}} \xi(x,t),$$

$$\hat{\alpha}(t) = \min_{x \in \overline{\Omega}} \alpha(x,t), \;\; \hat{\xi}(t) = \max_{x \in \overline{\Omega}} \xi(x,t).$$

The constant m will be chosen large enough, in particular such that

$$18\hat{\alpha} > 17\alpha^* \quad \text{in } (0,T). \tag{2.10}$$

We will present a Carleman estimate given by the following lemma:

Lemma 2.5. *For any sufficiently large s and λ, there exists a positive constant C (depending on T, s and λ) such that, for all $\varphi^T \in H$ and $\psi^T \in L^2(\Omega)$ and any $G_1 \in L^2(Q)^N$ and $G_2 \in L^2(Q)$, the solution to (2.8) verifies*

$$\begin{aligned}
\|\varphi(.,0)\|^2 + \|\psi(.,0)\|^2 + \iint_Q e^{-2s\alpha}[\xi^3(|\varphi|^2+|\psi|^2) + \xi(|\nabla\varphi|^2+|\nabla\psi|^2)]dx\,dt \\
\leq C\left(\iint_{\omega\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}\hat{\xi}^{16}(|\varphi|^2+|\psi|^2)dx\,dt\right. \\
\left. + \iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}(|G_1|^2+|G_2|^2)dx\,dt\right).
\end{aligned} \tag{2.11}$$

Proof. See, Lemma 2 in [Gue06]. □

2.3 Null controllability of linear system (2.7)

This section is dedicated to the null controllability of the linear system (2.7). We emphasize that two null controllability results will be obtained, since we will consider different cases for the initial data y^0, θ^0 and the functions F_1, F_2. More precisely, in the first case we will consider more common spaces in control theory, such as $H_0^1(\Omega)$ and $L^2(Q)$ while in the second case we will work with spaces less usual ones, like $W_0^{1,p}(\Omega)$ and $L^q(0,T;L^p(\Omega))$, for $3 < p \leq 6$ and $p < q < \infty$.

Let us set

$$\begin{cases}
\rho = e^{s\alpha}\xi^{-3/2}, \ \rho_1 = e^{2s\hat{\alpha}-s\alpha*}\hat{\xi}^{-15/4} \ \rho_2 = e^{4s\hat{\alpha}-3s\alpha^*}\hat{\xi}^{-8}, \ \rho_3 = e^{s\alpha^*}(\xi^*)^{-1/2}, \\
\mu_1 = e^{8s\hat{\alpha}-7s\alpha^*}\hat{\xi}^{-15}, \ \mu_2 = e^{8s\hat{\alpha}-7s\alpha^*}\hat{\xi}^{-16}, \ \mu_3 = e^{8s\hat{\alpha}-7s\alpha^*}\hat{\xi}^{-17}, \\
\kappa = e^{9s\hat{\alpha}-8s\alpha^*}\hat{\xi}^{-17},
\end{cases} \tag{2.12}$$

so that the values of s and λ satisfy the Lemma 2.5. By inequality (2.10), we can see that

$$\begin{cases}
\kappa \leq C\mu_3 \leq C\mu_2 \leq C\rho_2 \leq C\rho_3 \leq C\mu_2^2, \\
|\mu_{2,t}| \leq C\rho_1, |\mu_3\mu_{3,t}| \leq C\mu_2^2 \text{ and } \kappa_t \leq C\mu_3 \ \text{ in } (0,T).
\end{cases} \tag{2.13}$$

With Lemma 2.5 we will be able to obtain a null controllability result for (2.7), in which the right-hand side F_1 and F_2 decay sufficiently fast to zero as $t \to T$. In other words, the following propositions are valid:

Proposition 2.1. *Let us assume that*

- *if $N = 2$: $y^0 \in H$, $\theta^0 \in L^2(\Omega)$, $\rho_3 F_1 \in L^2(Q)^2$ and $\rho_3 F_2 \in L^2(0,T;L^{3/2}(\Omega))$.*
- *if $N = 3$: $y^0 \in H \cap L^4(\Omega)^3$, $\theta^0 \in L^2(\Omega)$, $\rho_3 F_1 \in L^2(Q)^3$ and $\rho_3 F_2 \in L^2(0,T;L^{3/2}(\Omega))$.*

Then, we can find state-controls (y, P, θ, v, v_0) for (2.7) such that

$$\iint\limits_{Q} \rho_1^2(|y|^2 + |\theta|^2)dx\,dt + \iint\limits_{\omega\times(0,T)} \rho_2^2(|v|^2 + |v_0|^2)dx\,dt \\ \leq C\left(\|y^0\|_H^2 + \|\theta^0\|^2 + \|\rho_3 F_1\|^2_{L^2(Q)^N} + \|\rho_3 F_2\|^2_{L^2(0,T;L^{3/2}(\Omega))}\right). \tag{2.14}$$

In particular, one has $y(x,T) = 0$ and $\theta(x,T) = 0$. Moreover, if $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$ then $y \in L^2(0,T;V) \cap C^0([0,T];H)$ and $\theta \in L^2(0,T;W^{2,3/2}(\Omega)) \cap C^0([0,T];L^{3/2}(\Omega))$.

Proof. It is enough to observe that if $N = 2$, by Sobolev embedding, we have $\rho_3 F_1 \in L^2(0,T;H^{-1}(\Omega)^2)$, $\rho_3 F_2 \in L^2(0,T;H^{-1}(\Omega)^2)$ and if $N = 3$, we have $\rho_3 F_1 \in L^2(0,T;W^{-1,6}(\Omega)^3)$, $\rho_3 F_2 \in L^2(0,T;H^{-1}(\Omega)^3)$. Hence, we can obtain the proof by following the ideas of Proposition 2 in [Gue06].

Indeed, let us introduce some notation:

- $P_0 = \{(\hat\varphi, \hat\pi, \hat\psi) \in C^\infty(\overline{Q})^{N+2}; \nabla\cdot\hat\varphi = 0 \text{ in } Q, \hat\varphi\,|_\Sigma = \hat\psi\,|_\Sigma = 0, \int\limits_\omega \hat\pi(x,t)dx = 0\}$;

- $$a((\varphi,\pi,\psi),(\hat\varphi,\hat\pi,\hat\psi)) = \iint\limits_Q \rho_1^{-2}(\mathcal{L}_1^*\varphi + \nabla\pi)(\mathcal{L}_1^*\hat\varphi + \nabla\hat\pi)dxdt \\ + \iint\limits_Q \rho_1^{-2}(\mathcal{L}_2^*\psi - \varphi_N)(\mathcal{L}_2^*\hat\psi - \hat\varphi_N)dxdt + \iint\limits_Q \chi_\omega\,\rho_2^{-2}(\varphi\hat\varphi + \psi\hat\psi)dxdt,\ \forall\,(\hat\varphi,\hat\pi,\hat\psi) \in P_0;$$

- P is the completion of P_0 for the norm associated to $a(.,.)$. Hence, it is possible to conclude that $a(.,.)$ is a continuous and coercive bilinear form in P;

- $\langle l, (\hat\varphi,\hat\pi,\hat\psi)\rangle = \int_0^T \langle F_1(t), \hat\varphi(t)\rangle dt + \int_0^T \langle F_2(t), \hat\psi(t)\rangle dt + \int\limits_\Omega y^0\hat\varphi(0)dx + \int\limits_\Omega \theta^0\tilde\psi(0)dxdt.$

Thus, due to Carleman inequality (2.11), we have

$$\|\hat\varphi(.,0)\|^2 + \|\hat\psi(.,0)\|^2 + \iint\limits_Q \rho_3^{-2}(|\hat\varphi|^2 + |\hat\psi|^2) + \rho_3^{-2}\xi^{-2}(|\nabla\hat\varphi|^2 + |\nabla\hat\psi|^2)dx\,dt \\ \leq C\,a((\hat\varphi,\hat\pi,\hat\psi),(\hat\varphi,\hat\pi,\hat\psi)),\ \forall\,(\hat\varphi,\hat\pi,\hat\psi) \in P_0 \tag{2.15}$$

from which it is possible to conclude, using the density of P_0 in P, that l is a bounded linear form on P. Therefore, applying Lax-Milgram's lemma, there exists one and only one $(\varphi,\pi,\psi) \in P$ satisfying

$$a((\varphi,\pi,\psi),(\hat\varphi,\hat\pi,\hat\psi)) = \langle l, (\hat\varphi,\hat\pi,\hat\psi)\rangle,\ \forall\,(\hat\varphi,\hat\pi,\hat\psi) \in P. \tag{2.16}$$

According, we can write

$$\begin{cases} y = \rho_1^{-2}(\mathcal{L}_1^*\varphi + \nabla\pi), \ \ \nabla\cdot\varphi = 0 & in \quad Q, \\ \theta = \rho_1^{-2}(\mathcal{L}_2^*\psi - \varphi e_N) & in \quad Q, \\ v = -\rho_2^{-2}\varphi\chi_\omega, \ \ v_0 = -\rho_2^{-2}\psi\chi_\omega & in \quad Q, \end{cases} \tag{2.17}$$

where (φ,π,ψ) is the unique solution of (2.16).

Next, just use the Sobolev embedding mentioned above and apply the arguments of [Gue06] to obtain the existence of controls $(v, v_0) \in L^2(\omega\times(0,T))^{N+1}$ and associated solutions to (2.7) satisfying

(2.14) and consequently (2.6). The regularity of the θ solution is justified by the maximum regularity for parabolic systems in spaces $L^p - L^q$, see Lemma 2.1, and consequently by the standard results for Stokes systems (Lemma 2.2) we obtain the regularity of y. $\square$

Proposition 2.2. *Consider $3 < p \leq 6$ and $p < q < \infty$. Let us assume that the functions F_1, F_2 in (2.7) satisfy $\rho_3 F_1 \in L^q(0,T;L^p(\Omega)^N)$, $\rho_3 F_2 \in L^q(0,T;L^p(\Omega))$ and $(y^0, \theta^0) \in V^p \times W_0^{1,p}(\Omega)$. Then (2.7) is null-controllable, and its control-state satisfy $(v, v_0) \in L^2(\omega \times (0,T))^{N+1}$, $y \in L^q(0,T;W^{2,p}(\Omega)^N) \cap C^0([0,T];L^p(\Omega)^N)$ and $\theta \in L^q(0,T;W^{2,p}(\Omega)) \cap C^0([0,T];L^p(\Omega))$.*

Proof. Indeed, since $V^p \times W_0^{1,p}(\Omega) \hookrightarrow H \cap L^4(\Omega)^N \times L^2(\Omega)$ and $L^q(0,T;L^p(\Omega)) \hookrightarrow L^2(Q) \hookrightarrow L^2(0,T;L^{3/2}(\Omega))$ then the Proposition 2.1 is verified and consequently (2.7) is null-controllable. The regularities of θ and y follow respectively from the Lemma 2.1 and the Lemma 2.3. $\square$

Estimates for the states solutions

In this subsection we will show estimates for the solutions associated with (2.7), that is, for both the velocity variable and the temperature variable. We will obtain estimates not only for y and θ, but also for $\nabla y, \Delta y, \nabla\theta, \Delta\theta$ and the controls v and v_0. The results obtained in this subsection will be fundamental to obtain the null controllability of the nonlinear systems (2.1) and (2.3).

Proposition 2.3. *Let the assumptions in Proposition 2.1 be satisfied. Let the state-control (y, P, θ, v, v_0) satisfy (2.7) and (2.14). Then, the following estimate holds:*

$$\begin{aligned} &\sup_{t\in[0,T]} \int_\Omega \mu_1^2 |y|^2 \, dx + \iint_Q \mu_1^2 |\nabla y|^2 \, dx \, dt \\ &\leq C \left(\|y^0\|_H^2 + \iint_Q [\rho_3^2 |F_1|^2 + \rho_1^2(|y|^2 + |\theta|^2)] \, dx \, dt + \iint_{\omega\times(0,T)} \rho_2^2 |v|^2 \, dx \, dt \right). \end{aligned} \tag{2.18}$$

Furthemore, if $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$, one also has

$$\begin{aligned} &\sup_{t\in[0,T]} \int_\Omega \mu_2^2 |\nabla y|^2 \, dx + \iint_Q \mu_2^2 (|y_t|^2 + |\Delta y|^2) \, dx \, dt \\ &\leq C \left(\|y^0\|_V^2 + \iint_Q [\rho_3^2 |F_1|^2 + \rho_1^2(|y|^2 + |\theta|^2)] \, dx \, dt + \iint_{\omega\times(0,T)} \rho_2^2 |v|^2 \, dx \, dt \right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\int_0^T \mu_2^2 \|\theta_t\|_{L^{3/2}(\Omega)}^2 \, dt + \int_0^T \mu_2^2 \|\Delta\theta\|_{L^{3/2}(\Omega)}^2 \, dt \\ &\leq C \left(\|\theta^0\|_{W_0^{1,3/2}(\Omega)}^2 + \iint_Q \rho_1^2 |\theta|^2 \, dx \, dt + \iint_{\omega\times(0,T)} \rho_2^2 |v_0|^2 \, dx \, dt + \int_0^T \|\rho_3 F_2\|_{L^{3/2}(\Omega)}^2 \, dt \right). \end{aligned} \tag{2.20}$$

Proof. The proofs of (2.18) and (2.19) can be obtained as in [FLM15] (just use the same arguments with the weights defined in (2.12)).

Let us prove (2.20). Denote by $\tilde{\theta} = \mu_2\theta$. Then, by (2.7), we have

$$\begin{cases} \tilde{\theta}_t - \nu_0\Delta\tilde{\theta} = \tilde{h} & \text{in} \quad Q, \\ \tilde{\theta}(x,t) = 0 & \text{on} \quad \Sigma \\ \tilde{\theta}(x,0) = \mu_2(0)\theta^0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.21}$$

where $\tilde{h} = \mu_2 v_0\chi_\omega + \mu_2 F_2 + \mu_{2,t}\theta$.

Note that, by (2.13) and (2.14) we have $\mu_2 v_0\chi_\omega, \mu_2 F_2 \in L^2(0,T;L^{3/2}(\Omega))$ and also

$$\begin{aligned} \|\mu_{2,t}\theta\|^2_{L^2(0,T;L^{3/2}(\Omega))} &\leq C\int_0^T \left(\int_\Omega |\rho_1\theta|^{3/2}\,dx\right)^{4/3} dt \\ &= C\,\|\rho_1\theta\|^2_{L^2(0,T;L^{3/2}(\Omega))} < +\infty. \end{aligned} \tag{2.22}$$

Then, $\tilde{h} \in L^2(0,T;L^{3/2}(\Omega))$. Therefore, from (2.9) we have

$$\begin{aligned} &\int_0^T \mu_2^2\|\theta_t\|^2_{L^{3/2}(\Omega)}\,dt + \int_0^T \mu_2^2\|\Delta\theta\|^2_{L^{3/2}(\Omega)}\,dt \\ &\leq C\left(\|\tilde{h}(t)\|^2_{L^2(0,T;L^{3/2}(\Omega))} + \|\mu_2(0)\theta^0\|^2_{W^{1,3/2}(\Omega)}\right) \end{aligned}$$

and consequently

$$\begin{aligned} &\int_0^T \mu_2^2\|\theta_t\|^2_{L^{3/2}(\Omega)}\,dt + \int_0^T \mu_2^2\|\Delta\theta\|^2_{L^{3/2}(\Omega)}\,dt \\ &\leq C\left(\|\theta^0\|^2_{W_0^{1,3/2}(\Omega)} + \iint_Q \rho_1^2|\theta|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_2^2|v_0|^2\,dx\,dt + \int_0^T \|\rho_3F_2\|^2_{L^{3/2}(\Omega)}\,dt\right), \end{aligned}$$

achieving the desired inequality. □

Proposition 2.4. *Let the assumptions in Proposition 2.2 be satisfied. Then, the controls verifies*

$$\kappa v \in L^2(0,T;[H^2(\omega)\cap H_0^1(\omega)]^N) \cap C^0([0,T];V),\ (\kappa v)_t \in L^2(\omega\times(0,T))^N. \tag{2.23}$$

$$\kappa v_0 \in L^2(0,T;H^2(\omega)) \cap C^0([0,T];H^1(\omega)),\ (\kappa v_0)_t \in L^2(\omega\times(0,T)). \tag{2.24}$$

with the estimate

$$\begin{aligned} &\int_0^T\int_\omega \left[|(\kappa v)_t|^2 + |(\kappa v_0)_t|^2 + |\kappa\Delta v|^2 + |\kappa\Delta v_0|^2\right] dxdt + \sup_{[0,T]}\|\kappa v\|^2_V \\ &+ \sup_{[0,T]}\|\kappa v_0\|^2_{H^1(\omega)} \leq C\left(\|y^0\|^2_{V^p} + \|\theta^0\|^2_{W_0^{1,p}(\Omega)} + \|\rho_3F_1\|^2_{L^q(0,T;L^p(\Omega)^N)}\right. \\ &\left.\quad + \|\rho_3F_2\|^2_{L^q(0,T;L^p(\Omega))}\right). \end{aligned}$$

Furthermore, the associated states satisfy

$$\begin{aligned} &\iint_Q \mu_3^2|y_t|^2dx\,dt + \sup_{[0,T]}\int_\Omega \mu_3^2|\nabla y|^2dx + \iint_Q \mu_3^2|\Delta y|^2dx\,dt + \sup_{[0,T]}\int_\Omega \mu_2^2|y|^2dx \\ &+ \iint_Q \mu_2^2|\nabla y|^2dx\,dt \leq C\left(\|y^0\|^2_{V^p} + \iint_Q \rho_1^2(|\theta|^2+|y|^2)\,dx\,dt + \iint_{\omega\times(0,T)} \rho_2^2|v|^2\,dx\,dt\right. \\ &\left.\quad + \|\rho_3F_1\|^2_{L^q(0,T;L^p(\Omega)^N)}\right) \end{aligned} \tag{2.25}$$

and

$$\iint\limits_Q \mu_3^2|\theta_t|^2 dx\,dt + \sup_{[0,T]} \int\limits_\Omega \mu_3^2|\nabla\theta|^2 dx + \iint\limits_Q \mu_3^2|\Delta\theta|^2 dx\,dt + \sup_{[0,T]} \int\limits_\Omega \mu_2^2|\theta|^2 dx$$
$$+ \iint\limits_Q \mu_2^2|\nabla\theta|^2 dx\,dt \le C \left(\|\theta^0\|^2_{W_0^{1,p}(\Omega)} + \iint\limits_Q \rho_1^2|\theta|^2\,dx\,dt + \iint\limits_{\omega\times(0,T)} \rho_2^2|v_0|^2\,dx\,dt \right. \tag{2.26}$$
$$\left. + \|\rho_3 F_2\|^2_{L^q(0,T;L^p(\Omega))} \right).$$

Proof. The first part of the proof will be dedicated to concluding (2.23) and (2.24).

Let us set $u = \rho_2^{-2}\varphi$. Hence by the definition given in (2.17) we have, after some computations,

$$\begin{aligned} \mathcal{L}_1^*(\kappa u) &= (\kappa\rho_2^{-2})\mathcal{L}_1^*\varphi - (\kappa\rho_2^{-2})_t\varphi \\ &= (\kappa\rho_2^{-2}\rho_1^2)y - (\kappa\rho_2^{-2})\nabla\pi - (\kappa\rho_2^{-2})_t\varphi. \end{aligned} \tag{2.27}$$

We notice that,

$$|\kappa\rho_2^{-2}\rho_1^2| \le C\rho_1,\ |\kappa\rho_2^{-2}| \le C,\ |(\kappa\rho_2^{-2})_t\rho\rho^{-1}| \le C\rho^{-1}. \tag{2.28}$$

And, from the Carleman estimate (2.11) and again (2.17), we get:

$$\iint\limits_Q \rho^{-2}(|\varphi|^2 + |\psi|^2)dxdt \le \iint\limits_Q \rho_1^2(|y|^2 + |\theta|^2)dxdt + \iint\limits_{\omega\times(0,T)} \rho_2^2(|v|^2 + |v_0|^2)dxdt < +\infty. \tag{2.29}$$

Therefore, by (2.27), (2.28) and (2.29) we obtain $(\tilde{u}, \tilde{\pi}) := (\kappa u, \kappa\rho_2^{-2}\pi)$ solution of the Stokes system

$$\begin{cases} \mathcal{L}_1^*\tilde{u} + \nabla\tilde{\pi} = \tilde{f},\ \ \nabla\cdot\tilde{u} = 0 & \text{in} \quad Q, \\ \tilde{u} = 0 & \text{on} \quad \Sigma, \\ \tilde{u}(.,T) = 0 & \text{in} \quad \Omega, \end{cases}$$

with $\tilde{f} = (\kappa\rho_2^{-2}\rho_1)\rho_1 y - (\kappa\rho_2^{-2})_t\rho\rho^{-1}\varphi \in L^2(Q)^N$.

By the standard regularity for solutions of Stokes systems, we can infer the regularity (2.23) for $\kappa v = -\tilde{u}\chi_\omega$.

Similarly, define $w = -\rho_2^{-2}\psi$ and note that $v_0 = w\chi_\omega$. Then,

$$\begin{aligned} \mathcal{L}_2^*(\kappa w) &= -(\kappa\rho_2^{-2})\mathcal{L}_2^*\psi + (\kappa\rho_2^{-2})_t\psi \\ &= -(\kappa\rho_2^{-2}\rho_1^2)\theta - (\kappa\rho_2^{-2})\varphi e_N + (\kappa\rho_2^{-2})_t\psi \\ &= N_1 + N_2 + N_3. \end{aligned} \tag{2.30}$$

Analyzing each N_i, $i = \{1, 2, 3\}$, we obtain

$$|N_1| \le e^{3s\hat{\alpha} - 3s\alpha^*}\hat{\xi}^{-19/4}\rho_1\theta,\ |N_2| \le Ce^{s\hat{\alpha} - s\alpha^*}\hat{\xi}^{-1}\rho^{-1}\varphi e_N,\ \ |N_3| \le Ce^{s\hat{\alpha} - s\alpha^*}\hat{\xi}^{7/4}\rho^{-1}\psi.$$

Thus, from (2.29), we deduce that $N_1 + N_2 + N_3 \in L^2(Q)$. Therefore, taking into consideration the PDE satisfied by κw and the fact that $(\kappa w)(.,T) = 0$, we concluded that

$$\kappa w \in L^2(0,T; H^2(\Omega)) \cap C^0([0,T]; H^1(\Omega)),\ \ (\kappa w)_t \in L^2(Q).$$

In particular, (2.24) holds.

Now, let's establish the second part of the proposition, that is, (2.25) and (2.26):

Firstly, multiplying the linear system $(2.7)_1$ by $\mu_2^2 y$ (as a test function), integrating in Ω, we have that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mu_2^2|y|^2dx + \nu_0\int_\Omega \mu_2^2|\nabla y|^2dx \leq C\left(\int_\Omega \rho_1^2(|\theta|^2+|y|^2)dx + \int_\omega \rho_2^2|v|^2dx + \int_\Omega \rho_3^2|F_1|^2dx\right)$$

thanks to $|\mu_2\mu_{2,t}| \leq C\rho_1^2$, $\mu_2^2 \leq C\rho_1^2$, $\mu_2^2 \leq C\rho_2^2$ and $\mu_2^2 \leq C\rho_3^2$. Then, integrating from 0 to t we find that

$$\sup_{[0,T]}\int_\Omega \mu_2^2|y|^2dx + \iint_Q \mu_2^2|\nabla y|^2dx\,dt \leq C\left(\|y^0\|_H^2 + \iint_Q \rho_1^2(|\theta|^2+|y|^2)\,dx\,dt + \iint_{\omega\times(0,T)} \rho_2^2|v|^2\,dx\,dt + \iint_Q \rho_3^2|F_1|^2dx\,dt\right). \tag{2.31}$$

Next, using $\mu_3^2 y_t$ as a test function in $(2.7)_1$, integrating in Ω and taking into account that $\mu_3^2 \leq C\rho_2^2 \leq C\rho_3^2$ and $|\mu_3\mu_{3,t}| \leq C\mu_2^2$, we obtain

$$\int_\Omega \mu_3^2|y_t|^2dx + \nu_0\frac{d}{dt}\int_\Omega \mu_3^2|\nabla y|^2dx \leq C\left(\int_\omega \rho_2^2|v|^2 + \int_\Omega \rho_3^2|F_1|^2dx + \int_\Omega \rho_1^2|\theta|^2dx + \int_\Omega \mu_2^2|\nabla y|^2dx\right).$$

Hence, integrating from 0 to t and making use of (2.31), we deduce that

$$\iint_Q \mu_3^2|y_t|^2dx\,dt + \sup_{[0,T]}\int_\Omega \mu_3^2|\nabla y|^2dx \leq C\left(\|y^0\|_V^2 + \iint_{\omega\times(0,T)} \rho_2^2|v|^2\,dx\,dt + \iint_Q \rho_1^2(|\theta|^2+|y|^2)\,dx\,dt + \iint_Q \rho_3^2|F_1|^2dx\,dt\right). \tag{2.32}$$

Finally, multiplying $(2.7)_1$ by $-\mu_3^2\Delta y$ and followed in a similar way to the previous estimates, we arrive at

$$\sup_{[0,T]}\int_\Omega \mu_3^2|\nabla y|^2dx + \iint_Q \mu_3^2|\Delta y|^2dx\,dt \leq C\left(\|y^0\|_V^2 + \iint_Q \rho_1^2(|\theta|^2+|y|^2)\,dx\,dt + \iint_{\omega\times(0,T)} \rho_2^2|v|^2\,dx\,dt + \iint_Q \rho_3^2|F_1|^2dx\,dt\right). \tag{2.33}$$

Therefore, from (2.31)-(2.33) and by Sobolev's immersions $V^p \hookrightarrow V$ and $L^q(0,T;L^p(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$, we see that (2.25) holds.

The estimate (2.26) is obtained by multiplying $\mu_2^2\theta$, $\mu_3^2\theta_t$ and $-\mu_3^2\Delta\theta$ one after the other in $(2.7)_2$ and using the same arguments as before.

□

The next result is a proposition from [Man+23] and will be of great importance for us to conclude our main theorems. For this section to be complete, we will give here the proof provided by [Man+23].

Proposition 2.5. *If $u \in L^q(0,T;W^{2,p}(\Omega))$, $u_t \in L^q(0,T;L^p(\Omega))$ then $u \in C^0([0,T];W^{1,p}(\Omega))$, $p > 2$ and $p < q < \infty$.*

Proof. Consider u a regular function with compact support contained in Ω, so we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla u|^p dx &= \int_\Omega \frac{d}{dt}(|\nabla u|^2)^{\frac{p}{2}}dx = \int_\Omega p\left((|\nabla u|^2)^{\frac{p}{2}-1}\right)(\nabla u \nabla u_t)dx \\
&= p\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_t dx \\
&= -p\int_\Omega \nabla(|\nabla u|^{p-2}\nabla u)u_t dx + \int_{\partial\Omega} p|\nabla u|^{p-2}\nabla u \cdot u_t \cdot \vec{\eta}\, d\Gamma \\
&= \int_\Omega (p(p-2)|\nabla u|^{p-4}\nabla u \nabla(u_{x_i})u_{x_i} + p|\nabla u|^{p-2}\Delta u)u_t dx \\
&\leq C\int_\Omega (p(p-2)|\nabla u|^{p-2}|D^2 u| + p|\nabla u|^{p-2}|\Delta u|)|u_t| dx.
\end{aligned}$$

Since $\frac{1}{p} + \frac{1}{p} + \frac{1}{\frac{p}{p-2}} = 1$ then

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla u|^p dx &\leq C(\int_\Omega (|\nabla u|^{p-2})^{\frac{p}{p-2}}dx)^{\frac{p-2}{p}}(\int_\Omega |D^2u|^p dx)^{\frac{1}{p}}(\int_\Omega |u_t|^p dx)^{\frac{1}{p}} \\
&\leq C(\int_\Omega |\nabla u|^p dx)^{\frac{p-2}{p}}\|u\|_{W^{2,p}(\Omega)}\|u_t\|_{L^p(\Omega)}
\end{aligned}$$

integrating in $[0,t]$, like $\frac{1}{q} + \frac{1}{q} + \frac{1}{\frac{q}{q-2}} = 1$, and using that $\frac{q(p-2)}{q-2} < q$ there is,

$$\begin{aligned}
\int_\Omega |\nabla u(t)|^p dx &\leq C\left(\left(\int_0^T (\int_\Omega |\nabla u|^p)^{\frac{p-2}{p}\frac{q}{q-2}}dxdt\right)^{\frac{q-2}{q}}\|u\|_{L^q(0,T;W^{2,p}(\Omega))}\|u_t\|_{L^q(0,T;L^p} \right. \\
&\quad \left. + \int_\Omega |\nabla u(0)|^p dx(\Omega)\right) \\
&\leq C\|u\|^{q-2}_{L^q(0,T;W^{1,p}(\Omega))}\|u\|_{L^q(0,T;W^{2,p}(\Omega))}\|u_t\|_{L^q(0,T;L^p(\Omega))} + \|u^0\|^p_{W_0^{1,p}(\Omega)}.
\end{aligned}$$

The result follows by density. □

The following proposition will be fundamental to guarantee the null controllability of the system (2.3) and its proof will be acquired from the previous results of this section.

Proposition 2.6. *Let the assumptions in Proposition 2.4 be satisfied. Then, the following estimates are valid*

$$\begin{aligned}
&\|(\kappa\theta)_t\|_{L^q(0,T;L^p(\Omega))} + \|\kappa\theta\|_{L^q(0,T;W^{2,p}(\Omega))} + \|\kappa\theta\|_{C^0(0,T;W^{1,p}(\Omega))} \\
&\leq C\left(\|y^0\|_{V^p} + \|\theta^0\|_{W_0^{1,p}(\Omega)} + \|\rho_3 F_1\|_{L^q(0,T;L^p(\Omega)^N)} + \|\rho_3 F_2\|_{L^q(0,T;L^p(\Omega))}\right)
\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}
&\|(\kappa y)_t\|_{L^q(0,T;L^p(\Omega)^N)} + \|\kappa y\|_{L^q(0,T;W^{2,p}(\Omega)^N)} + \|\kappa y\|_{C^0(0,T;W^{1,p}(\Omega)^N)} \\
&\leq C\left(\|y^0\|_{V^p} + \|\theta^0\|_{W_0^{1,p}(\Omega)} + \|\rho_3 F_1\|_{L^q(0,T;L^p(\Omega)^N)} + \|\rho_3 F_2\|_{L^q(0,T;L^p(\Omega))}\right)
\end{aligned} \tag{2.35}$$

Proof. Define $\tau = \kappa\theta$. Then, by (2.7), we have

$$\begin{cases} \tau_t - \nu_0 \Delta \tau = b & \text{in} \quad Q, \\ \tau(x,t) = 0 & \text{on} \quad \Sigma, \\ \tau(x,0) = \kappa(0)\theta^0(x) & \text{in} \quad \Omega, \end{cases}$$

where $b = \kappa v_0 \chi_\omega + \kappa F_2 + \kappa_t \theta$. Thus, as consequence of (2.13), (2.24) and (2.26) together with the fact that $C^0(0,T;H^1(\Omega)) \hookrightarrow L^q(0,T;L^p(\Omega))$ we have $b \in L^q(0,T;L^p(\Omega))$. Then, applying the Lemma 2.1 and Proposition 2.5, we get (2.34).

Now, notice that, $(\gamma, \bar{P}) = (\kappa y, \kappa P)$ solve the Stokes equation

$$\begin{cases} \gamma_t - \nu_0 \Delta \gamma + \nabla \bar{P} = \bar{b}, \ \nabla \cdot \gamma = 0 & \text{in} \quad Q, \\ \gamma(x,t) = 0 & \text{on} \quad \Sigma, \\ \gamma(x,0) = \kappa(0) y^0(x) & \text{in} \quad \Omega, \end{cases}$$

where $\bar{b} = \kappa v \chi_\omega + \kappa_t y + \nu_0 \kappa \theta e_N + \kappa F_1$. Hence, applying (2.13), (2.23), (2.25) and (2.34) we have $\bar{b} \in L^q(0,T;L^p(\Omega)^N)$. Then, from the Lemma 2.3 and again by Proposition 2.5, (2.35) is acquired. □

2.4 Proofs of the main theorems

In this section we will prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1

Here, we will proved the local null controllability for the system (2.1). Let us consider the Stokes operator $A : D(A) \to H$, where $D(A) := V \cap H^2(\Omega)^N$, $Aw = P(-\Delta w)$ for all $w \in D(A)$ and $P : L^2(\Omega)^N \to H$ is the orthogonal projector.

Let $\mathcal{E}_N$ be (for $N = 2$ or $N = 3$) the following space:

$$\begin{gathered} \mathcal{E}_N = \{(y,P,\theta,v,v_0) : \rho_1 y, \rho_2 v \chi_\omega \in L^2(Q)^N, y \in L^2(0,T;D(A)), P \in L^2(0,T;H^1(\Omega)), \\ \rho_1\theta, \rho_2 v_0 \chi_\omega \in L^2(Q), \theta \in L^2(0,T;W^{2,3/2}(\Omega)), \text{ for } F_1 := \mathcal{L}_1 y + \nabla P - \nu_0 \theta e_N - v\chi_\omega \\ \text{and } F_2 := \mathcal{L}_2\theta - v_0\chi_\omega, \rho_3 F_1 \in L^2(Q)^N, \rho_3 F_2 \in L^2(0,T;L^{3/2}(\Omega)), \\ \nabla \cdot y \equiv 0, y(.,0) \in V, \theta(.,0) \in W_0^{1,3/2}(\Omega), \theta|_\Sigma = 0\}, \end{gathered} \tag{2.36}$$

emphasizing that $\mathcal{L}_1 y = y_t - \nu_0 \Delta y$ and $\mathcal{L}_2 \theta = \theta_t - \nu_0 \Delta\theta$. Thus, it's clear that $\mathcal{E}_N$ is a Banach space for the norm $\|.\|_{\mathcal{E}_N}$, where

$$\begin{aligned} &\|(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_N} := \|y\|^2_{L^2(0,T;D(A))} + \|\theta\|^2_{L^2(0,T;W^{2,3/2}(\Omega))} \\ &+ \|\rho_1 y\|^2_{L^2(Q)^N} + \|\rho_1\theta\|^2_{L^2(Q)} + \|p\|^2_{L^2(0,T;H^1(\Omega))} + \|\rho_2 v\|^2_{L^2(\omega\times(0,T))^N} \\ &+ \|\rho_2 v_0\|^2_{L^2(\omega\times(0,T))} + \|\rho_3(\mathcal{L}_1 y + \nabla p - \nu_0\theta e_N - v 1_\omega)\|^2_{L^2(Q)^N} \\ &+ \|\rho_3(\mathcal{L}_2\theta - v_0 1_\omega)\|^2_{L^2(0,T;L^{3/2}(\Omega))} + \|\theta(.,0)\|^2_{W^{1,3/2}(\Omega)}. \end{aligned}$$

Due to Proposition 2.3 and linear system (2.7) we get:

$$\begin{aligned} &\|\mu_1 y\|_{L^\infty(0,T;L^2(\Omega)^N)} + \|\mu_1 y\|_{L^2(0,T;H^1(\Omega)^N)} + \|\mu_2 y\|_{L^\infty(0,T;H^1(\Omega)^N)} \\ &+ \|\mu_2 y\|_{L^2(0,T;H^2(\Omega)^N)} + \|\mu_2 y_t\|_{L^2(0,T;L^2(\Omega)^N)} + \|\mu_2\theta_t\|_{L^2(0,T;L^{3/2}(\Omega))} \\ &+ \|\mu_2\theta\|_{L^2(0,T;W^{2,3/2}(\Omega))} \le C\|(y,p,\theta,v,v_0)\|_{\mathcal{E}_N}. \end{aligned} \tag{2.37}$$

Furthermore, if $(y, p, \theta, v, v_0) \in \mathcal{E}_N$, then $y_t \in L^2(Q)^N$, whence $y : [0, T] \to V$ is continuous (see, [Eva10]) and we have $y(., 0) \in V$, with

$$\|y(., 0)\|_V \leq C\|(y, p, \theta, v, v_0)\|_{\mathcal{E}_N}, \tag{2.38}$$

Now, let us introduce the Banach Space

$$\mathcal{Z}_N = L^2(\rho_3^2; Q)^N \times V \times L^2(\rho_3^2(0, T); L^{3/2}(\Omega)) \times W_0^{1,3/2}(\Omega), \tag{2.39}$$

where $L^2(\rho_3^2(0, T); L^{3/2}(\Omega))$ be the Hilbert space formed by the measurable functions $u = u(x; t)$ such that $\rho_3 u \in L^2(0, T; L^{3/2}(\Omega))$, i.e.,

$$\|u\|^2_{L^2(\rho_3^2(0,T);L^{3/2}(\Omega))} = \int_0^T \rho_3^2 \|u(t)\|^2_{L^{3/2}(\Omega)} dt < +\infty.$$

Replacing $L^{3/2}(\Omega)$ by $L^2(\Omega)$ in $L^2(\rho_3^2(0, T); L^{3/2}(\Omega))$, we get $L^2(\rho_3^2; Q)$.

Finally, consider also the mapping $\mathcal{F} : \mathcal{E}_N \to \mathcal{Z}_N$, such that

$$\mathcal{F}(y, p, \theta, v, v_0) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)(y, p, \theta, v, v_0) \tag{2.40}$$

where

$$\begin{cases} \mathcal{F}_1(y, p, \theta, v, v_0) := y_t - \nu(\nabla y)\Delta y + (y \cdot \nabla)y + \nabla p - \nu_0 \theta e_N - v 1_\omega, \\ \mathcal{F}_2(y, p, \theta, v, v_0) := y(., 0), \\ \mathcal{F}_3(y, p, \theta, v, v_0) := \theta_t - \nu(\nabla y)\Delta\theta + y \cdot \nabla\theta - \nu(\nabla y)Dy : \nabla y - v_0 1_\omega, \\ \mathcal{F}_4(y, p, \theta, v, v_0) := \theta(., 0). \end{cases} \tag{2.41}$$

Note that, in $(2.41)_1$ we used the definition of $\nabla \cdot (\nu(\nabla y)Dy)$ to rewrite in the form $\nu(\nabla y)\Delta y$, since $\nabla \cdot y = 0$.

We are interested in apply the Mapping Inverse Theorem in infinite dimensional spaces, that can be found in [ATF87], and is given below, where $B_r(0)$ and $B_\delta(\zeta_0)$ are open ball, respectively of radius r and δ.

Theorem 2.4 (Liusternik's Inverse Mapping Theorem)**.** *Let $\mathcal{E}$ and $\mathcal{Z}$ be Banach spaces and let $\mathcal{F} : B_r(0) \subset \mathcal{E} \to \mathcal{Z}$ be a $\mathcal{C}^1$ mapping. Let as assume that $\mathcal{F}'(0)$ is onto and let us set $\mathcal{F}(0) = \zeta_0$. Then, there exist $\delta > 0$, a mapping $W : B_\delta(\zeta_0) \subset \mathcal{Z} \to \mathcal{E}$ and a constant $K > 0$ such that*

$$W(z) \in B_r(0),\ \mathcal{F}(W(z)) = z \text{ and } \|W(z)\|_{\mathcal{E}} \leq K\|z - \mathcal{F}(0)\|_{\mathcal{Z}}\ \forall z \in B_\delta(\zeta_0).$$

In particular, W is a local inverse-to-the-right of $\mathcal{F}$.

Thus, we will prove that we can apply this Theorem 2.4 to the mapping $\mathcal{F}$ in (2.40)-(2.41), through the following three lemmas:

Lemma 2.6. *Let $\mathcal{F} : \mathcal{E}_N \to \mathcal{Z}_N$ be given by (2.40)-(2.41). Then, $\mathcal{F}$ is well defined, and is continuous around the origin.*

Proof. We will do the proof for the $N = 3$ case, the $N = 2$ case is similar.

We want to show that $\mathcal{F}(y,p,\theta,v,v_0)$ belongs to $\mathcal{Z}_3$, for every $(y,p,\theta,v,v_0)\in\mathcal{E}_3$. To do this, we will show that each $\mathcal{F}_i(y,p,\theta,v,v_0)$, with $i=\{1,2,3,4\}$, defined in (2.41) belongs to its respective space. Note that,

$$\|\mathcal{F}_1(y,p,\theta,v,v_0)\|^2_{L^2(\rho_3^2;Q)^3} \leq 3\|\rho_3(\mathcal{L}_1 y+\nabla p-\nu_0\theta e_N - v1_\omega)\|^2_{L^2(Q)^3} + 3\|\rho_3(y.\nabla)y)\|^2_{L^2(Q)^3} + 3\|\rho_3\nu_1\|\nabla y\|^2\Delta y\|^2_{L^2(Q)^3}.$$

By the definition of $\mathcal{E}_3$ we have that

$$\|\rho_3(\mathcal{L}_1 y+\nabla p-\nu_0\theta e_N - v1_\omega)\|^2_{L^2(Q)^3} \leq C\,\|(y,v,\theta,v,v_0)\|^2_{\mathcal{E}_3}. \tag{2.42}$$

Also, from (2.13), in view of (2.37) and by continuous immersion $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ we have

$$\begin{aligned}\|\rho_3(y.\nabla)y)\|^2_{L^2(Q)^3} &\leq C\iint_Q \mu_2^4|y|^2|\nabla y|^2 dx\,dt\\ &\leq C\left(\sup_{[0,T]}\int_\Omega \mu_2^2|\nabla y|^2\,dx\right)\left(\int_0^T \mu_2^2\|y\|^2_{L^\infty(\Omega)^3}dt\right)\\ &\leq C\,\|\mu_2 y\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|\mu_2 y\|^2_{L^2(0,T;H^2(\Omega)^3)}\\ &\leq C\,\|(y,v,\theta,v,v_0)\|^4_{\mathcal{E}_3}.\end{aligned} \tag{2.43}$$

And, since $\mu_2^{-1}\leq C$,

$$\begin{aligned}\|\rho_3\nu_1\|\nabla y\|^2\Delta y\|^2_{L^2(Q)^3} &\leq C\int_0^T \mu_2^{-2}\mu_2^4\|\nabla y\|^4\int_\Omega|\mu_2\Delta y|^2dx\,dt\\ &\leq C\left(\sup_{[0,T]}\mu_2^2\|\nabla y\|^2\right)^2\int_\Omega|\mu_2\Delta y|^2dx\\ &\leq C\,\|\mu_2 y\|^4_{L^\infty(0,T;H^1(\Omega)^3)}\|\mu_2 y\|^2_{L^2(0,T;H^2(\Omega)^3)}\\ &\leq C\,\|(y,p,\theta,v,v_0)\|^6_{\mathcal{E}_3}.\end{aligned} \tag{2.44}$$

Therefore, by (2.42), (2.43) and (2.44) we get $\mathcal{F}_1(y,p,\theta,v,v_0)\in L^2(\rho_3^2;Q)^3$. From the inequality (2.38) it follows that $\mathcal{F}_2(y,p,\theta,v,v_0)\in V$.

Now, for $\mathcal{F}_3(y,p,\theta,v,v_0)$ note the following:

$$\begin{aligned}&\|\mathcal{F}_3(y,p,\theta,v,v_0)\|^2_{L^2(\rho_3^2(0,T);L^{3/2}(\Omega))} \leq C\,\Big(\|\rho_3(\mathcal{L}_2\theta - v_0 1_\omega)\|^2_{L^2(0,T;L^{3/2}(\Omega))}\\ &+\|\rho_3 y\cdot\nabla\theta\|^2_{L^2(0,T;L^{3/2}(\Omega))} + \|\rho_3\nu_1\|\nabla y\|^2\Delta\theta\|^2_{L^2(0,T;L^{3/2}(\Omega))}\\ &+\|\rho_3(\nu_0+\nu_1\|\nabla y\|^2)Dy:\nabla y\|^2_{L^2(0,T;L^{3/2}(\Omega))}\Big) = C\textstyle\sum_{s=1}^4 X_s.\end{aligned} \tag{2.45}$$

Let's analyze each $X_s, s=\{1,2,3,4\}$:

- $X_1\leq C\,\|(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_3}$;

Using the continuous embedding $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$,

$$\bullet\ X_2 \le \int_0^T \rho_3^2 \left[\left(\int_\Omega |y|^3\,dx \right)^{1/2} \left(\int_\Omega |\nabla\theta|^3\,dx \right)^{1/2} \right]^{4/3} dt$$

$$= \int_0^T \rho_3^2 \|y\|^2_{L^3(\Omega)^3} \|\nabla\theta\|^2_{L^3(\Omega)^3}\,dt \le C \int_0^T \mu_2^4 \|\nabla y\|^2 \|\nabla\theta\|^2_{W^{1,3/2}(\Omega)^3}\,dt$$

$$\le C \int_0^T \mu_2^4 \|\nabla y\|^2 \|\theta\|^2_{W^{2,3/2}(\Omega)^3}\,dt$$

$$\le C\|\mu_2 y\|^2_{L^\infty(0,T;H^1(\Omega)^3)} \|\mu_2\theta\|^2_{L^2(0,T;W^{2,3/2}(\Omega))}$$

$$\le C\,\|(y,p,\theta,v,v_0)\|^4_{\mathcal{E}_3};$$

$$\bullet\ X_3 \le C\int_0^T \left(\int_\Omega \mu_2^3 \|\nabla y\|^3 |\Delta\theta|^{3/2} dx \right)^{4/3} dt$$

$$\le C\int_0^T \mu_2^4 \mu_2^{-2} \|\nabla y\|^4 \left(\int_\Omega \mu_2^{3/2} |\Delta\theta|^{3/2} dx \right)^{4/3} dt$$

$$\le C\left(\sup_{[0,T]} \mu_2^2 \|\nabla y\|^2 \right)^2 \int_0^T \left(\int_\Omega \mu_2^{3/2} |\Delta\theta|^{3/2}\,dx \right)^{4/3} dt$$

$$\le C\,\|\mu_2 y\|^4_{L^\infty(0,T;H^1(\Omega)^3)} \|\mu_2\theta\|^2_{L^2(0,T;W^{2,3/2}(\Omega))}$$

$$\le C\,\|(y,p,\theta,v,v_0)\|^6_{\mathcal{E}_3};$$

$$\bullet\ X_4 \le C\int_0^T \left(\int_\Omega \mu_2^3 (\nu_0 + \nu_1\|\nabla y\|^2)^{3/2} |\nabla y|^3\,dx \right)^{4/3} dt$$

$$\le C\int_0^T (\nu_0 + \nu_1\|\nabla y\|^2)^2 \left(\int_\Omega \mu_2^3 |\nabla y|^3 dx \right)^{4/3} dt$$

$$\le C\int_0^T 2(\nu_0^2 + \nu_1^2\|\nabla y\|^4) \left(\int_\Omega \mu_2^3 |\nabla y|^3 dx \right)^{4/3} dt$$

$$\le C\int_0^T \left(\int_\Omega \mu_2^3 |\nabla y|^3\,dx \right)^{4/3} dt + C\int_0^T \mu_2^{-4}\mu_2^4 \|\nabla y\|^4 \left(\int_\Omega \mu_2^3 |\nabla y|^3\,dx \right)^{4/3} dt$$

$$= C\,(\tilde{K}_1 + \tilde{K}_2).$$

First let's analyze $\tilde{K}_1$. Since $\nabla(\mu_2 y)$ belongs to $L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^1(\Omega)^3)$ then using the Lemma 6.7, in Chapter 1, from [Lio69], which ensures continuous embedding

$$L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;H^1(\Omega)^3) \hookrightarrow L^4(0,T;L^3(\Omega)^3), \tag{2.46}$$

we have $\nabla(\mu_2 y) \in L^4(0,T;L^3(\Omega)^3)$. Even more,

$$\|\nabla(\mu_2 y)\|_{L^4(0,T;L^3(\Omega)^3)} \le C\|\nabla(\mu_2 y)\|^{1/2}_{L^\infty(0,T;L^2(\Omega)^3)} \|\nabla(\mu_2 y)\|^{1/2}_{L^2(0,T;H^1(\Omega)^3)}.$$

Thereat,

$$\begin{array}{rcl}\tilde{K}_1 & \leq & C\,\|\nabla(\mu_2 y)\|^2_{L^\infty(0,T;L^2(\Omega)^3)}\|\nabla(\mu_2 y)\|^2_{L^2(0,T;H^1(\Omega)^3)}.\\ & \leq & C\,\|\mu_2 y\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|\mu_2 y\|^2_{L^2(0,T;H^2(\Omega)^3)}\\ & \leq & C\|(y,p,\theta,v,v_0)\|^4_{\mathcal{E}_3}.\end{array}$$

And again using the fact that $\mu_2^{-1} \leq C$, we get

$$\begin{array}{rcl}\tilde{K}_2 & \leq & C\left(\sup_{[0,T]}\int_\Omega \mu_2^2|\nabla y|^2\,dx\right)^2 \int_0^T \left(\int_\Omega \mu_2^3|\nabla y|^3\,dx\right)^{4/3} dt\\ & \leq & C\,\|(y,p,\theta,v,v_0)\|^8_{\mathcal{E}_3}.\end{array}$$

Thus,

$$\bullet\, X_4 \leq C(1+\|(y,p,\theta,v,v_0)\|^4_{\mathcal{E}_3})\|(y,p,\theta,v,v_0)\|^4_{\mathcal{E}_3} \tag{2.47}$$

and consequently, from what we saw for each $X_s, s=\{1,2,3,4\}$, we obtain by (2.45) that

$$\mathcal{F}_3(y,p,\theta,v,v_0) \in L^2(\rho_3^2(0,T);L^{3/2}(\Omega)).$$

Finally, without difficulties, we can see that $\mathcal{F}_4(y,p,\theta,v,v_0) \in W_0^{1,3/2}(\Omega)$. This prove that $\mathcal{F}$ is well defined.

The verification that $\mathcal{F}$ is continuous around the origin is done in a similar way. With this, we have the proof of the lemma. □

Lemma 2.7. *The mapping $\mathcal{F} : \mathcal{E}_N \longrightarrow \mathcal{Z}_N$ is continuously differentiable.*

Proof. We will the proof for $N=3$ (the case $N=2$ is similar). Let us first prove that $\mathcal{F}$ is Gâteaux-differentiable at any $(y,p,\theta,v,v_0) \in \mathcal{E}_3$ and let us compute the *G-derivative* $\mathcal{F}'(y,p,\theta,v,v_0)$.

Let us fix $(y,p,\theta,v,v_0) \in \mathcal{E}_3$ and let us take $(y',p',\theta',v',v_0') \in \mathcal{E}_3$ and $\sigma > 0$. Also, by the decomposition made in (2.41), we introduce the linear mapping $\mathcal{DF} : \mathcal{E}_3 \longrightarrow \mathcal{Z}_3$ with $\mathcal{DF}(y,p,\theta,v,v_0) = \mathcal{DF} = (\mathcal{DF}_1, \mathcal{DF}_2, \mathcal{DF}_3, \mathcal{DF}_4)$ where

$$\begin{cases} \mathcal{DF}_1(y',p',\theta',v',v_0') := y_t' - (\nu_0+\nu_1\|\nabla y\|^2)\Delta y' - 2\nu_1(\nabla y,\nabla y')\Delta y + \nabla p' \\ \qquad\qquad + (y'\cdot\nabla)y + (y\cdot\nabla)y' - \nu_0\theta' e_3 - v' 1_\omega, \\ \mathcal{DF}_2(y',p',\theta',v',v_0') := y'(.,0), \\ \mathcal{DF}_3(y',p',\theta',v',v_0') := \theta_t' - (\nu_0+\nu_1\|\nabla y\|^2)\Delta\theta' - 2\nu_1(\nabla y,\nabla y')\Delta\theta \\ \qquad\qquad + y'\cdot\nabla\theta + y\cdot\nabla\theta' - v_0' 1_\omega - (\nu_0+\nu_1\|\nabla y\|^2)Dy : \nabla y' \\ \qquad\qquad - \left[(\nu_0+\nu_1\|\nabla y\|^2)Dy' + 2\nu_1(\nabla y,\nabla y')Dy\right] : \nabla y, \\ \mathcal{DF}_4(y',p',\theta',v',v_0') := \theta'(.,0). \end{cases} \tag{2.48}$$

From the definition of the spaces $\mathcal{E}_3, \mathcal{Z}_3$ and (2.48), it becomes clear that $\mathcal{DF} \in \mathcal{L}(\mathcal{E}_3, \mathcal{Z}_3)$. Furthermore, for each $j=\{1,2,3,4\}$ we have

$$\begin{array}{l}\frac{1}{\sigma}[\mathcal{F}_j\left((y,p,\theta,v,v_0)+\sigma(y',p',\theta',v',v_0')\right) - \mathcal{F}_j(y,p,\theta,v,v_0)]\\ \text{converges to}\, \mathcal{DF}_j(y',p',\theta',v',v_0') \text{ strong in } \mathcal{Z}_3, \text{ as } \sigma \longrightarrow 0.\end{array} \tag{2.49}$$

Let us show that (2.49) is true. Firstly, we state that,

$$\begin{aligned}&\frac{1}{\sigma}[\mathcal{F}_1\left((y,p,\theta,v,v_0)+\sigma(y',p',\theta',v',v_0')\right)-\mathcal{F}_1(y,p,\theta,v,v_0)]\\ &\text{converges to}\, \mathcal{D}\mathcal{F}_1(y',p',\theta',v',v_0')\text{ strong in } L^2(\rho_3^2;Q)^3,\ \text{as}\,\sigma\longrightarrow 0.\end{aligned}\tag{2.50}$$

Indeed,

$$\begin{aligned}&\|\frac{1}{\sigma}[\mathcal{F}_1\left((y,p,\theta,v,v_0)+\sigma(y',p',\theta',v',v_0')\right)-\mathcal{F}_1(y,p,\theta,v,v_0)]\\ &-\mathcal{D}\mathcal{F}_1(y',p',\theta',v',v_0')\|_{L^2(\rho_3^2;Q)^3}\leq\sigma\|(y'\cdot\nabla)y'\|_{L^2(\rho_3^2;Q)^3}\\ &+\|\frac{\nu_1}{\sigma}\left(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2\right)\Delta y-2\nu_1(\nabla y,\nabla y')\Delta y\|_{L^2(\rho_3^2;Q)^3}\\ &+\|\nu_1(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2)\Delta y'\|_{L^2(\rho_3^2;Q)^3}=H_1+H_2+H_3.\end{aligned}$$

Note that, as proved in the Lemma 2.6,

$$H_1^2\leq C\sigma\|(y',p',\theta',v',v_0')\|_{\mathcal{E}_3}^4$$

and we see that, $H_1\longrightarrow 0$, as $\sigma\longrightarrow 0$.

Also, as $\sigma\longrightarrow 0$,

$$H_2^2=\nu_1^2\iint\limits_Q\rho_3^2|\frac{1}{\sigma}\left(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2\right)\Delta y-2\nu_1(\nabla y,\nabla y')\Delta y|^2dxdt\longrightarrow 0$$

and

$$H_3^2=\nu_1^2\iint\limits_Q\rho_3^2|\,\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2\,|^2|\Delta y'|^2dxdt\longrightarrow 0.$$

Thus, (2.50) holds.

Now, that the difference quotient

$$\frac{1}{\sigma}[\mathcal{F}_j\left((y,p,\theta,v,v_0)+\sigma(y',p',\theta',v',v_0')\right)-\mathcal{F}_j(y,p,\theta,v,v_0)]\tag{2.51}$$

converges to $\mathcal{D}\mathcal{F}_j(y',p',\theta',v',v_0')$ strong for $j=2$ and $j=4$, respectively, in V and $W_0^{1,3/2}(\Omega)$, as $\sigma\longrightarrow 0$, is immediate.

Finally, let's see that

$$\begin{aligned}&\frac{1}{\sigma}[\mathcal{F}_3\left((y,p,\theta,v,v_0)+\sigma(y',p',\theta',v',v_0')\right)-\mathcal{F}_3(y,p,\theta,v,v_0)]\\ &\text{converges to}\,\mathcal{D}\mathcal{F}_3(y',p',\theta',v',v_0')\text{ strong in } L^2(\rho_3^2(0,T);L^{3/2}(\Omega)),\ \text{as}\,\sigma\longrightarrow 0.\end{aligned}\tag{2.52}$$

For simplicity, we omit the notation of inequality norms below but let it be clear that they are all norms in $L^2(\rho_3^2(0,T);L^{3/2}(\Omega))$. More precisely,

$$\begin{aligned}&\|\frac{1}{\sigma}[\mathcal{F}_3\left((y,p,\theta,v,v_0)+\sigma(y',p',\theta',v',v_0')\right)-\mathcal{F}_3(y,p,\theta,v,v_0)]\\ &-\mathcal{D}\mathcal{F}_3(y',p',\theta',v',v_0')\|\leq\sigma\|y'\cdot\nabla\theta'\|\\ &+\sigma\|(\nu_0+\nu_1\|\nabla(y+\sigma y')\|^2)Dy':\nabla y'\|\\ &+\|\frac{\nu_1}{\sigma}\left(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2\right)\Delta\theta-2\nu_1(\nabla y,\nabla y')\Delta\theta\|\\ &+\|\nu_1(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2)\Delta\theta'\|\\ &+\|\frac{\nu_1}{\sigma}\left(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2\right)Dy:\nabla y-2\nu_1(\nabla y,\nabla y')Dy:\nabla y\|\\ &+\|\nu_1(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2)Dy:\nabla y'\|\\ &+\|\nu_1(\|\nabla(y+\sigma y')\|^2-\|\nabla y\|^2)Dy':\nabla y\|=\sum_{j=1}^{7}I_j.\end{aligned}$$

By the same arguments from the proof of Lemma 2.6 (see, X_2, X_3 and X_4) together with those used in (2.50), we have

$$\sum_{j=1}^{5} I_j \longrightarrow 0, \text{ as } \sigma \longrightarrow 0.$$

Also, since

$$\begin{aligned}
I_6^2 &\leq C \int_0^T \left(2\sigma\|\nabla y\|\|\nabla y'\| + \sigma^2\|\nabla y'\|^2\right)^2 \mu_2^4 \left[\int_\Omega |\nabla y|^{3/2}|\nabla y'|^{3/2}dx\right]^{4/3} dt \\
&\leq C\,\sigma^2 \left(\|y\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|y'\|^2_{L^\infty(0,T;H^1(\Omega)^3)} + \sigma^2\|y'\|^4_{L^\infty(0,T;H^1(\Omega)^3)}\right) \\
&\qquad \int_0^T \mu_2^4 \left[\left(\int_\Omega |\nabla y|^3 dx\right)^{1/2} \left(\int_\Omega |\nabla y'|^3 dx\right)^{1/2}\right]^{4/3} dt \\
&= C\,\sigma^2 \left(\|y\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|y'\|^2_{L^\infty(0,T;H^1(\Omega)^3)} + \sigma^2\|y'\|^4_{L^\infty(0,T;H^1(\Omega)^3)}\right) \\
&\qquad \left(\int_0^T \mu_2^4 \|\nabla y\|^2_{L^3(\Omega)^N}\|\nabla y'\|^2_{L^3(\Omega)^N} dt\right) \\
&\leq C\,\sigma^2 \left(\|y\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|y'\|^2_{L^\infty(0,T;H^1(\Omega)^3)} + \sigma^2\|y'\|^4_{L^\infty(0,T;H^1(\Omega)^3)}\right) \\
&\qquad \frac{1}{2}\left(\int_0^T \|\nabla(\mu_2 y)\|^4_{L^3(\Omega)^3} dt + \int_0^T \|\nabla(\mu_2 y')\|^4_{L^3(\Omega)^3} dt\right) \\
&= C\,\sigma^2 \left(\|y\|^2_{L^\infty(0,T;H^1(\Omega)^3)}\|y'\|^2_{L^\infty(0,T;H^1(\Omega)^3)} + \sigma^2\|y'\|^4_{L^\infty(0,T;H^1(\Omega)^3)}\right) \\
&\qquad \left(\|\nabla(\mu_2 y)\|^4_{L^4(0,T;L^3(\Omega)^3)} + \|\nabla(\mu_2 y')\|^4_{L^4(0,T;L^3(\Omega)^3)}\right)
\end{aligned}$$

and from continuous embedding (2.46), we have that I_6 is bounded and therefore $I_6 \longrightarrow 0$, as $\sigma \longrightarrow 0$. By the same arguments we also have $I_7 \longrightarrow 0$, as $\sigma \longrightarrow 0$. Consequently, (2.52) is hold.

Therefore, from (2.50), (2.51) and (2.52) we have (2.49) and $\mathcal{F}$ is Gâteaux-differentiable at any $(y, p, \theta, v, v_0) \in \mathcal{E}_3$, with *G-derivative* $\mathcal{F}'(y, p, \theta, v, v_0) = \mathcal{DF}(y, p, \theta, v, v_0)$.

Now, let us prove that $(y, p, \theta, v, v_0) \longmapsto \mathcal{F}'(y, p, \theta, v, v_0)$ is a continuous mapping. Thus, we will prove that $\mathcal{F}$ is not only Gâteaux-differentiable, but also Fréchet-differentiable and $\mathcal{C}^1$. For that, suppose that

$$(y_m, p_m, \theta_m, v_m, v_{0m}) \longrightarrow (y, p, \theta, v, v_0) \text{ in } \mathcal{E}_3$$

and we will prove the existence of $\varepsilon_m(y, p, \theta, v, v_0)$ such that

$$\begin{aligned}
&\|\left(\mathcal{F}'(y_m, p_m, \theta_m, v_m, v_{0m}) - \mathcal{F}'(y, p, \theta, v, v_0)\right)(y', p', \theta', v', v_0')\|^2_{\mathcal{Z}_3} \\
&\qquad \leq \varepsilon_m \|(y', p', \theta', v', v_0')\|^2_{\mathcal{E}_3},
\end{aligned} \tag{2.53}$$

for all $(y', p', \theta', v', v_0') \in \mathcal{E}_3$ and $\lim\limits_{m\to\infty} \varepsilon_m = 0$.

In order to simplify the notation, we will consider

$$\mathcal{D}_{j,m} := \mathcal{F}_j'(y_m, p_m, \theta_m, v_m, v_{0m}) - \mathcal{F}_j'(y, p, \theta, v, v_0).$$

So, notice that

- $\|\mathcal{D}_{1,m}(y',p',\theta',v',v_0')\|^2_{L^2(\rho_3^2;Q)^3}$
$\leq 3\|\nu_1\left(\|\nabla y\|^2-\|\nabla y_m\|^2\right)\Delta y'\|^2_{L^2(\rho_3^2;Q)^3}$
$+3\|2\nu_1(\nabla y,\nabla y')\Delta y-2\nu_1(\nabla y_m,\nabla y')\Delta y_m\|^2_{L^2(\rho_3^2;Q)^3}$
$+3\|(y'\cdot\nabla)(y_m-y)+((y_m-y)\cdot\nabla)y'\|^2_{L^2(\rho_3^2;Q)^3}$
$=3K_1+12K_2+3K_3.$

Since,

$$K_1\leq C\Big(\|\|\nabla(y-y_m)\|\|\nabla y\|\Delta y'\|^2_{L^2(\rho_3^2;Q)^3} + \|\|\nabla(y-y_m)\|\|\nabla y_m\|\Delta y'\|^2_{L^2(\rho_3^2;Q)^3}\Big)$$

then, using the same arguments as (2.44), we conclude that

$$K_1\leq \varepsilon_{1,m}\|(y',p',\theta',v',v_0')\|^2_{\mathcal{E}_3}$$

where

$$\varepsilon_{1,m}=C\|(y_m,p_m,\theta_m,v_m,v_{0m})-(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_3}\left(\|(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_3} +\|(y_m,p_m,\theta_m,v_m,v_{0m})\|^2_{\mathcal{E}_3}\right).$$

For K_2 let's first see the following, adding and subtracing $\nu_1(\nabla y_m,\nabla y')\Delta y$ in K_2, we have

$$\begin{aligned}K_2&=\|-\nu_1(\nabla(y_m-y),\nabla y')\Delta y-\nu_1(\nabla y_m,\nabla y')\Delta(y_m-y)\|^2_{L^2(\rho_3^2;Q)^3}\\
&\leq C\left(\|\nu_1(\nabla(y_m-y),\nabla y')\Delta y\|^2_{L^2(\rho_3^2;Q)^3}+\|\nu_1(\nabla y_m,\nabla y')\Delta(y_m-y)\|^2_{L^2(\rho_3^2;Q)^3}\right)\\
&\leq \varepsilon_{2,m}\|(y',p',\theta',v',v_0')\|^2_{\mathcal{E}_3},\end{aligned}$$

where

$$\varepsilon_{2,m}=C\|(y_m,p_m,\theta_m,v_m,v_{0m})-(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_3}\left(\|(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_3} +\|(y_m,p_m,\theta_m,v_m,v_{0m})\|^2_{\mathcal{E}_3}\right).$$

And, by the same reasoning as (2.43),

$$\begin{aligned}K_3&\leq C\left(\|(y'\cdot\nabla)(y_m-y)\|^2_{L^2(\rho_3^2;Q)^3}+\|((y_m-y)\cdot\nabla)y'\|^2_{L^2(\rho_3^2;Q)^3}\right)\\
&\leq \varepsilon_{3,m}\|(y',p',\theta',v',v_0')\|^2_{\mathcal{E}_3},\end{aligned}$$

with

$$\varepsilon_{3,m}=C\|(y_m,p_m,\theta_m,v_m,v_{0m})-(y,p,\theta,v,v_0)\|^2_{\mathcal{E}_3}.$$

It is easy to check that $\mathcal{D}_{j,m}$ for $j=2$ and $j=4$ satisfy similar inequalities.

Again, all inequality norms below are norms in $L^2(\rho_3^2(0,T);L^{3/2}(\Omega))$, we will omit them for simplicity. For $\mathcal{D}_{3,m}$ after some calculations we get the following:

- $\|\mathcal{D}_{3,m}(y',p',\theta',v',v_0')\|^2\leq C\left[\|\,\|\nabla(y_m-y)\|\,\|\nabla y\|\,\Delta\theta'\,\|^2\right.$
$+\|\,\|\nabla(y_m-y)\|\,\|\nabla y_m\|\Delta\theta'\,\|^2+\|\nu_1(\nabla y_m,\nabla y')\Delta(\theta_m-\theta)\|^2$
$+\|\nu_1(\nabla(y_m-y),\nabla y')\Delta\theta\|^2+\|(\nu_0+\nu_1\|\nabla y_m\|^2)D(y_m-y):\nabla y'\|^2$
$+\|\nu_1(\|\nabla y_m\|^2-\|\nabla y\|^2)Dy:\nabla y'\|^2+\|(\nu_0+\nu_1\|\nabla y_m\|^2)Dy':\nabla(y_m-y)\|^2$
$+\|\nu_1(\|\nabla y_m\|^2-\|\nabla y\|^2)Dy':\nabla y\|^2+\|\nu_1(\nabla y_m,\nabla y')Dy_m:\nabla(y_m-y)\|^2$
$+\|\nu_1(\nabla(y_m-y),\nabla y')Dy_m:\nabla y\|^2+\|\nu_1(\nabla y,\nabla y')D(y_m-y):\nabla y\|^2$
$\left.+\|y'\cdot\nabla(\theta_m-\theta)\|^2+\|(y_m-y)\cdot\nabla\theta'\|^2\right]=C\displaystyle\sum_{s=4}^{16}K_s.$

Let's check some terms,

$$
\begin{aligned}
K_4 &\leq C \int_0^T \left(\int_\Omega \mu_2^3 \|\nabla (y_m - y)\|^{3/2} \|\nabla y\|^{3/2} |\Delta \theta'|^{3/2} dx \right)^{4/3} dt \\
&\leq C \|\mu_2 (y_m - y)\|^2_{L^\infty(0,T;H^1(\Omega)^3)} \|\mu_2 y\|^2_{L^\infty(0,T;H^1(\Omega)^3)} \|\mu_2 \theta'\|^2_{L^2(0,T;W^{2,3/2}(\Omega))} \\
&\leq \varepsilon_{4,m} \|(y', p', \theta', v', v_0')\|^2_{\mathcal{E}_3},
\end{aligned}
$$

where

$$
\varepsilon_{4,m} = C\|(y_m, p_m, \theta_m, v_m, v_{0m}) - (y, p, \theta, v, v_0)\|^2_{\mathcal{E}_3} \|(y, p, \theta, v, v_0)\|^2_{\mathcal{E}_3}.
$$

Also,

$$
\begin{aligned}
K_8 &\leq C \int_0^T \left(\int_\Omega \mu_2^3 |(\nu_0 + \nu_1 \|\nabla y_m\|^2)|^{3/2} |\nabla (y_m - y)|^{3/2} |\nabla y'|^2 dx \right)^{4/3} dt \\
&\leq C \left(1 + \|\mu_2 y_m\|^4_{L^\infty(0,T;H^1(\Omega)^3)} \right) \||\nabla(\mu_2 y')|^2\|_{L^2(0,T;L^{3/2}(\Omega))} \\
&\quad \||\nabla(\mu_2 (y_m - y))|^2\|_{L^2(0,T;L^{3/2}(\Omega))} \\
&\leq \varepsilon_{8,m} \|(y', p', \theta', v', v_0')\|^2_{\mathcal{E}_3},
\end{aligned}
$$

where

$$
\varepsilon_{8,m} = C(1 + \|(y_m, p_m, \theta_m, v_m, v_{0m})\|^4_{\mathcal{E}_3}) \|(y_m, p_m, \theta_m, v_m, v_{0m}) - (y, p, \theta, v, v_0)\|^2_{\mathcal{E}_3}.
$$

And,

$$
K_{16} \leq \varepsilon_{16,m} \|(y', p', \theta', v', v_0')\|^2_{\mathcal{E}_3},
$$

where

$$
\varepsilon_{16,m} = C\|(y_m, p_m, \theta_m, v_m, v_{0m}) - (y, p, \theta, v, v_0)\|^2_{\mathcal{E}_3}.
$$

The other terms follow analogously.

Thus, we have $\lim\limits_{m\to\infty} \varepsilon_{s,m} = 0$ for all $s \in \{1, ..., 16\}$ and consequently follows (2.53). This ends the proof. □

Lemma 2.8. *Let $\mathcal{F}$ be the mapping in (2.40)-(2.41). Then, $\mathcal{F}'(0,0,0,0,0)$ is onto.*

Proof. Let $(F_1, y^0, F_2, \theta^0) \in \mathcal{Z}_N$. From Proposition 2.1, Proposition 2.3 and the regularity indicated in (2.5) we know that there exist (y, p, θ, v, v_0) solution of

$$
\begin{cases}
y_t - \nu_0 \Delta y + \nabla p = v 1_\omega + \nu_0 \theta e_N + F_1, \;\; \nabla \cdot y = 0 & \text{in} \quad Q, \\
\theta_t - \nu_0 \Delta \theta = v_0 1_\omega + F_2 & \text{in} \quad Q, \\
y(x,t) = 0, \;\; \theta(x,t) = 0 & \text{on} \quad \Sigma, \\
y(x,0) = y^0(x), \;\; \theta(x,0) = \theta^0(x) & \text{in} \quad \Omega.
\end{cases}
$$

satisfying the definition of $\mathcal{E}_N$. Therefore, $(y, p, \theta, v, v_0) \in \mathcal{E}_N$ and

$$
\mathcal{F}'(0,0,0,0,0)(y, p, \theta, v, v_0) = (F_1, y^0, F_2, \theta^0).
$$

Consequently, Lemma (2.8) holds. □

Proof of Theorem 2.1 We conclude from Lemmas 2.6-2.8 that the Inverse Mapping Theorem (Theorem 2.4) can be applied to the spaces $\mathcal{E}_N$ and $\mathcal{Z}_N$ together with the mapping $\mathcal{F}$ introduced at the beginning of this Section. Thus, there exists $\delta > 0$ such that, for every $(y^0, \theta^0) \in V \times W_0^{1,3/2}(\Omega)$ satisfying $\|(y^0,\theta^0)\|_{V\times W_0^{1,3/2}} < \delta$, there exists controls $v \in L^2(\omega \times (0,T))^N$ and $v_0 \in L^2(\omega\times(0,T))$ and associated solutions (y, p, θ) to (2.1) such that $y(x,T) = 0$ and $\theta(x,T) = 0$ in Ω.

This proves that, the nonlinear system (2.1) is locally null-controllable at time $T > 0$.

Proof of Theorem 2.2

Let

$$
\begin{aligned}
\mathcal{U}_N = (y, P, \theta, v, v_0) : \rho_1 y \in L^2(Q)^N, \rho_2 v, (\kappa v)_t \in L^2(\omega\times(0,T))^N, y \in L^q(0,T; W^{2,p}(\Omega)^N),\\
P \in L^q(0,T;L^p(\Omega)), \rho_1\theta \in L^2(Q), \rho_2 v_0, (\kappa v_0)_t \in L^2(\omega\times(0,T)), \theta \in L^q(0,T;W^{2,p}(\Omega)),\\
\text{for}\, F_1 := \mathcal{L}_1 y + \nabla P - \nu_0\theta e_N - v\chi_\omega \,\text{and}\, F_2 := \mathcal{L}_2\theta - v_0\chi_\omega, \rho_3 F_1 \in L^q(0,T;L^p(\Omega)^N),\\
\rho_3 F_2 \in L^q(0,T;L^p(\Omega)), \nabla\cdot y \equiv 0, y(.,0) \in V^p, \theta(.,0) \in W_0^{1,p}(\Omega),\\
y|_\Sigma = 0, \theta|_\Sigma = 0, \text{where}\, 3 < p \le 6 \,\text{and}\, p < q < \infty\},
\end{aligned}
\tag{2.54}
$$

It's clear that $\mathcal{U}_N$ is a Banach space for the norm $\|.\|_{\mathcal{U}_N}$, with

$$
\begin{aligned}
&\|(y,P,\theta,v,v_0)\|^q_{\mathcal{U}_N} := \|y\|^q_{L^q(0,T;W^{2,p}(\Omega)^N)} + \|\theta\|^q_{L^q(0,T;W^{2,p}(\Omega))} + \|\rho_1 y\|^q_{L^2(Q)^N}\\
&+ \|\rho_1\theta\|^q_{L^2(Q)} + \|P\|^q_{L^q(0,T;L^p(\Omega))} + \|\rho_2 v\|^q_{L^2(\omega\times(0,T))^N} + \|\rho_2 v_0\|^q_{L^2(\omega\times(0,T))}\\
&+ \|(\kappa v)_t\|^q_{L^2(\omega\times(0,T))^N} + \|\kappa\Delta v\|^q_{L^2(\omega\times(0,T))^N} + \|(\kappa v_0)_t\|^q_{L^2(\omega\times(0,T))}\\
&+ \|\kappa\Delta v_0\|^q_{L^2(\omega\times(0,T))} + \|\rho_3 F_1\|^q_{L^q(0,T;L^p(\Omega)^N)} + \|\rho_3 F_2\|^q_{L^q(0,T;L^p(\Omega))}\\
&+ \|y(.,0)\|^q_{V^p} + \|\theta(.,0)\|^q_{W_0^{1,p}(\Omega)}.
\end{aligned}
$$

Now, let us introduce the Banach space

$$
\mathcal{R}_N := L^q(\rho_3^q(0,T); L^p(\Omega)^N) \times V^p \times L^q(\rho_3^q(0,T); L^p(\Omega)) \times W_0^{1,p}(\Omega), \tag{2.55}
$$

and the mapping $\mathcal{I} : \mathcal{U}_N \to \mathcal{R}_N$, such that

$$
\mathcal{I}(y,P,\theta,v,v_0) = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4)(y,P,\theta,v,v_0) \tag{2.56}
$$

where

$$
\begin{cases}
\mathcal{I}_1(y,P,\theta,v,v_0) := y_t - \bar{\nu}(\nabla y)\Delta y + (y\cdot\nabla)y + \nabla P - \nu_0\theta e_N - v\chi_\omega,\\
\mathcal{I}_2(y,P,\theta,v,v_0) := y(.,0),\\
\mathcal{I}_3(y,P,\theta,v,v_0) := \theta_t - \bar{\nu}(\nabla\theta)\Delta\theta + y\cdot\nabla\theta - \bar{\nu}(\nabla y)Dy : \nabla y - v_0\chi_\omega,\\
\mathcal{I}_4(y,P,\theta,v,v_0) := \theta(.,0).
\end{cases}
\tag{2.57}
$$

To simplify the notation, in the norms of $L^p(\Omega)^N$ we will just write $L^p(\Omega)$. That said, we have the following results:

Lemma 2.9. *Let $\mathcal{I} : \mathcal{U}_N \to \mathcal{R}_N$ be given by (2.56)-(2.57). Then, $\mathcal{I}$ is well defined, and is continuous around the origin.*

Proof. Let's prove that, for each $(y, P, \theta, v, v_0) \in \mathcal{U}_N$ we have $\mathcal{I}(y, P, \theta, v, v_0) \in \mathcal{R}_N$.

That $\mathcal{I}_2$ and $\mathcal{I}_4$ are well defined follows immediately from the definition of $\mathcal{U}_N$. So let's find out $\mathcal{I}_1$ and $\mathcal{I}_3$.

Analysis of $\mathcal{I}_1$:

$$\bullet\ \|\rho_3 F_1\|^q_{L^q(0,T;L^p(\Omega))} \leq C\|(y, P, \theta, v, v_0)\|^q_{\mathcal{U}_N}.$$

Taking into account (2.10) we have $\rho_3\kappa^{-2} \leq C$. Moreover, using the fact that $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ (since $p > N$) and the estimate (2.35) from the Proposition 2.6, we obtain

$$\begin{aligned}
&\bullet\ \|\rho_3(y\cdot\nabla)y\|^q_{L^q(0,T;L^p(\Omega))} \leq \int_0^T \left(\int_\Omega \rho_3^p|y|^p|\nabla y|^p dx\right)^{q/p} dt \\
&= \int_0^T \left(\int_\Omega \rho_3^p\kappa^{-2p}\kappa^{2p}|y|^p|\nabla y|^p dx\right)^{q/p} dt \\
&\leq C\int_0^T \|\kappa y\|^q_{L^\infty(\Omega)} \left(\int_\Omega |\kappa\nabla y|^p dx\right)^{q/p} dt \\
&\leq C\|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{1,p}(\Omega))} \leq C\|(y, P, \theta, v, v_0)\|^{2q}_{\mathcal{U}_N}.
\end{aligned} \tag{2.58}$$

In a similar way

$$\begin{aligned}
&\bullet\ \|\rho_3\nu_1\|\nabla y\|^2_{L^p}\Delta y\|^q_{L^q(0,T;L^p(\Omega))} \leq C\int_0^T \rho_3^q\kappa^{-3q}\|\kappa\nabla y\|^{2q}_{L^p(\Omega)}\|\kappa\Delta y\|^q_{L^p(\Omega)} \\
&\leq C\|\kappa y\|^{2q}_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{2,p}(\Omega))} \leq C\|(y, P, \theta, v, v_0)\|^{3q}_{\mathcal{U}_N}.
\end{aligned} \tag{2.59}$$

Hence, $\mathcal{I}_1(y, P, \theta, v, v_0) \in L^q(\rho_3^q(0,T); L^p(\Omega))$.

Analysis of $\mathcal{I}_3$:

$$\bullet\ \|\rho_3 F_2\|^q_{L^q(0,T;L^p(\Omega))} \leq C\|(y, P, \theta, v, v_0)\|^q_{\mathcal{U}_N}.$$

Using the same previous arguments together with the estimates (2.34) and (2.35) from the Proposition 2.6, we get

$$\begin{aligned}
&\bullet\ \|\rho_3 y\cdot\nabla\theta\|^q_{L^q(0,T;L^p(\Omega))} \leq C\int_0^T \|\kappa\nabla\theta\|^q_{L^\infty(\Omega)}\left(\int_\Omega |\kappa y|^p dx\right)^{q/p} dt \\
&\leq C\int_0^T \|\kappa\nabla\theta\|^q_{W^{1,p}(\Omega)}\|\kappa y\|^q_{L^p(\Omega)} dt \leq C\|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa\theta\|^q_{L^q(0,T;W^{2,p}(\Omega))} \\
&\leq C\|(y, P, \theta, v, v_0)\|^{2q}_{\mathcal{U}_N};
\end{aligned}$$

$$\begin{aligned}
&\bullet\ \|\rho_3\nu_1\|\nabla\theta\|^2_{L^p}\Delta\theta\|^q_{L^q(0,T;L^p(\Omega))} \leq C\int_0^T \rho_3^q\kappa^{-3q}\|\kappa\nabla\theta\|^{2q}_{L^p(\Omega)}\|\kappa\Delta\theta\|^q_{L^p(\Omega)} dt \\
&\leq C\|\kappa\theta\|^{2q}_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa\theta\|^q_{L^q(0,T;W^{2,p}(\Omega))} \leq C\|(y, P, \theta, v, v_0)\|^{3q}_{\mathcal{U}_N};
\end{aligned}$$

and, using that $\rho_3\kappa^{-4} \leq C$,

$$\begin{aligned}
&\bullet\ \|\rho_3\bar{\nu}(\nabla y)Dy : \nabla y\|^q_{L^q(0,T;L^p(\Omega))} \leq C\|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{2,p}(\Omega))} \\
&+ C\|\kappa y\|^{3q}_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y\|^q_{L^q(0,T;W^{2,p}(\Omega))} \leq C\|(y, P, \theta, v, v_0)\|^{4q}_{\mathcal{U}_N}.
\end{aligned}$$

Consequently we have $\mathcal{I}_3(y, P, \theta, v, v_0) \in L^q(\rho_3^q(0,T); L^p(\Omega))$.

Using similar arguments it is easy to check the $\mathcal{I}$ is continuous around the origin. This proves the Lemma. □

Lemma 2.10. *The mapping $\mathcal{I} : \mathcal{U}_N \longrightarrow \mathcal{R}_N$ is continuously differentiable.*

Proof. Let us first prove that $\mathcal{I}$ is Gâteaux-differentiable at any $(y, P, \theta, v, v_0) \in \mathcal{U}_N$ and let us compute the *G-derivative* $\mathcal{I}'(y, P, \theta, v, v_0)$.

Let us fix $(y, P, \theta, v, v_0) \in \mathcal{U}_N$ and let us take $(y', P', \theta', v', v_0') \in \mathcal{U}_N$ and $\sigma > 0$. Also, by the decomposition made in (2.57), we introduce the linear mapping $\mathcal{I} : \mathcal{U}_N \longrightarrow \mathcal{R}_N$ with $\mathcal{DI}(y, P, \theta, v, v_0) = \mathcal{DI} = (\mathcal{DI}_1, \mathcal{DI}_2, \mathcal{DI}_3, \mathcal{DI}_4)$ where

$$
\begin{cases}
\mathcal{DI}_1(y', P', \theta', v', v_0') := y_t' - \bar{\nu}(\nabla y)\Delta y' - \left(2\nu_1 \|\nabla y\|_{L^p}^{2-p} \displaystyle\int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx\right) \Delta y \\
\qquad\qquad + \nabla P' + (y' \cdot \nabla) y + (y \cdot \nabla) y' - \nu_0 \theta' e_3 - v' \chi_\omega, \\
\mathcal{DI}_2(y', P', \theta', v', v_0') := y'(., 0), \\
\mathcal{DI}_3(y', P', \theta', v', v_0') := \theta_t' - \bar{\nu}(\nabla \theta)\Delta \theta' - \left(2\nu_1 \|\nabla \theta\|_{L^p}^{2-p} \displaystyle\int_\Omega |\nabla \theta|^{p-2} \nabla \theta \nabla \theta' dx\right) \Delta \theta \\
\qquad\qquad + y' \cdot \nabla \theta + y \cdot \nabla \theta' - v_0' \chi_\omega - \bar{\nu}(\nabla y) Dy : \nabla y' \\
\qquad\qquad - \left[\bar{\nu}(\nabla y) Dy' + \left(2\nu_1 \|\nabla y\|_{L^p}^{2-p} \displaystyle\int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx\right) Dy\right] : \nabla y, \\
\mathcal{DI}_4(y', P', \theta', v', v_0') := \theta'(., 0).
\end{cases}
\tag{2.60}
$$

From the definition of the spaces $\mathcal{U}_N, \mathcal{R}_N$ and (2.60), it becomes clear that $\mathcal{DI} \in \mathcal{L}(\mathcal{U}_N, \mathcal{R}_N)$. Furthermore, for each $j = \{1, 2, 3, 4\}$ we have

$$
\begin{aligned}
&\frac{1}{\sigma}[\mathcal{I}_j\left((y, P, \theta, v, v_0) + \sigma(y', P', \theta', v', v_0')\right) - \mathcal{I}_j(y, P, \theta, v, v_0)] \\
&\text{converges to } \mathcal{DI}_j(y', P', \theta', v', v_0') \text{ strong in } \mathcal{R}_N, \text{ as } \sigma \longrightarrow 0.
\end{aligned}
\tag{2.61}
$$

Firstly, notice that,

$$
\begin{aligned}
&\|\frac{1}{\sigma}[\mathcal{I}_1\left((y, P, \theta, v, v_0) + \sigma(y', P', \theta', v', v_0')\right) - \mathcal{I}_1(y, P, \theta, v, v_0)] \\
&-\mathcal{DI}_1(y', P', \theta', v', v_0')\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \leq \sigma \|(y' \cdot \nabla) y'\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \\
&+ \|\frac{\nu_1}{\sigma}\left(\|\nabla(y + \sigma y')\|_{L^p}^2 - \|\nabla y\|_{L^p}^2\right) \Delta y \\
&- \left(2\nu_1 \|\nabla y\|_{L^p}^{2-p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx\right) \Delta y\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \\
&+ \|\nu_1(\|\nabla(y + \sigma y')\|_{L^p}^2 - \|\nabla y\|_{L^p}^2) \Delta y'\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} = \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3.
\end{aligned}
$$

That $\tilde{H}_1 \to 0$, as $\sigma \to 0$, is immediate. Let's analyze $\tilde{H}_3$, using first order Taylor expansion and discarding terms of higher order than σ, we have

$$
\begin{aligned}
&\left(\int_\Omega |\nabla(y + \sigma y')|^p dx\right)^{2/p} = \left(\int_\Omega |\nabla y|^p dx + \sigma \int_\Omega p |\nabla y|^{p-2} \nabla y \nabla y' dx\right)^{2/p} \\
&= \left(\int_\Omega |\nabla y|^p dx\right)^{2/p} + \frac{2\sigma}{p}\left(\int_\Omega |\nabla y|^p dx\right)^{(2-p)/p} \int_\Omega p |\nabla y|^{p-2} \nabla y \nabla y' dx \\
&= \|\nabla y\|_{L^p}^2 + 2\sigma \|\nabla y\|_{L^p}^{2-p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx.
\end{aligned}
$$

Then,

$$\begin{aligned}
&\lim_{\sigma\to 0}\left(\|\nabla(y+\sigma y')\|_{L^p}^2-\|\nabla y\|_{L^p}^2\right)\Delta y'\\
&=\lim_{\sigma\to 0}\left(\|\nabla y\|_{L^p}^2+2\sigma\|\nabla y\|_{L^p}^{2-p}\int_\Omega |\nabla y|^{p-2}\nabla y\nabla y' dx-\|\nabla y\|_{L^p}^2\right)\Delta y'\\
&=\lim_{\sigma\to 0}\sigma\left(2\|\nabla y\|_{L^p}^{2-p}\int_\Omega |\nabla y|^{p-2}\nabla y\nabla y' dx\right)\Delta y'=0.
\end{aligned}$$

Therefore, by the arguments used in the analysis of $\mathcal{I}_1$ in Lemma 2.9 and by Lebesgue's dominated convergence theorem we obtain $\tilde{H}_3\to 0$ as $\sigma\to 0$. In a similar way, $\tilde{H}_2\to 0$ as $\sigma\to 0$.

For $j=2$ and $j=3$ the convergence (2.61) is prompt.

Finally, let's see that

$$\begin{aligned}
&\frac{1}{\sigma}[\mathcal{I}_3\left((y,P,\theta,v,v_0)+\sigma(y',P',\theta',v',v_0')\right)-\mathcal{I}_3(y,P,\theta,v,v_0)]\\
&\text{converges to}\,\mathcal{D}\mathcal{I}_3(y',P',\theta',v',v_0')\text{ strong in }L^q(\rho_3^q(0,T);L^p(\Omega)),\text{ as}\,\sigma\longrightarrow 0.
\end{aligned}\tag{2.62}$$

Here, for simplicity, we will also omit the notation of norms but make it clear that they are all norms in $L^q(\rho_3^q(0,T);L^p(\Omega))$. Therefore,

$$\begin{aligned}
&\|\frac{1}{\sigma}[\mathcal{I}_3\left((y,P,\theta,v,v_0)+\sigma(y',P',\theta',v',v_0')\right)-\mathcal{I}_3(y,P,\theta,v,v_0)]\\
&-\mathcal{D}\mathcal{I}_3(y',P',\theta',v',v_0')\|\le\sigma\|y'\cdot\nabla\theta'\|\\
&+\sigma\|(\nu_0+\nu_1\|\nabla(y+\sigma y')\|_{L^p}^2)Dy':\nabla y'\|\\
&+\|\frac{\nu_1}{\sigma}\left(\|\nabla(\theta+\sigma\theta')\|_{L^p}^2-\|\nabla\theta\|_{L^p}^2\right)\Delta\theta-\left(2\nu_1\|\nabla\theta\|_{L^p}^{2-p}\int_\Omega|\nabla\theta|^{p-2}\nabla\theta\nabla\theta' dx\right)\Delta\theta\|\\
&+\|\nu_1(\|\nabla(\theta+\sigma\theta')\|_{L^p}^2-\|\nabla\theta\|_{L^p}^2)\Delta\theta'\|\\
&+\|\frac{\nu_1}{\sigma}\left(\|\nabla(y+\sigma y')\|_{L^p}^2-\|\nabla y\|_{L^p}^2\right)Dy:\nabla y\\
&-\left(2\nu_1\|\nabla y\|_{L^p}^{2-p}\int_\Omega|\nabla y|^{p-2}\nabla y\nabla y' dx\right)Dy:\nabla y\|\\
&+\|\nu_1(\|\nabla(y+\sigma y')\|_{L^p}^2-\|\nabla y\|_{L^p}^2)Dy:\nabla y'\|\\
&+\|\nu_1(\|\nabla(y+\sigma y')\|_{L^p}^2-\|\nabla y\|_{L^p}^2)Dy':\nabla y\|=\sum_{j=1}^{7}\tilde{I}_j.
\end{aligned}$$

By the same arguments from the proof of Lemma 2.6 together with Lebesgue's dominated convergence theorem, we have

$$\sum_{j=1}^{5}\tilde{I}_j\longrightarrow 0,\ \text{as}\ \sigma\longrightarrow 0.$$

Also, using Hölder's inequality for p and $\frac{p}{p-1}$ and $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$,

$$
\begin{aligned}
\tilde{I}_6^q &\leq C \int_0^T \left(\int_\Omega \rho_3^p (\|\nabla(y+\sigma y')\|^2_{L^p} - \|\nabla y\|^2_{L^p})^p |\nabla y|^p |\nabla y'|^p dx \right)^{q/p} dt \\
&\leq C \int_0^T \sigma^q \left(\|\nabla y\|^{2-p}_{L^p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx \right)^q \left(\int_\Omega \rho_3^p |\nabla y|^p |\nabla y'|^p dx \right)^{q/p} dt \\
&\leq C\,\sigma^q \int_0^T \|\nabla y\|^{q(2-p)}_{L^p(\Omega)} \left(\int_\Omega |\nabla y|^{p-1} |\nabla y'| dx \right)^q \left(\int_\Omega \rho_3^p |\nabla y|^p |\nabla y'|^p dx \right)^{q/p} dt \\
&\leq C\sigma^q \left(\sup_{[0,T]} \|\nabla y\|^q_{L^p(\Omega)} \right)^{(2-p)} \int_0^T \left(\|\nabla y\|^{q(p-1)}_{L^p(\Omega)} \|\nabla y'\|^q_{L^p(\Omega)} \right) \left(\int_\Omega \rho_3^p |\nabla y|^p |\nabla y'|^p dx \right)^{q/p} dt \\
&\leq C\sigma^q \|\kappa y\|^{q(2-p)+q(p-1)}_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa y'\|^q_{L^\infty(0,T;W^{1,p}(\Omega))} \int_0^T \|\kappa \nabla y'\|^q_{L^\infty(\Omega)} \left(\int_\Omega |\kappa \nabla y|^p dx \right)^{q/p} dt \\
&\leq C\sigma^q \|\kappa y\|^q_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa y'\|^q_{L^\infty(0,T;W^{1,p}(\Omega))} \int_0^T \|\kappa \nabla y'\|^q_{W^{1,p}(\Omega)} \|\kappa \nabla y\|^q_{L^p(\Omega)} dt \\
&\leq C\sigma^q \|\kappa y\|^{2q}_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa y'\|^q_{L^\infty(0,T;W^{1,p}(\Omega))} \int_0^T \|\kappa \nabla y'\|^q_{W^{1,p}(\Omega)} dt \\
&\leq C\sigma^q \|\kappa y\|^{2q}_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa y'\|^q_{L^\infty(0,T;W^{1,p}(\Omega))} \|\kappa y'\|^q_{L^q(0,T;W^{2,p}(\Omega))}.
\end{aligned}
\tag{2.63}
$$

From which we can conclude that $\tilde{I}_6 \longrightarrow 0$, as $\sigma \longrightarrow 0$. By the same arguments we also have $\tilde{I}_7 \longrightarrow 0$, as $\sigma \longrightarrow 0$. Consequently, (2.62) is true.

Then we can conclude that (2.61) holds and $\mathcal{I}$ is Gâteaux-differentiable at any $(y, p, \theta, v, v_0) \in \mathcal{U}_N$, with *G-derivative* $\mathcal{I}'(y, p, \theta, v, v_0) = \mathcal{D}\mathcal{I}(y, p, \theta, v, v_0)$.

Now, we will show that $(y, P, \theta, v, v_0) \longmapsto \mathcal{I}'(y, P, \theta, v, v_0)$ is continuous from $\mathcal{U}_N$ into $\mathcal{L}(\mathcal{U}_N, \mathcal{R}_N)$ and as consequently, in view of classical results, we will have that $\mathcal{I}$ is Fréchet-differentiable and $\mathcal{C}^1$. Thus, suppose that

$$
(y_m, P_m, \theta_m, v_m, v_{0m}) \longrightarrow (y, P, \theta, v, v_0) \text{ in } \mathcal{U}_N
$$

and let us check that

$$
\begin{aligned}
&\|\left(\mathcal{I}'(y_m, P_m, \theta_m, v_m, v_{0m}) - \mathcal{I}'(y, P, \theta, v, v_0)\right)(y', P', \theta', v', v_0')\|_{\mathcal{R}_N} \\
&\qquad \leq \chi_m \|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N},
\end{aligned}
\tag{2.64}
$$

for all $(y', P', \theta', v', v_0') \in \mathcal{U}_N$, for some $\lim\limits_{m\to\infty} \chi_m = 0$.

In order to simplify the notation, we will consider

$$
\mathbb{D}_{j,m} := \mathcal{I}_j'(y_m, P_m, \theta_m, v_m, v_{0m}) - \mathcal{I}_j'(y, p, \theta, v, v_0).
$$

So, notice that

$$
\begin{aligned}
\bullet\ & \|\mathbb{D}_{1,m}(y', P', \theta', v', v_0')\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \\
&\leq C \Big(\|\nu_1(\|\nabla y\|^2_{L^p(\Omega)} - \|\nabla y_m\|^2_{L^p(\Omega)}) \Delta y'\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \\
&+ \| \left(2\nu_1 \|\nabla y\|^{2-p}_{L^p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx \right) \Delta y \\
&- \left(2\nu_1 \|\nabla y_m\|^{2-p}_{L^p} \int_\Omega |\nabla y_m|^{p-2} \nabla y_m \nabla y' dx \right) \Delta y_m \|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \\
&+ \|(y' \cdot \nabla)(y_m - y) + ((y_m - y) \cdot \nabla) y'\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \Big) \\
&= C(\tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3).
\end{aligned}
\tag{2.65}
$$

Since,

$$\tilde{K}_1 \le C\left(\| \|\nabla(y-y_m)\|_{L^p} \|\nabla y\|_{L^p} \Delta y' \|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \right.$$
$$\left. + \| \|\nabla(y-y_m)\|_{L^p} \|\nabla y_m\|_{L^p} \Delta y' \|_{L^q(\rho_3^q(0,T);L^p(\Omega))} \right)$$

then, using the same arguments as (2.59), we conclude that

$$\tilde{K}_1 \le \chi_{1,m} \|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N}$$

where

$$\chi_{1,m} = C\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N} \left(\|(y, P, \theta, v, v_0)\|_{\mathcal{U}_N} \right.$$
$$\left. + \|(y_m, P_m, \theta_m, v_m, v_{0m})\|_{\mathcal{U}_N} \right).$$

Now, adding and subtracing $\left(2\nu_1 \|\nabla y\|_{L^p}^{2-p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx\right) \Delta y_m$ in $\tilde{K}_2$, we have

$$\tilde{K}_2 \le \| \left(2\nu_1 \|\nabla y\|_{L^p}^{2-p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx \right) \Delta(y_m - y) \|$$
$$+ \|2\nu_1 \left(\|\nabla y_m\|_{L^p}^{2-p} \int_\Omega |\nabla y_m|^{p-2} \nabla y_m \nabla y' dx - \|\nabla y\|_{L^p}^{2-p} \int_\Omega |\nabla y|^{p-2} \nabla y \nabla y' dx \right) \Delta y_m \|$$
$$= \tilde{K}_{2,1} + \tilde{K}_{2,2}.$$

Using arguments similar to those applied in (2.63), we have

$$\begin{aligned} (\tilde{K}_{2,1})^q &\le C \left(\sup_{[0,T]} \|\nabla y\|_{L^p(\Omega)}^q \right)^{(2-p)} \int_0^T \left(\|\nabla y\|_{L^p(\Omega)}^{q(p-1)} \|\nabla y'\|_{L^p(\Omega)}^q \right) \left(\int_\Omega \rho_3^p |\Delta(y_m - y)|^p dx \right)^{q/p} dt \\ &\le C \|\kappa y\|_{L^\infty(0,T;W^{1,p}(\Omega))}^q \|\kappa y'\|_{L^\infty(0,T;W^{1,p}(\Omega))}^q \|\kappa(y_m - y)\|_{L^q(0,T;W^{2,p}(\Omega))}^q \\ &\le C \left(\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N}^q \|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N}^q \right. \\ &\quad \left. \|(y, P, \theta, v, v_0)\|_{\mathcal{U}_N}^q \right). \end{aligned} \tag{2.66}$$

And, adding and subtracing $2\nu_1 \int_\Omega \frac{|\nabla y|^{p-2}}{\|\nabla y\|_{L^p}^{p-2}} \nabla y_m \nabla y' dx \Delta y_m$ in $\tilde{K}_{2,2}$, we get

$$\begin{aligned} \tilde{K}_{2,2} &= \|2\nu_1 \left[\int_\Omega \left(\frac{|\nabla y_m|^{p-2}}{\|\nabla y_m\|_{L^p}^{p-2}} - \frac{|\nabla y|^{p-2}}{\|\nabla y\|_{L^p}^{p-2}} \right) \nabla y_m \nabla y' dx \right] \Delta y_m \| \\ &\quad + \|2\nu_1 \left(\int_\Omega \frac{|\nabla y|^{p-2}}{\|\nabla y\|_{L^p}^{p-2}} \nabla(y_m - y) \nabla y' dx \right) \Delta y_m \| \\ &= \tilde{K}_{2,2}^1 + \tilde{K}_{2,2}^2. \end{aligned} \tag{2.67}$$

Let's analyze the integrals of (2.67) separately. First, denote by $z_m = \frac{|\nabla y_m|}{\|\nabla y_m\|_{L^p}}$ and $z = \frac{|\nabla y|}{\|\nabla y\|_{L^p}}$. Applying in order Holder's inequality for $\frac{p-2}{p} + \frac{1}{p} + \frac{1}{p} = 1$, the Mean Value Theorem and again Hölder for $\frac{1}{p-2}+\frac{p-3}{p-2}=1$, we obtain

$$
\begin{aligned}
&\bullet \int_\Omega (z_m^{p-2}-z^{p-2})\nabla y_m \nabla y' dx \\
&\le \left(\int_\Omega (z_m^{p-2}-z^{p-2})^{p/(p-2)}dx\right)^{(p-2)/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le \left[\int_\Omega \left((p-2)(|z|+|z_m|)^{p-3}|z_m-z|\right)^{p/(p-2)} dx\right]^{(p-2)/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\left(\int_\Omega (|z|+|z_m|)^{\frac{(p-3)p}{p-2}}|z_m-z|^{p/(p-2)}dx\right)^{(p-2)/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\left[\left(\int_\Omega (|z|+|z_m|)^p dx\right)^{\frac{(p-3)}{p-2}} \left(\int_\Omega |z_m-z|^p dx\right)^{1/(p-2)}\right]^{(p-2)/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\left(\int_\Omega (|z|+|z_m|)^p dx\right)^{\frac{(p-3)}{p}} \left(\int_\Omega |z_m-z|^p dx\right)^{1/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C(\|z\|_{L^p}+\|z_m\|_{Lp})^{p-3}\|z_m-z\|_{L^p}\|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\, 2^{p-3}\|z_m-z\|_{L^p}\|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\left(\int_\Omega \left|\frac{|\nabla y_m|}{\|\nabla y_m\|_{L^p}}-\frac{|\nabla y|}{\|\nabla y_m\|_{L^p}}+\frac{|\nabla y|}{\|\nabla y_m\|_{L^p}}-\frac{|\nabla y|}{\|\nabla y\|_{L^p}}\right|^p dx\right)^{1/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\left[\int_\Omega \left(\left|\frac{\nabla(y_m-y)}{\|\nabla y_m\|_{L^p}}\right|+\left|\frac{|\nabla y|(\|\nabla y\|_{L^p}-\|\nabla y_m\|_{L^p})}{\|\nabla y_m\|_{L^p}\|\nabla y\|_{L^p}}\right|\right)^p dx\right]^{1/p} \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\left(\frac{\|\nabla(y_m-y)\|_{L^p}}{\|\nabla y_m\|_{L^p}}+\frac{\|\nabla y\|_{L^p}\|\nabla(y_m-y)\|_{L^p}}{\|\nabla y_m\|_{L^p}\|\nabla y\|_{L^p}}\right) \|\nabla y_m\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\|\nabla(y_m-y)\|_{L^p}\|\nabla y'\|_{L^p}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\tilde{K}^1_{2,2})^q &\le C\int_0^T \|\nabla(y_m-y)\|^q_{L^p}\|\nabla y'\|^q_{L^p}\left(\int_\Omega \rho_3^p|\Delta y_m|^p dx\right)^{q/p} \\
&\le C\|\kappa(y_m-y)\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y'\|^q_{L^\infty(0,T;W^{1,p}(\Omega))}\|\kappa y_m\|^q_{L^q(0,T;W^{2,p}(\Omega))} \\
&\le C\Big(\|(y_m,P_m,\theta_m,v_m,v_{0m})-(y,P,\theta,v,v_0)\|^q_{\mathcal{U}_N}\|(y',P',\theta',v',v_0')\|^q_{\mathcal{U}_N} \\
&\quad \|(y_m,P_m,\theta_m,v_m,v_{0m})\|^q_{\mathcal{U}_N}\Big).
\end{aligned}
\tag{2.68}
$$

And, again using Holder's inequality for $\frac{p-2}{p}+\frac{1}{p}+\frac{1}{p}=1$

$$
\begin{aligned}
&\bullet \int_\Omega \frac{|\nabla y|^{p-2}}{\|\nabla y\|^{p-2}_{L^p}}\nabla(y_m-y)\nabla y' dx \\
&\le C\left(\int_\Omega \frac{|\nabla y|_p}{\|\nabla y\|^p_{L^p}}dx\right)^{(p-2)/p} \|\nabla(y_m-y)\|_{L^p}\|\nabla y'\|_{L^p} \\
&\le C\|\nabla(y_m-y)\|_{L^p}\|\nabla y'\|_{L^p}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
(\tilde{K}^2_{2,2})^q &\le C\Big(\|(y_m,P_m,\theta_m,v_m,v_{0m})-(y,P,\theta,v,v_0)\|^q_{\mathcal{U}_N}\|(y',P',\theta',v',v_0')\|^q_{\mathcal{U}_N} \\
&\quad \|(y,P,\theta,v,v_0)\|^q_{\mathcal{U}_N}\Big).
\end{aligned}
\tag{2.69}
$$

From (2.68) and (2.69) in (2.67), we conclude that

$$\tilde{K}_{2,2} \leq C\left(\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N} \|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N} \right.$$
$$\left. \|(y_m, P_m, \theta_m, v_m, v_{0m})\|_{\mathcal{U}_N}\right). \tag{2.70}$$

And as a consequence of (2.66) and (2.70)

$$\tilde{K}_2 \leq \chi_{2,m}\|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N},$$

with

$$\chi_{2,m} = C\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N}\|(y_m, P_m, \theta_m, v_m, v_{0m})\|_{\mathcal{U}_N}.$$

Moreover, by the same reasoning as (2.58),

$$\begin{aligned}\tilde{K}_3 &\leq C\left(\|(y'\cdot\nabla)(y_m - y)\|_{L^q(\rho_3^q(0,T);L^p(\Omega))} + \|((y_m - y)\cdot\nabla)y'\|_{L^q(\rho_3^q(0,T);L^p(\Omega))}\right)\\ &\leq \chi_{3,m}\|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N},\end{aligned}$$

with

$$\chi_{3,m} = C\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N}.$$

It is easy to check that $\mathbb{D}_{j,m}$ for $j = 2$ and $j = 4$ satisfy similar inequalities.

Again, all inequality norms below are norms in $L^q(\rho_3^q(0,T); L^p(\Omega))$, we will omit them for simplicity. For $\mathbb{D}_{3,m}$ after some manipulations we get the following:

- $$\begin{aligned}&\|\mathbb{D}_{3,m}(y', P', \theta', v', v_0')\| \leq C\,[\|\ \|\nabla(y_m - y)\|_{L^p}\ \|\nabla y\|_{L^p}\ \Delta\theta'\ \|\\ &+\|\ \|\nabla(y_m - y)\|_{L^p}\ \|\nabla y_m\|_{L^p}\Delta\theta'\ \| + \|\left(2\nu_1\|\nabla\theta\|_{L^p}^{2-p}\int_\Omega|\nabla\theta|^{p-2}\nabla\theta\nabla\theta' dx\right)\Delta(\theta_m - \theta)\|\\ &+\|2\nu_1\left(\|\nabla\theta_m\|_{L^p}^{2-p}\int_\Omega|\nabla\theta_m|^{p-2}\nabla\theta_m\nabla\theta' dx - \|\nabla\theta\|_{L^p}^{2-p}\int_\Omega|\nabla\theta|^{p-2}\nabla\theta\nabla\theta' dx\right)\Delta\theta_m\|\\ &+\|(\bar{\nu}(\nabla y_m)D(y_m - y) : \nabla y'\| + \|\nu_1(\|\nabla y_m\|_{L^p}^2 - \|\nabla y\|_{L^p}^2)Dy : \nabla y'\|\\ &+\|\bar{\nu}(\nabla y_m)Dy' : \nabla(y_m - y)\|\ + \|\nu_1(\|\nabla y_m\|_{L^p}^2 - \|\nabla y\|_{L^p}^2)Dy' : \nabla y\|\\ &+\|\left(2\nu_1\|\nabla y_m\|_{L^p}^{2-p}\int_\Omega|\nabla y_m|^{p-2}\nabla y_m\nabla y' dx\right)Dy_m : \nabla(y_m - y)\|\\ &+\|\left(2\nu_1\|\nabla y_m\|_{L^p}^{2-p}\int_\Omega|\nabla y_m|^{p-2}\nabla y_m\nabla y' dx\right)D(y_m - y) : \nabla y\|\\ &+\|2\nu_1\left(\|\nabla y_m\|_{L^p}^{2-p}\int_\Omega|\nabla y_m|^{p-2}\nabla y_m\nabla y' dx - \|\nabla y\|_{L^p}^{2-p}\int_\Omega|\nabla y|^{p-2}\nabla y\nabla y' dx\right)Dy : \nabla y\|\\ &+\|y'\cdot\nabla(\theta_m - \theta)\| + \|(y_m - y)\cdot\nabla\theta'\|] = C\sum_{s=4}^{16}\tilde{K}_s.\end{aligned}$$

Applying arguments similar to those used in Lemma 2.9 and in (2.65) we can conclude that $\tilde{K}_s \leq \chi_{s,m}$ for $k = \{4, 5, \ldots, 16\}$. Indeed, let us evaluate $\tilde{K}_{14}$, from the calculations performed for (2.67) we have

$$\begin{aligned}&(\tilde{K}_{14})^q \leq C\int_0^T \|\nabla(y_m - y)\|_{L^p}^q\|\nabla y'\|_{L^p}^q\left(\int_\Omega \rho_3^p|\nabla y|^{2p}dx\right)^{q/p}\\ &\leq C\|\kappa(y_m - y)\|_{L^\infty(0,T;W^{1,p}(\Omega))}^q\|\kappa y'\|_{L^\infty(0,T;W^{1,p}(\Omega))}^q\|\kappa y\|_{L^\infty(0,T;W^{1,p}(\Omega))}^q\|\kappa y\|_{L^q(0,T;W^{2,p}(\Omega))}^q\\ &\leq C\left(\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N}^q\|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N}^q\ \|(y, P, \theta, v, v_0)\|_{\mathcal{U}_N}^{2q}\right).\end{aligned}$$

Thus

$$\tilde{K}_{14} \leq \chi_{15,m}\|(y', P', \theta', v', v_0')\|_{\mathcal{U}_N},$$

with

$$\chi_{14,m} = C\|(y_m, P_m, \theta_m, v_m, v_{0m}) - (y, P, \theta, v, v_0)\|_{\mathcal{U}_N} \|(y, P, \theta, v, v_0)\|^2_{\mathcal{U}_N}.$$

Thus, we have $\lim_{m\to\infty} \chi_{s,m} = 0$ for all $s \in \{1, ..., 16\}$ and consequently (2.64) is obtained. This ends the proof. □

Lemma 2.11. *Let $\mathcal{I}$ be the mapping in* (2.56)-(2.57). *Then, $\mathcal{I}'(0,0,0,0,0)$ is onto.*

Proof. Let $(F_1, y^0, F_2, \theta^0) \in \mathcal{R}_N$. From Proposition 2.2 we know there exists (y, P, θ, v, v_0) satisfying (2.7) and (2.14). Furthermore, we have $y \in L^q(0,T; W^{2,p}(\Omega)^N) \cap C^0([0,T]; L^p(\Omega)^N)$ and $\theta \in L^q(0,T; W^{2,p}(\Omega)) \cap C^0([0,T]; L^p(\Omega))$. Consequently, $(y, P, \theta, v, v_0) \in \mathcal{U}_N$ and

$$\mathcal{I}'(0,0,0,0,0)(y, P, \theta, v, v_0) = (F_1, y^0, F_2, \theta^0).$$

□

Proof of Theorem 2.2. According to Lemmas 2.9-2.11, we can apply the Inverse Mapping Theorem (Theorem 2.4), then, there exists $\delta > 0$ and a mapping $W : B_\delta(0) \subset \mathcal{R}_N \to \mathcal{U}_N$ such that

$$W(z) \in B_\delta(0) \text{ and } \mathcal{I}(W(z)) = z, \ \forall z \in B_\delta(0).$$

Taking $(0, y^0, 0, \theta^0) \in B_\delta(0)$ and $(y, P, \theta, v, v_0) = W(0, y^0, 0, \theta^0) \in \mathcal{U}_N$, we have

$$\mathcal{I}(y, P, \theta, v, v_0) = (0, y^0, 0, \theta^0).$$

Thus, we conclude that (2.3) is locally null controllable at time $T > 0$.

2.5 Large time null-controllability

This section is devoted to the proof of Theorem 2.3. Following the ideas of [Car+23; Le 20], we will make the system (2.1) evolve without control and certify an asymptotic behavior according to $t \to \infty$ of its solutions, when $N = 2$. That is, we will deal with the energy decay of the solutions of the system complete Ladyzhenskaya-Boussinesq. Having verified this analysis, we will take a time $T^* > 0$ such that the solutions $y(T^*, .)$ and $\theta(T^*, .)$ related to the null local controllability of (2.1) (Theorem 2.1). Thus, by setting $y(T^*, .)$ and $\theta(T^*, .)$ as the initial data in (2.1), Theorem 2.1 gives us the v and v_0 controls that drive the solutions to zero in some sufficiently large time.

Accordingly we state the following lemma, which will be fundamental for the demonstration of Theorem 2.3.

Lemma 2.12. *For $N = 2$, any $T > 0$ and $(y^0, \theta^0) \in V \times H^1_0(\Omega)$, if there is positive constant $r > 0$ such that*

$$\|y^0\|_V + \|\theta^0\|_{H^1_0} < r$$

and (y, p, θ) is a solution of (2.1) *with $v \equiv v_0 \equiv 0$, so this solution has asymptotic behavior as $t \to \infty$. More precisely, for*

$$E(t) := \|\nabla y(t, .)\|^2 + \|\theta(t, .)\|^2 + \|\nabla\theta(t.,)\|^2$$

there are positive constants $\mathrm{C}_1, \mathrm{C}_2$ such that

$$E(t) \leq \mathrm{C}_2\, e^{-\mathrm{C}_1 t} E(0) \; a.e \; in \, (0, T). \tag{2.71}$$

For the convenience, we will give the proof for inequality (2.71) in Lemma 2.12 in Appendix B.1.

Proof of Theorem 2.3 First, let's fix $T_0 > 0$. Applying the Theorem 2.1 there exists $\delta > 0$ such that the system (2.1), with any initial data $(\bar{y}^0, \bar{\theta}^0) \in V \times W_0^{1,3/2}(\Omega)$ satisfying $\|(\bar{y}^0, \bar{\theta}^0)\|_{V \times W_0^{1,3/2}} < \delta$, is locally null controllable at T_0.

Determine $(y^0, \theta^0) \in V \times H_0^1(\Omega)$ and consider $\mathrm{C}_1, \mathrm{C}_2 > 0$ as defined in the statement of Lemma 2.12. Let then T^* be a positive time satisfying

$$T^* > \frac{-1}{C_1} \ln \left(\frac{\delta}{\mathrm{C}_2(\|\nabla y^0\|^2 + \|\theta^0\|^2 + \|\nabla \theta^0\|^2)} \right) \tag{2.72}$$

and consider a solution (y, p, θ) of the system (2.1), with $T = T^* + T_0$, $v \equiv v_0 \equiv 0$ and (y^0, θ^0) as the initial data.

From (2.71) and (2.72), $y(., T^*)$, $\theta(., T^*)$ are such that

$$\| \,(y(., T^*), \theta(., T^*))\, \|_{V \times W_0^{1,3/2}} \leq \mathrm{C}_2\, e^{-\mathrm{C}_1 T^*} (\|\nabla y^0\|^2 + \|\theta^0\|^2 + \|\nabla \theta^0\|^2) < \delta.$$

Consequently, by Theorem 2.1, (2.1) is locally null controllable at $T^* + T_0$.

CHAPTER **3**

Strong solution of the Navier-Stokes equations in non-cylindrical domains

3.1 Problem Formulation

Let us denote by W an open, bounded, and nonempty subset of $\mathbb{R}_x^N \times \mathbb{R}_t$, with $N \leq 3$. Suppose also $\Omega_s = W \cap \{t = s; s \in \mathbb{R}\}$ are open, bounded, and nonempty sets with boundaries Γ_s. We fix the interval $[0, T]$ of $\mathbb{R}_t$ and consider $\widehat{Q} = \bigcup_{0<s<T} \Omega_s \times \{s\}$ the non-cylindrical domain contained in $\mathbb{R}_x^N \times \mathbb{R}_t$ with lateral boundary defined by $\widehat{\Sigma} = \bigcup_{0<s<T} \Gamma_s \times \{s\}$ and its boundary by $\partial\widehat{Q} = \Omega_0 \cup \widehat{\Sigma} \cup \Omega_T$ in these conditions, we are concerned with the existence of solutions for the Navier-Stokes equations

$$\begin{cases} u' - \nu\Delta u + (u \cdot \nabla)u = f - \nabla p & \text{in} \quad \widehat{Q}, \\ \nabla \cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.1}$$

The methodology we will employ to solve the problem (3.1) consist in transforming it into a cylindrical problem by means of a perturbation of equation (3.1) adding two singular terms, depending on a parameter $\epsilon > 0$ which is destined tend to zero. This method was idealized by Lions (see, for example [Lio69]) adding a singular term and is called by him a penalty method. To apply the Lions' method, some restrictive hypotheses on $\widehat{Q}$ are necessary. In fact we suppose $\widehat{Q} \subset Q$ with $\Omega_0 \subset \Omega$ where $Q = \Omega \times [0, T)$. Moreover, we consider hypotheses about geometry and regularity of $\widehat{Q}$.

(H1) (Geometry of $\widehat{Q}$) If $t_1 \leq t_2$ then $proj|_{t=0}\, \Omega_{t_1} \subset proj|_{t=0}\, \Omega_{t_2}$. It means, the family $\{\Omega_t\}_{0\leq t\leq T}$ is increasing.

(H2) (Regularity of $\widehat{Q}$) If $v \in H_0^m(\Omega)$ and $D^\beta v = 0$ on $\Omega \setminus \Omega_t$ for almost all $t \in [0, T)$ and $|\beta| \leq m - 1$ then $v \in H_0^m(\Omega_t)$.

(H3) (Bounded data) There is $\rho > 0$ such that

$$\begin{aligned} \kappa(f, u_0) &= \|f\|^2_{L^2(0,T;L^2(\Omega_t)^N)} + \|f\|^2_{L^2(0,T;V(\Omega_t))} + \|u_0\|_{V(\Omega_0)} + |\Delta u_0|^2_{L^2(\Omega_0)^N} \\ &< \min\left\{\frac{\nu^2}{4\tilde{C}_1}, \left(\frac{\nu}{4C_1}\right)^2\right\} = \rho, \end{aligned}$$

where C_1, $\tilde{C}_1$, $V(\Omega_t)$ and $V(\Omega_0)$ will be justified throughout the text.

Many real-world problems involve partial differential equations where the domain of interest changes with time. For example, in fluid dynamics, PDEs are employed to describe the flow and evolution of fluid interfaces, such as free surface flows, multiphase flows, or droplet dynamics, see [GPW22], [SKR22].

We organize this chapter as follows: in Section 3.2, we introduce the notations and definitions that formed the Penalized problem and the declaration of results. In Section 3.3 we demonstrate the main results, on the existence and uniqueness of strong solutions. Section 3.4 is dedicated to the proof of a decay result for the solution of the system (3.1).

3.2 Penalized problem and statement of results

This section is dedicated to presenting the formulation of the penalized problem, as well as enunciating the main results of this work about the existence and uniqueness of a strong solution, for the (3.1) problem, and also stating a theorem that under some conditions guarantees us energy decay for the solution found.

Let $\beta : Q \to \mathbb{R}$ be a function defined by

$$\beta(x,t) = \begin{cases} 1 & \text{in} \quad Q \setminus \widehat{Q} \cup (\Omega_0 \times \{0\}), \\ 0 & \text{in} \quad \widehat{Q} \cup (\Omega_0 \times \{0\}). \end{cases} \tag{3.2}$$

Consider $\tilde{\alpha}(x,t)$ solution to the problem

$$\begin{cases} -\Delta\tilde{\alpha}(x,t) = 1 & \text{in} \quad \Omega \setminus \Omega_t, \\ \tilde{\alpha}(x,t) = 0 & \text{in} \quad \partial(\Omega \setminus \Omega_t) = \partial\Omega_t \cup \partial\Omega, \end{cases} \tag{3.3}$$

have up for almost every t in $[0,T]$, then $\tilde{\alpha}(\cdot,t) \in H^2(\Omega\setminus\Omega_t) \cap H_0^1(\Omega\setminus\Omega_t)$ and the principle of maximum gives us, $\tilde{\alpha}(\cdot,t) \geq 0$ in $\overline{\Omega - \bar{\Omega}_t}$.

Let $\alpha : Q \to \mathbb{R}$ be a function defined by

$$\alpha(x,t) = \begin{cases} \tilde{\alpha}(x,t) & \text{in} \quad Q \setminus \widehat{Q}, \\ 0 & \text{in} \quad \widehat{Q}. \end{cases} \tag{3.4}$$

From the above definitions, we can conclude that $\alpha(x,t) = \tilde{\alpha}(x,t)\beta(x,t)$ in $Q \setminus \widehat{Q}$ and $-\Delta\alpha(x,t) = \beta(x,t)$ in Q.

Denoting by $\tilde{u}_0$ the extension of u_0 to Ω defined zero in $\Omega \setminus \Omega_0$ it implies $\tilde{u}_0 \in V(\Omega) \cap (H^2(\Omega))^N$ and $\tilde{f} \in L^2(0,T;V(\Omega))$ the extension of f to Q defined zero in $Q \setminus \widehat{Q}$, where $V(\Omega)$ will be defined below. For $\epsilon > 0$ consider the problem penalized

$$\begin{cases} u_\epsilon' - \nu\Delta u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon + \frac{1}{\epsilon}\alpha(x,t)u_\epsilon' - \frac{1}{\epsilon}\beta(x,t)\Delta u_\epsilon' = \tilde{f} - \nabla p_\epsilon & \text{in} \quad Q, \\ \nabla \cdot u_\epsilon = 0 & \text{in} \quad Q, \\ u_\epsilon = 0 & \text{on} \quad \partial\Omega \times [0,T), \\ u_\epsilon(\cdot,0) = \tilde{u}_0 & \text{in} \quad \Omega. \end{cases} \tag{3.5}$$

Let $\widehat{Q}_t = \Omega_t \times \{t\}$, if $u' = 0$ in $Q \setminus \widehat{Q}_t$ and the domain is increasing then

$$u(x,t) - u(x,0) = \int_0^t u'(x,\sigma)\, d\sigma = 0, \tag{3.6}$$

and

$$u(x,t) = u(x,0) = \tilde{u}(x,0) = 0. \tag{3.7}$$

Thus, $u = 0$ in $Q \setminus \widehat{Q}_t$.

Now, let us recall the definition of some vector spaces in the context of incompressible fluids. For $\mathcal{O} \subset \mathbb{R}^N$, consider

$$\begin{aligned}
\mathcal{V}(\mathcal{O}) &= \{\varphi \in D(\mathcal{O})^N : \nabla \cdot \varphi = 0 \text{ in } \mathcal{O}\};\\
H(\mathcal{O}) &= \overline{\mathcal{V}(\mathcal{O})}^{L^2(\mathcal{O})^N} = \{\varphi \in L^2(\mathcal{O})^N : \nabla \cdot \varphi = 0 \text{ in } \mathcal{O}, \varphi \cdot n = 0 \text{ on } \partial\Omega\};\\
V(\mathcal{O}) &= \overline{\mathcal{V}(\mathcal{O})}^{H^1(\mathcal{O})^N} = \{\varphi \in H_0^1(\mathcal{O})^N : \nabla \cdot \varphi = 0 \text{ in } \mathcal{O}\},
\end{aligned}$$

so to $\mathcal{O} = \Omega$ there are spaces $L^p(0,T;V(\Omega))$, $L^p(0,T;V(\Omega) \cap H^2(\Omega)^N)$, $1 \leq p \leq \infty$. Since $\widehat{Q} = \bigcup_{0<s<T} \Omega_s \times \{s\} \subset \Omega \times [0,T)$ we define the following L^p spaces:

$$\begin{aligned}
L^p(0,T;V(\Omega_t)) &= \{u \in L^p(0,T;V(\Omega)) : \text{ a.e. } t \in [0,T], u(t) \in V(\Omega_t)\};\\
L^p(0,T;H(\Omega_t)) &= \{u \in L^p(0,T;H(\Omega)) : \text{ a.e. } t \in [0,T], u(t) \in H(\Omega_t)\};\\
L^p(0,T;H^2(\Omega_t)^N \cap V(\Omega_t)) &= \{u \in L^p(0,T;H^2(\Omega)^N \cap V(\Omega)) : \text{ a.e. } t \in [0,T], \\
&\qquad u(t) \in H^2(\Omega_t)^N \cap V(\Omega_t)\};\\
L_0^2(\Omega_t) &= \{u \in L^2(\Omega) : \text{a.e. } t \in [0,T], \frac{1}{\text{med}(\Omega_t)} \int_{\Omega_t} u(x)dx = 0\},
\end{aligned} \tag{3.8}$$

where $\text{med}(\Omega_t)$ means the measure of Ω_t.

Definition 3.1. *A strong solution for* (3.1) *is a function* $u : \widehat{Q} \to \mathbb{R}$ *in the class*

$$u \in L^\infty(0,T;H^2(\Omega_t)^N \cap V(\Omega_t)), \ u' \in L^2(0,T;V(\Omega_t))$$

satisfying the integral identity

$$\int_{\widehat{Q}} \big(u' - \nu\Delta u + u\nabla u\big)\, \varphi \, dx\, dt = \int_{\widehat{Q}} f\varphi \, dx\, dt, \ \forall\, \varphi \in L^2(0,T;H(\Omega_t)). \tag{3.9}$$

Moreover, u verifies the initial condition $u(.,0) = u_0$.

That said, we are in a position to present the main results of this part on the existence and uniqueness of a strong solution, according to the definition 3.1, for the problem (3.1). And assuming a condition under the Ω_t domain, a decay result for such a solution. We state these theorems as follows:

Theorem 3.1. *Suppose $u_0 \in V(\Omega_0) \cap H^2(\Omega_0)^N$, $f \in L^2(0,T,V(\Omega_t))$ such that (H1), (H2) and (H3) hold. Then, the problem* (3.1) *admits a strong solution in the class $u \in L^\infty(0,T;H^2(\Omega_t)^N \cap V(\Omega_t))$, $u' \in L^2(0,T;V(\Omega_t))$ and $p \in L^2(0,T;H^1(\Omega_t)) \cap L^2(0,T;L_0^2(\Omega_t))$.*

Theorem 3.2. *Assuming the same hypotheses as the Theorem 3.1, the solution to* (3.1) *is unique.*

Theorem 3.3. *Let u be the system solution* (3.1) *for $f = 0$. Then u has asymptotic behavior as $t \to \infty$. In other words,*

$$|u(t)|^2_{L^2(\Omega_t)^N} \leq e^{1-t/M} |u_0|^2_{L^2(\Omega_0)^N}, \tag{3.10}$$

where $M = \frac{1}{c_1}$.

3.3 Proof of main results

This section is dedicated to demonstrating the main results of this problem which are about the existence and uniqueness of strong solutions.

Proof for Theorem 3.1

In order to obtain Theorem 3.1, we will now prove a Lemma that will contribute to this goal.

Lemma 3.1. *Let* $\epsilon > 0$, $\tilde{u}_0 \in V(\Omega) \cap H^2(\Omega)^N$, $\tilde{f} \in L^2(0,T;V(\Omega))$ *exist* $u_\epsilon \in L^\infty(0,T;H^2(\Omega)^N \cap V(\Omega))$, $u'_\epsilon \in L^2(0,T;V(\Omega))$ *solution to problem (3.5).*

Proof of Lemma 3.1. Let (w_i) be the eigenfunctions of the Stokes operator such that they are orthonormal in $H(\Omega)$ and orthogonal in $V(\Omega)$ and let λ_i be their respective eigenvalues. For every $m \geq 1$, whether $E_m = span\{w_1, \cdots w_m\}$, we look for the $u_{\epsilon m} = \sum_{i=1}^m g_{\epsilon i m}(t) w_i$ solution of

$$\begin{cases} (u'_{\epsilon m}, w_i) + \nu(\nabla u_{\epsilon m}, \nabla w_i) + ((u_{\epsilon m} \cdot \nabla) u_{\epsilon m}, w_i) + \frac{1}{\epsilon}(\alpha(x,t) u'_{\epsilon m}, w_i) \\ -\frac{1}{\epsilon}(\beta(x,t)\Delta u'_{\epsilon m}, w_i) = (\tilde{f}, w_i) \text{ in } Q, \\ u_{\epsilon m}(\cdot, 0) = u_{0m} \to \tilde{u}_0 \text{ in } V(\Omega) \cap H^2(\Omega)^N. \end{cases} \tag{3.11}$$

By Carathéodory's theorem, the equation (3.11) has a local solution, and by the a priori estimate I, one can extend the solution to interval $[0,T]$ for all $T > 0$.

Estimate I. Multiplying (3.11) by $\lambda_i g'_{\epsilon i m}(t)$ and adding from $i = 1$ to m, we get

$$\begin{aligned} &(u'_{\epsilon m}, -\Delta u'_{\epsilon m}) + \nu(\nabla u_{\epsilon m}, \nabla(-\Delta u'_{\epsilon m})) + ((u_{\epsilon m} \cdot \nabla) u_{\epsilon m}, -\Delta u'_{\epsilon m}) + \\ &\quad \frac{1}{\epsilon}(\alpha(x,t) u'_{\epsilon m}, -\Delta u'_{\epsilon m}) - \frac{1}{\epsilon}(\beta(x,t)\Delta u'_{\epsilon m}, -\Delta u'_{\epsilon m}) = (\tilde{f}, -\Delta u'_{\epsilon m}) \end{aligned}$$

also using the fact that for $N \leq 3$, $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$, i.e. there are C_1, C_2, C_3 dependent on Ω such that

$$|z|_{L^\infty(\Omega)} \leq C_1 |z|_{H^2(\Omega)}, \quad |z|_{L^6(\Omega)} \leq C_2 |z|_{H^1(\Omega)}, \quad |z|_{L^3(\Omega)} \leq C_3 |z|_{H^1(\Omega)}.$$

We also have that in $H_0^1(\Omega) \cap H^2(\Omega)$, $|z|_{H^2(\Omega)}$ and $|\Delta z|_{L^2(\Omega)}$ are equivalent norms, so we can calculate the following inequalities:

$$
\begin{aligned}
\bullet\ |((u_{\epsilon m}\cdot\nabla)u_{\epsilon m},-\Delta u'_{\epsilon m})| &= \left|-\int_\Omega u_{\epsilon mi}\frac{\partial u_{\epsilon mj}}{\partial x_i}\frac{\partial^2 u'_{\epsilon mj}}{\partial x_k^2}\,dx\right| \\
&\le \left|\int_\Omega\left(\frac{\partial u_{\epsilon mi}}{\partial x_k}\frac{\partial u_{\epsilon mj}}{\partial x_i}\frac{\partial u'_{\epsilon mj}}{\partial x_k}+u_{\epsilon mi}\frac{\partial^2 u_{\epsilon mj}}{\partial x_i\partial x_k}\frac{\partial u'_{\epsilon mj}}{\partial x_k}\right)dx-\int_{\partial\Omega}u_{\epsilon mi}\frac{\partial u_{\epsilon mj}}{\partial x_i}\frac{\partial u'_{\epsilon mj}}{\partial x_k}\,d\Gamma\right| \\
&\le C\left(\left|\frac{\partial u_{\epsilon mi}}{\partial x_k}\right|_{L^6(\Omega)^N}\left|\frac{\partial u_{\epsilon mj}}{\partial x_i}\right|_{L^3(\Omega)^N}\left|\frac{\partial u'_{\epsilon m}}{\partial x_k}\right|_{L^2(\Omega)^N}\right. \\
&\quad\left.+|u_{\epsilon mi}|_{L^\infty(\Omega)^N}\left|\frac{\partial^2 u_{\epsilon mj}}{\partial x_i\partial x_k}\right|_{L^2(\Omega)^N}\left|\frac{\partial u'_{\epsilon mj}}{\partial x_k}\right|_{L^2(\Omega)^N}\right) \\
&\le \left|\frac{\partial u_{\epsilon mi}}{\partial x_k}\right|_{H^1(\Omega)^N}\left|\frac{\partial u_{\epsilon mj}}{\partial x_i}\right|_{H^1(\Omega)^N}\|u'_{\epsilon m}\|_{V(\Omega)} \\
&\quad+C\,|u_{\epsilon mi}|_{H^2(\Omega)^N}\,|u_{\epsilon mj}|_{H^2(\Omega)^N}\left\|u'_{\epsilon mj}\right\|_{V(\Omega)} \\
&\le C|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}\|u'_{\epsilon m}\|_{V(\Omega)}\le\frac14\|u'_{\epsilon m}\|^2_{V(\Omega)}+\tilde C_1|\Delta u_{\epsilon m}|^4_{L^2(\Omega)^N};
\end{aligned}
\tag{3.12}
$$

Since $-\Delta\alpha(x,t)=\beta(x,t)$ in Q,

$$
\begin{aligned}
\bullet\ \frac1\epsilon(\alpha(x,t)u_{\epsilon m},-\Delta u'_{\epsilon m}) &= \frac1\epsilon\int_\Omega\alpha(x,t)u'_{\epsilon m}(-\Delta u'_{\epsilon m})\,dx \\
&= \frac1\epsilon\int_\Omega\nabla(\alpha(x,t)u'_{\epsilon m})\nabla u'_{\epsilon m}\,dx-\frac1\epsilon\int_{\partial\Omega}\alpha(x,t)u'_{\epsilon m}\nabla u'_{\epsilon m}d\Gamma \\
&= \frac1\epsilon\int_\Omega\nabla\alpha(x,t)u'_{\epsilon m}\nabla u'_{\epsilon m}\,dx+\frac1\epsilon\int_\Omega\alpha(x,t)|\nabla u'_{\epsilon m}|^2\,dx \\
&= \frac1{2\epsilon}\int_\Omega\nabla\alpha(x,t)\nabla((u'_{\epsilon m})^2)\,dx+\frac1\epsilon\int_\Omega\alpha(x,t)|\nabla u'_{\epsilon m}|^2\,dx \\
&\ge \frac1{2\epsilon}\int_\Omega-\Delta\alpha(x,t)(u'_{\epsilon m})^2\,dx+\frac1{2\epsilon}\int_{\partial\Omega}\nabla\alpha(x,t)|u'_{\epsilon m}|^2d\Gamma \\
&\ge \frac1{2\epsilon}\int_\Omega\beta(x,t)|u'_{\epsilon m}|^2\,dx;
\end{aligned}
\tag{3.13}
$$

$$
\bullet\ |(\tilde f,-\Delta u'_{\epsilon m})|\le\|\tilde f\|_{V(\Omega)}\|u'_{\epsilon m}\|^2_{V(\Omega)}\le\frac14\|u'_{\epsilon m}\|^2_{V(\Omega)}+\|\tilde f\|^2_{V(\Omega)}. \tag{3.14}
$$

From (3.12), (3.13) and (3.14) in (3.11) we have

$$
\begin{aligned}
\|u'_{\epsilon m}\|^2_{V(\Omega)}+\nu\frac{d}{dt}|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}+\frac1\epsilon\int_\Omega\beta(x,t)|u'_{\epsilon m}|^2dx+\frac2\epsilon\int_\Omega\beta(x,t)|\Delta u'_{\epsilon m}|^2dx \\
\le\tilde C_1|\Delta u_{\epsilon m}|^4_{L^2(\Omega)^N}+\|\tilde f\|^2_{V(\Omega)}.
\end{aligned}
\tag{3.15}
$$

Before we get the next estimate, we claim that for any z that satisfies the assumptions of this lemma we have

$$
\int_0^t\int_{\Omega\setminus\Omega_s}|z(x,s)|^2dx\,ds\le T^2\int_0^t\int_{\Omega\setminus\Omega_s}|z'(x,s)|^2dx\,ds. \tag{3.16}
$$

Indeed, since $z(x,0)=0$ in $\Omega\setminus\Omega_t$ for any $t\in\mathbb{R}$ then

$$
z(x,t)-z(x,0)=\int_0^t z'(x,s)ds
$$

so

$$
|z(x,t)|\le\left(\int_0^t 1\,ds\right)^{1/2}\left(\int_0^t|z'(x,s)|^2ds\right)^{1/2}.
$$

Integrating in $\Omega \setminus \Omega_s \times [0,t)$, with $t \in [0,T]$, we have

$$\begin{aligned}
\int_0^t \int_{\Omega\setminus\Omega_s} |z(x,\sigma)|^2 dx\, d\sigma &\leq \int_0^t t \left(\int_{\Omega\setminus\Omega_s} \int_0^t |z'(x,s)|^2 ds\, dx \right) d\sigma \\
&\leq \int_0^T T \left(\int_0^t \int_{\Omega\setminus\Omega_s} |z'(x,s)|^2 dx\, ds \right) d\sigma \\
&\leq T^2 \int_0^t \int_{\Omega\setminus\Omega_s} |z'(x,s)|^2 dx\, ds.
\end{aligned}$$

Estimate II. Multiplying (3.11) by $\lambda_i g_{im}(t)$ and adding from i=1 to m, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{\epsilon m}\|^2_{V(\Omega)} + \nu|\Delta u_{\epsilon m}|^2_{L^2(\Omega)} + ((u_{\epsilon m}\cdot\nabla)u_{\epsilon m}, -\Delta u_{\epsilon m}) + \frac{1}{\epsilon}(\alpha(x,t)u'_{\epsilon m}, -\Delta u_{\epsilon m}) \\
+\frac{1}{\epsilon}(\beta(x,t)\Delta u'_{\epsilon m}, \Delta u_{\epsilon m}) = (\tilde{f}, -\Delta u_{\epsilon m}).
\end{aligned} \tag{3.17}$$

Using the same arguments as Estimate I, we compute

$$|((u_{\epsilon m}\cdot\nabla)u_{\epsilon m}, -\Delta u_{\epsilon m})| \leq C_1\|u_{\epsilon m}\|_{V(\Omega)}|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}. \tag{3.18}$$

By the ideas of [NN78] (see too, [Rab94]), we have

$$\frac{1}{\epsilon}\int_0^t (\beta(x,s)\Delta u'_{\epsilon m}, \Delta u_{\epsilon m})ds \geq \frac{1}{2\epsilon}|\beta(t)\Delta u_{\epsilon m}(t)|^2_{L^2(\Omega)^N} - \frac{1}{2\epsilon}|\beta(0)\Delta u_{\epsilon m}(0)|^2_{L^2(\Omega)^N}, \tag{3.19}$$

and from definitions of $\tilde{u}_0$ and $\beta(x,t)$, we easily get that

$$\frac{1}{\epsilon}\beta(0)\Delta u_{\epsilon m}(0) = \frac{1}{\epsilon}\beta(0)\Delta u_{0m} \longrightarrow \frac{1}{\epsilon}\beta(0)\Delta\tilde{u}_0 = 0 \text{ strongly in } L^2(\Omega)^N \text{ as } m\to\infty.$$

Thus, consider $\sigma_1 > 0$ such that $\kappa(u_0,f) + \sigma_1 < \min\{\frac{\nu^2}{4\tilde{C}_1}, \left(\frac{\nu}{4C_1}\right)^2\}$ and

$$\frac{1}{\epsilon}|\beta(0)\Delta u_{\epsilon m}(0)|^2_{L^2(\Omega)^N} \leq \sigma_1, \text{ for each } m \geq m_0(\epsilon,\sigma_1). \tag{3.20}$$

Furthermore, since $\alpha(x,t) = \tilde{\alpha}(x,t)\beta(x,t)$ in $Q\setminus\widehat{Q}$ we have

$$\frac{2}{\epsilon}(\alpha(x,t)u'_{\epsilon m}, -\Delta u_{\epsilon m}) \leq \frac{2}{\epsilon}\int_{\Omega\setminus\Omega_t} |\alpha(x,t)|_{L^\infty(\Omega)^N}\,\beta(x,t)\,|u'_{\epsilon m}|\,|\Delta u_{\epsilon m}|\,dx \tag{3.21}$$

and, by definition, α satisfies problem (3.3) then $|\Delta\alpha|_{H^2(\Omega\setminus\Omega_t)} \leq |1|_{L^2(\Omega\setminus\Omega_t)} \leq \text{med}(\Omega\setminus\Omega_t) \leq \text{med}(\Omega)$. Since $N \leq 3$ we have the embedding continuous $H^2(\Omega\setminus\Omega_t) \hookrightarrow L^\infty(\Omega\setminus\Omega_t)$, from which we can conclude that

$$|\alpha(x,t)|_{L^\infty(\Omega\setminus\Omega_t)^N} \leq C(\Omega), \text{ with } C \text{ independent of } t. \tag{3.22}$$

Hence, as a consequence of (3.16), we obtain for (3.21) that

$$\left|\frac{2}{\epsilon}\int_0^t (\alpha(x,s)u'_{\epsilon m}, -\Delta u_{\epsilon m})ds\right| \leq TC_1\left(\frac{1}{\epsilon}\int_0^t\int_\Omega \beta(x,s)(|u'_{\epsilon m}(s)|^2 + |\Delta u'_{\epsilon m}(s)|^2)dx\,ds\right). \tag{3.23}$$

And,

$$|(\tilde{f}, -\Delta u_{\epsilon m})| \leq \frac{\nu}{2}|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N} + \frac{1}{2\nu}|\tilde{f}|^2_{L^2(\Omega)^N}. \tag{3.24}$$

Then, from (3.18) and (3.24) in (3.17) we get

$$
\begin{aligned}
&\frac{d}{dt}\|u_{\epsilon m}\|^2_{V(\Omega)} + \frac{\nu}{2}|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N} + |\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}\left(\frac{\nu}{2} - C_1\|u_{\epsilon m}\|_{V(\Omega)}\right)\\
&\quad + \frac{2}{\epsilon}(\alpha(x,t)u'_{\epsilon m}, -\Delta u_{\epsilon m}) + \frac{1}{\epsilon}(\beta(x,t)\Delta u'_{\epsilon m}, \Delta u_{\epsilon m}) \leq \frac{1}{\nu}|\tilde{f}|^2_{L^2(\Omega)^N}.
\end{aligned} \tag{3.25}
$$

Thus, from (3.15) and (3.25),

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_{\epsilon m}\|^2_{V(\Omega)} + \nu|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}\right) + \|u'_{\epsilon m}\|^2_{V(\Omega)} + \frac{\nu}{2}|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}\\
&+\frac{2}{\epsilon}(\alpha(x,t)u'_{\epsilon m}, -\Delta u_{\epsilon m}) \ + \ \frac{1}{\epsilon}(\beta(x,t)\Delta u'_{\epsilon m}, \Delta u_{\epsilon m}) + \frac{1}{\epsilon}\int_\Omega \beta(x,t)|u'_{\epsilon m}|^2 dx\\
&+\frac{2}{\epsilon}\int_\Omega \beta(x,t)|\Delta u'_{\epsilon m}|^2 dx \ + |\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}\left(\frac{\nu}{2} - C_1\|u_{\epsilon m}\|_{V(\Omega)} - \tilde{C}_1|\Delta u_{\epsilon m}|^2_{L^2(\Omega)^N}\right)\\
&\leq \|\tilde{f}\|^2_{V(\Omega)} + \frac{1}{\nu}|\tilde{f}|^2_{L^2(\Omega)^N}.
\end{aligned} \tag{3.26}
$$

By hypothesis (H3) and (3.20),

$$
\left\{
\begin{aligned}
&C_1\|\tilde{u}_0\|_{V(\Omega)} + \tilde{C}_1|\Delta\tilde{u}_0|^2_{L^2(\Omega)^N} = C_1\|u_0\|_{V(\Omega_0)} + \tilde{C}_1|\Delta u_0|^2_{L^2(\Omega_0)^N} < \frac{\nu}{2},\\
&\|u_0\|^2_{V(\Omega_0)} + \nu|\Delta u_0|^2_{L^2(\Omega_0)^N} + \|f\|^2_{L^2(0,T;L^2(\Omega_t)^N)} + \|f\|^2_{L^2(0,T;V(\Omega_t))}\\
&+\sigma_1 < \min\left\{\frac{\nu^2}{4\tilde{C}_1}, \left(\frac{\nu}{4C_1}\right)^2\right\}.
\end{aligned}
\right. \tag{3.27}
$$

Thus,

$$
C_1\|u_{\epsilon m}(0)\|_{V(\Omega)} + \tilde{C}_1|\Delta u_{\epsilon m}(0)|^2_{L^2(\Omega)^N} \leq C_1\|u_0\|_{V(\Omega_0)} + \tilde{C}_1|\Delta u_0|^2_{L^2(\Omega_0)^N} < \frac{\nu}{2}.
$$

That said, for continuity we make the following claim:
Claim 1. For each $\epsilon > 0$ fixed and $m \geq m_0 = m_0(\epsilon, \sigma_1)$ given in (3.20), we have

$$
B(t) := C_1\|u_{\epsilon m}(t)\|_{V(\Omega)} + \tilde{C}_1|\Delta u_{\epsilon m}(t)|^2_{L^2(\Omega)^N} < \nu/2, \ \forall\, t \in [0, t_m], \text{ where } 0 < t_m \leq T_0.
$$

We shall prove by contradiction. Suppose that there exists t^*_m minimum such that

$$
B(t) < \frac{\nu}{2} \text{ in } 0 \leq t < t^*_m, \text{ and } B(t^*_m) = \frac{\nu}{2}. \tag{3.28}
$$

Integrating (3.26) from 0 to t^*_m and using (3.19), (3.20) and (3.23) for t^*_m, with $t^*_m < t_m < T_0 C_1 < 1/2$ we calculate that

$$
\begin{aligned}
&\|u_{\epsilon m}(t^*_m)\|^2_{V(\Omega)} + \nu|\Delta u_{\epsilon m}(t^*_m)|^2_{L^2(\Omega)^N} + \frac{1}{2\epsilon}|\beta(t^*_m)\Delta u_{\epsilon m}(t^*_m)|^2_{L^2(\Omega)^N}\\
&+\int_0^{t^*_m}\|u'_{\epsilon m}(s)\|^2_{V(\Omega)}ds \ + \int_0^{t^*_m}\frac{\nu}{2}|\Delta u_{\epsilon m}(s)|^2_{L^2(\Omega)^N}ds\\
&+\frac{1}{2\epsilon}\int_0^{t^*_m}\int_\Omega \beta(x,t)|u'_{\epsilon m}(s)|^2 dx\, ds + \frac{3}{2\epsilon}\int_0^{t^*_m}\int_\Omega \beta(x,t)|\Delta u'_{\epsilon m}(s)|^2 dx\, ds\\
&\leq \|u_{\epsilon m}(0)\|^2_{V(\Omega)} + \nu|\Delta u_{\epsilon m}(0)|^2_{L^2(\Omega)^N} + \frac{1}{2\epsilon}|\beta(0)\Delta u_{\epsilon m}(0)|^2_{L^2(\Omega)^N}\\
&+\int_0^{t^*_m}(\|\tilde{f}(s)\|^2_{V(\Omega)} + \frac{1}{\nu}|\tilde{f}(s)|^2_{L^2(\Omega)^N})ds\\
&\leq \|u_0\|^2_{V(\Omega_0)} + \nu|\Delta u_0|^2_{L^2(\Omega_0)^N} + \int_0^{T_0}(\|f(s)\|^2_{V(\Omega_t)} + \frac{1}{\nu}|f(s)|^2_{L^2(\Omega_t)^N})ds + \sigma_1.
\end{aligned} \tag{3.29}
$$

Therefore, from (3.27) and (3.29) we have

$$B(t_m^*) < \frac{\nu}{2}.$$

hence, comparing with (3.28), we have a contradiction.

As a result of (3.19), (3.23) and of claim 1 in (3.26), for each $\epsilon > 0$ fixed and $m \geq m_0(\epsilon, \sigma_1)$, we compute

$$\begin{aligned}
&\|u_{\epsilon m}(t)\|^2_{V(\Omega)} + \nu|\Delta u_{\epsilon m}(t)|^2_{L^2(\Omega)^N} + \int_0^{T_0} \|u'_{\epsilon m}(s)\|^2_{V(\Omega)} ds + \frac{1}{2\epsilon}|\beta(T^0)\Delta u_{\epsilon m}(T^0)|^2_{L^2(\Omega)^N} \\
&+ \int_0^{T_0} \frac{\nu}{2}|\Delta u_{\epsilon m}(s)|^2_{L^2(\Omega)^N} ds + \frac{1}{2\epsilon}\int_0^{T_0}\int_\Omega \beta(x,t)|u'_{\epsilon m}(s)|^2 dx\, ds \\
&+ \frac{3}{2\epsilon}\int_0^{T_0}\int_\Omega \beta(x,t)|\Delta u'_{\epsilon m}(s)|^2 dx\, ds \\
&\leq \|u_0\|^2_{V(\Omega_0)} + \nu|\Delta u_0|^2_{L^2(\Omega_0)^N} + \frac{1}{2\epsilon}|\beta(0)\Delta u_{\epsilon m}(0)|^2_{L^2(\Omega)^N} \\
&+ \int_0^{T_0} (\|f(s)\|^2_{V(\Omega_t)} + \frac{1}{\nu}|f(s)|^2_{L^2(\Omega_t)^N}) ds.
\end{aligned} \tag{3.30}$$

Applying Aubin-Lions Theorem (see, Chapter 1, Theorem 5.1 in [Lio69]) in (3.30) we can extract a subsequence of $\{u_{\epsilon m}\}$ denoted equal such that making $m \to \infty$, we have

$$\begin{cases}
u'_{\epsilon m} \longrightarrow u'_\epsilon & \text{weak in} \quad L^2(0, T_0; V(\Omega)), \\
u_{\epsilon m} \longrightarrow u_\epsilon & \text{weak}^* \text{ in} \quad L^\infty(0, T_0; H^2(\Omega)^N \cap V(\Omega)), \\
\frac{1}{\epsilon}\beta(x,t)u'_{\epsilon m} \longrightarrow \frac{1}{\epsilon}\beta(x,t)u'_\epsilon & \text{weak in} \quad L^2(\Omega \times (0, T_0))^N, \\
\frac{1}{\epsilon}\beta(x,t)\Delta u'_{\epsilon m} \longrightarrow \frac{1}{\epsilon}\beta(x,t)\Delta u'_\epsilon & \text{weak in} \quad L^2(\Omega \times (0, T_0))^N, \\
u_{\epsilon m} \longrightarrow u_\epsilon & \text{strong in} \quad L^2(0, T_0; H(\Omega)), \\
u_{\epsilon m i} \longrightarrow u_{\epsilon i} & \text{a.e. in} \quad \Omega \times (0, T_0), \\
\frac{\partial u_{\epsilon m j}}{\partial x_i} \longrightarrow \frac{\partial u_{\epsilon j}}{\partial x_i} & \text{a.e. in} \quad \Omega \times (0, T_0).
\end{cases} \tag{3.31}$$

Thus, $u_{\epsilon m i}\partial u_{\epsilon m j}/\partial x_i \longrightarrow u_{\epsilon i}\partial u_{\epsilon j}/\partial x_i$ a.e. in $\Omega \times (0, T_0)$. And, since $N \leq 3$,

$$\sum_{i,j=1}^N \int_\Omega \left| u_{\epsilon m i}\frac{\partial u_{\epsilon m j}}{\partial x_i}\right|^2 \leq C|u_{\epsilon m}|^2_{H^2(\Omega)}|u_{\epsilon m}|^2_{V(\Omega)}.$$

Consequently, by the Lions Lemma, we obtain

$$u_{\epsilon m i}\frac{\partial u_{\epsilon m j}}{\partial x_i} \longrightarrow u_{\epsilon i}\frac{\partial u_{\epsilon j}}{\partial x_i} \text{ weak in } L^2(\Omega \times (0, T_0)). \tag{3.32}$$

Therefore, applying the same reasoning when considering $u(., T_0) = u_{T_0}$ in Ω as initial data, we conclude (3.31) and (3.32) in $\Omega \times [T_0, 2T_0)$. Thus, repeating the process recursively until $nT_0 > T$, for $n \in \mathbb{N}$, we have (3.31) and (3.32) in $\Omega \times [0, T)$.

In this way, we can pass to the limit at (3.11) and then

$$\begin{aligned}
&\int_Q (u'_\epsilon - \nu\Delta u_\epsilon + u_\epsilon \nabla u_\epsilon + \frac{1}{\epsilon}\alpha(x,t)u'_\epsilon - \frac{1}{\epsilon}\beta(x,t)\Delta u'_\epsilon)\varphi\, dx\, dt \\
&\qquad = \int_Q \tilde{f}\varphi\, dx\, dt, \forall\, \varphi \in L^2(0, T; H(\Omega)),
\end{aligned}$$

proving that u_ϵ is a solution to the problem (3.5) such that $u_\epsilon \in L^\infty(0,T;H^2(\Omega)^N \cap V(\Omega))$, $u'_\epsilon \in L^2(0,T;V(\Omega))$. $\square$

Proof of Theorem 3.1. From Lemma 3.1 and the definitions of α and β given by (3.4) and (3.2), respectively, we have

$$\int_{\widehat{Q}} \big(u'_\epsilon - \nu\Delta u_\epsilon + u_\epsilon\nabla u_\epsilon\big)\,\varphi\,dx\,dt = \int_{\widehat{Q}} f\varphi\,dx\,dt,\ \forall\,\varphi \in L^2(0,T;H(\Omega_t)). \tag{3.33}$$

Furthermore, by weak convergence property of (3.30) and (3.31) we can deduce that

$$\begin{aligned} &\|u_\epsilon(t)\|^2_{V(\Omega)} + \nu|\Delta u_\epsilon(t)|^2_{L^2(\Omega)^N} + \int_0^t \|u'_\epsilon(s)\|^2_{V(\Omega)}ds + \int_0^t \frac{\nu}{2}|\Delta u_\epsilon(s)|^2_{L^2(\Omega)^N}ds \\ &\le \|u_0\|^2_{V(\Omega_0)} + \nu|\Delta u_0|^2_{L^2(\Omega_0)^N} + \int_0^T (\|f(s)\|^2_{V(\Omega_t)} + \frac{1}{\nu}|f(s)|^2_{L^2(\Omega_t)^N})ds + \sigma_1. \end{aligned} \tag{3.34}$$

and by arguments analogous to those applied to obtain (3.31) and (3.32), we have

$$\left\{\begin{array}{llll} u'_\epsilon \longrightarrow u' & \text{weak in} & L^2(0,T_0;V(\Omega)), \\ u_\epsilon \longrightarrow u & \text{weak}^* \text{ in} & L^\infty(0,T_0;H^2(\Omega)^N \cap V(\Omega)), \\ u_\epsilon \longrightarrow u & \text{strong in} & L^2(0,T_0;H(\Omega)), \\ u_{\epsilon i}\dfrac{\partial u_{\epsilon j}}{\partial x_i} \longrightarrow u_i\dfrac{\partial u_j}{\partial x_i} & \text{weak in} & L^2(\Omega\times(0,T_0)), \end{array}\right. \tag{3.35}$$

as $\epsilon \to 0$. Therefore, we can pass the limit in (3.33) as $\epsilon \to 0$ and then

$$\int_{\widehat{Q}} \big(u' - \nu\Delta u + u\nabla u\big)\,\varphi\,dx\,dt = \int_{\widehat{Q}} f\varphi\,dx\,dt,\ \forall\,\varphi \in L^2(0,T;H(\Omega_t)), \tag{3.36}$$

i.e., u is a strong solution to the problem (3.1) in the class $u \in L^\infty(0,T;H^2(\Omega_t)^N \cap V(\Omega_t))$, $u' \in L^2(0,T;V(\Omega_t))$.

Finally, we will show that it is possible to recover the pressure term. In fact, by the Du Bois-Reymond lemma ([MM19], Proposition 1.4) in (3.36) we get

$$u' - \nu\Delta u + u\nabla u - f = 0 \text{ in } L^2(0,T;H(\Omega_t)),$$

consequently, for almost ever $s \in [0,T]$,

$$u' - \nu\Delta u + u\nabla u - f = 0 \text{ in } H(\Omega_s).$$

By duality,

$$\langle u' - \nu\Delta u + u\nabla u - f, \psi\rangle = 0\ \forall\ \psi \in V(\Omega_s)$$

and from Rham's theorem ([BF13], Theorem IV.2.3) there exists a unique $p(s) \in L^2_0(\Omega_s)^N$ such that

$$u' - \nu\Delta u + u\nabla u - f = -\nabla p(s). \tag{3.37}$$

So, $\nabla p(s) \in L^2(\Omega_s)^N$ and consequently $p(s) \in H^1(\Omega_s)^N$.

Therefore, let $\varphi \in \mathcal{D}(\widehat{Q})$ such that

$$\big(u' - \nu\Delta u + u\nabla u + \nabla p(s) - f, \varphi(x,s)\big)_{L^2(\Omega_s)^N \times L^2(\Omega_s)^N} = 0.$$

Then,

$$\int_0^T \int_{\Omega_s} (u' - \nu\Delta u + u\nabla u + \nabla p - f)\varphi \, dx \, dt = 0$$

thus implying the Navier-Stokes equation

$$u' - \nu\Delta u + u\nabla u + \nabla p - f = 0 \text{ in } L^2(\widehat{Q}),$$

with $p \in L^2(0,T;H^1(\Omega_t)) \cap L^2(0,T;L^2_0(\Omega_t))$. Concluding the proof of the Theorem 3.1. □

Proof for Theorem 3.2

Proof of Theorem 3.2. Let u_1 and u_2 be two solutions of Theorem 3.1, then $u = u_1 - u_2$ satisfy

$$\begin{cases} u' - \nu\Delta u = u_2 \cdot \nabla u_2 - u_1 \cdot \nabla u_1 & \text{in} \quad \widehat{Q}, \\ \nabla \cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot, 0) = 0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.38}$$

Multiplying by $\varphi = u$, integrating over $\widehat{Q}_t$ we have

$$\int_{\widehat{Q}_t} u'u \, dxdt - \int_{\widehat{Q}_t} \nu\Delta uu \, dxdt = \int_{\widehat{Q}_t} (u_2 \cdot \nabla u_2 - u_1 \cdot \nabla u_1)u \, dxdt \tag{3.39}$$

So, we use the Gauss-Green theorem, and since u vanishes on $\widehat{\Sigma}$.

$$\begin{aligned} \int_{\widehat{Q}_t} u'u \, dxdt &= \frac{1}{2}\int_{\widehat{Q}_t} \frac{d}{dt}|u|^2 \, dxdt = \frac{1}{2}\int_{\widehat{Q}_t} div(0,0,u^2) \, dxdt \\ &= \frac{1}{2}\int_{\partial\widehat{Q}_t} (0,0,u^2) \cdot n \, d\Gamma \\ &= \frac{1}{2}\int_{\Omega_t} (0,0,u^2)(0,0,1) d\Gamma + \int_{\Sigma_t} (0,0,u^2) \cdot n \, d\Gamma \\ &\quad + \int_{\Omega_0} (0,0,u^2)(0,0,-1) d\Gamma \\ &= \frac{1}{2}\int_{\Omega_t} |u|^2 \, dx; \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} -\int_{\widehat{Q}_t} \Delta uu \, dxdt &= \int_{\widehat{Q}_t} [-div_{(x,t)}(\nabla uu, 0) + |\nabla u|^2] \, dxdt \\ &= -\int_{\partial\widehat{Q}_t} (\nabla uu, 0) \cdot n d\Gamma dt + \int_{\widehat{Q}_t} |\nabla u|^2 \, dxdt \\ &= \int_{\widehat{Q}_t} |\nabla u|^2 \, dxdt; \end{aligned} \tag{3.41}$$

therefore, from (3.40) and (3.41) we have

$$\begin{aligned} \frac{1}{2}\int_{\Omega_t} |u|^2 \, dx + \int_{\widehat{Q}_t} |\nabla u|^2 \, dxdt &= \int_{\widehat{Q}_t} (u_2 \cdot \nabla u_2 - u_1 \cdot \nabla u_1)u \, dxdt \\ &= -\int_{\widehat{Q}_t} (u_2 \cdot \nabla uu + u \cdot \nabla u_1 u) \, dxdt \\ &= J_1 + J_2; \end{aligned} \tag{3.42}$$

then

$$
\begin{aligned}
J_1 &= \sum_{i,j=1}^{3} \int_{\widehat{Q}_t} u_{2i} \frac{\partial u_j}{\partial x_i} u_j \, dxdt = \sum_{i,j=1}^{3} \int_{\widehat{Q}_t} \frac{1}{2} u_{2i} \frac{\partial}{\partial x_i} (u_j^2) \, dxdt \\
&= \frac{1}{2} \int_{\widehat{Q}_t} div_{(x,t)} (u_2 (\sum_{j=1}^{3} u_j^2), 0) \, dxdt \\
&= \frac{1}{2} \int_{\partial \widehat{Q}_t} (u_2 (\sum_{j=1}^{3} u_j^2), 0) \cdot n d\Gamma dt = 0.
\end{aligned} \tag{3.43}
$$

If $N = 2$, we have $|z|_{L^4(\Omega_t)^N} \leq C|z|^{1/2}_{L^2(\Omega_t)^N} |\nabla z|^{1/2}_{L^2(\Omega_t)^N}$, we get to $\epsilon > 0$

$$
\begin{aligned}
J_2 &\leq \int_0^t |u|_{L^4(\Omega_t)^N} |\nabla u_1|_{L^2(\Omega_t)^N} |u|_{L^4(\Omega_t)^N} \, ds \\
&\leq C \int_0^t |u|_{L^2(\Omega_t)^N} |\nabla u|_{L^2(\Omega_t)^N} |\nabla u_1|_{L^2(\Omega_t)^N} \, ds \\
&\leq \epsilon \int_0^t |\nabla u|^2_{L^2(\Omega_t)^N} + C(\epsilon) \int_0^t |\nabla u_1|^2_{L^2(\Omega_t)^N} |u|^2_{L^2(\Omega_t)^N} \, ds,
\end{aligned} \tag{3.44}
$$

or if $N = 3$, we have $|z|_{L^3(\Omega_t)^N} \leq C|z|^{1/2}_{L^2(\Omega_t^N)} |\nabla z|^{1/2}_{L^2(\Omega_t)^N}$, we can get

$$
\begin{aligned}
J_2 &\leq \int_0^t |u|_{L^3(\Omega_t)^N} |\nabla u_1|_{L^2(\Omega_t)^N} |u|_{L^6(\Omega_t)^N} \, ds \\
&\leq C \int_0^t |u|^{1/2}_{L^2(\Omega_t)^N} |\nabla u|^{1/2}_{L^2(\Omega_t)^N} |\nabla u_1|_{L^2(\Omega_t)^N} |u|_{H^1(\Omega_t)^N} \, ds \\
&\leq C \int_0^t |u|^{1/2}_{L^2(\Omega_t)^N} |\nabla u|^{3/2}_{L^2(\Omega_t)^N} |\nabla u_1|_{L^2(\Omega_t)^N} \, ds \\
&\leq \epsilon \int_0^t |\nabla u|^2_{L^2(\Omega_t)^N} + C(\epsilon) \int_0^t |\nabla u_1|^4_{L^2(\Omega_t)^N} |u|^2_{L^2(\Omega_t)^N} \, ds.
\end{aligned} \tag{3.45}
$$

Now, replacing (3.43), (3.44) and (3.45) in (3.42), we obtain

$$
\frac{1}{2} \int_{\Omega_t} |u|^2 dx + \int_{\widehat{Q}_t} |\nabla u|^2 dxdt \leq C(\epsilon) \int_0^t m(s) |u|^2_{L^2(\Omega_t)^N} \, ds, \tag{3.46}
$$

where

$$
m(s) = \begin{cases} |\nabla u_1|^2_{L^2(\Omega_t)^N}, & \text{if} \quad N = 2, \\ |\nabla u_1|^4_{L^2(\Omega_t)^N}, & \text{if} \quad N = 3. \end{cases} \tag{3.47}
$$

Using Grönwall's inequality in (3.46), we have $u = 0$ and consequently $u_1 = u_2$. □

3.4 Decay of solutions

In this section we shall consider the decay of solutions. We will assume that Ω_t tends to a bounded domain $\tilde{\Omega}$, as $t \to \infty$. Thus, the constant in the Poincaré inequality can be considered independent of t, which we will denote by c_1. For more details, see [Sal88].

Proof of Theorem 3.3. Multiplying the first line of the system (3.1) by u and integrating in Ω_τ we have

$$
\frac{d}{d\tau} |u(\tau)|^2_{L^2(\Omega_\tau)^N} + |\nabla u(\tau)|^2_{L^2(\Omega_\tau)^N} \leq 0, \tag{3.48}
$$

in this account we are using the fact that

$$\int_{\Omega_\tau} (u\cdot\nabla)u\,u\,dx = \sum_{i,j=1}^{3}\int_{\Omega_\tau} u_i \frac{\partial u_j}{\partial x_i} u_j\,dx = \sum_{i,j=1}^{3}\int_{\Omega_\tau} \frac{1}{2} u_i \frac{\partial}{\partial x_i}(u_j^2)\,dx$$

$$= \frac{1}{2}\int_{\Omega_\tau} div_{(x)} u(\sum_{j=1}^{3} u_j^2) = \frac{1}{2}\int_{\Gamma_\tau} u(\sum_{j=1}^{3} u_j^2)\cdot n\,d\Gamma = 0.$$

Then by the Poincaré inequality, $\|u\|^2_{H_0^1(\Omega_\tau)^N} \geq \mathrm{c}_1 |u|^2_{L^2(\Omega_\tau)^N}$, and integrating (3.48) from s to t, for $s < t$,

$$|u(t)|^2_{L^2(\Omega_t)^N} + \mathrm{c}_1 \int_s^t |u(\tau)|^2_{L^2(\Omega_\tau)^N} d\tau \leq |u(s)|^2_{L^2(\Omega_s)^N}$$

Hence, applying Theorema 8.1 of [Kom94] with $M = \frac{1}{\mathrm{c}_1}$,

$$|u(t)|^2_{L^2(\Omega_t)^N} \leq e^{1-t/M} |u_0|^2_{L^2(\Omega_0)^N}.$$

Proving that (3.10) holds. □

Some additional comments and open questions

In the context of the result obtained in this thesis, we will give some comments on the systems studied and we will also expose some problems in the context addressed here that, as far as we know, are open.

About Chapter 1:

(1) **System** (1.1) **with fewer controls acting.** Note that, when $N = 2$, it immediately follows from Theorem 1.1 that only a scalar control insensitizes the functional (1.2). However, when $N = 3$, to follow the same techniques used here we would need a Carleman estimate of type (1.24) with only one φ_j on the right side, and this is a more complex assignment. A Carleman estimate for just one scalar control was proved in [CL14], where the authors proved a local null controllability result for the three-dimensional Navier–Stokes system which has two vanishing components.

(2) **Ladyzhenskaya–Boussinesq system.** An important observation is that, following the techniques of this work, it is possible to prove the existence of insensitizing control for the Ladyzhenskaya–Boussinesq system, described by

$$\begin{cases} y_t - \nabla\cdot\big((\nu_0+\nu_1\|\nabla y\|^2)Dy\big) + (y\cdot\nabla)y + \nabla p = f + v\chi_\omega + \theta e_N & \text{in} \quad Q, \\ \nabla\cdot y = 0 & \text{in} \quad Q, \\ \theta_t - \nabla\cdot\big((\nu_0+\nu_1\|\nabla y\|^2)\nabla\theta\big) + y\cdot\nabla\theta = f_0 + v_0\chi_\omega & \text{in} \quad Q, \\ y(x,t) = 0,\ \theta(x,t) = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0 + \tau\widehat{y}^0,\ \theta(x,0) = \theta^0 + \tau\widehat{\theta}^0 & \text{in} \quad \Omega. \end{cases} \tag{3.49}$$

where

$$e_N = \begin{cases} (0,1) \text{ if } N = 2, \\ (0,0,1) \text{ if } N = 3. \end{cases}$$

More accurately, such techniques would lead us to obtain regularities similar to those established in Lemmas 1.2–1.4 for the velocity variable y and for the temperature variable θ of the corresponding linearized system of (3.49). For this, the starting point would be to consider the linearized system of [CGG15], in which the weights considered for the velocity variable are different from the weights defined here.

Extending equation (3.49), the same can be done for the complete Ladyzhenskaya–Boussinesq system (2.1) since ∇y is L^4 in time.

(3) **Problems with the K_τ and I_τ functionals.** As indicated in Section 1.2, it would be interesting to verify whether it is possible to prove the existence of insensitizing control for (1.1) considering a functional that depends on the state gradient, that is, considering the sentinel functional given by

$$K_\tau(y) = \frac{1}{2}\iint\limits_{\mathcal{O}\times(0,T)} |\nabla y|^2\,dx\,dt$$

or by the L^2 norm of its curl$(\nabla\times y)$. In other words,

$$I_\tau(y) = \frac{1}{2}\iint\limits_{\mathcal{O}\times(0,T)} |\nabla\times y|^2\,dx\,dt.$$

(4) **Problems with boundary controls.** The other relevant point to considered is the case in which the boundary data is partially unknown. For example, for simplicity, assume $\nu_0 = 1$ and $\nu_1 = 0$ in (1.1) and consider

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)\, y + \nabla p = f + v\chi_\omega, \;\; \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = g + \tau_1 \hat{g} & \text{on} \quad \Sigma, \\ y(0) = 0 & \text{in} \quad \Omega, \end{cases}$$

where $g, \hat{g} \in L^2(\Sigma)^N$ with $\|\hat{g}\|_{L^2(\Sigma)^N} = 1$ and τ_1 is a real and small number. In this case, $\hat{g}$ and τ_1 are unknown. Defining the sentinel functional as

$$\Phi_{\tau_1}(y) = \|y\|^2_{L^2(\mathcal{O}\times(0,T))^N},$$

we obtain by the same arguments applied in Section 1.2 that

$$\left.\frac{\partial \Phi_{\tau_1}(y)}{\partial \tau_1}\right|_{\tau_1=0} = 0 \;\text{ if and only if }\; \frac{\partial r}{\partial \nu} = 0 \text{ a.e. on } \Sigma,$$

where r corresponds to a formal adjoint of the equation governed by the derivative of y with respect to τ_1 at $\tau_1 = 0$ and $\frac{\partial}{\partial \nu}$ denotes the outward normal derivation. In other words, the insensitizing control condition is equivalent to $\frac{\partial r}{\partial \nu} = 0$ a.e. on Σ. For a better understanding of this case, see [BF95] in which such a study is carried out for a semilinear heat equation.

As far as we know, results related to the existence (or non-existence) of insensitive boundary controls for fluid equations are still unknown. Note that, considering our system (1.1), it does not seem clear that there are controls that insensitize the energy in an open $\mathcal{O}$ of the system

$$\begin{cases} y_t - \nabla \cdot ((\nu_0 + \nu_1 \|\nabla y\|^2) Dy) + (y \cdot \nabla)\, y + \nabla p = f & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = v\chi_\gamma & \text{on} \quad \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in} \quad \Omega, \end{cases} \tag{3.50}$$

where $v \in L^2(\Sigma)^N$ is the control to be determined acting on $\gamma \subset \partial\Omega$, an open non-empty on the boundary. However, an interesting open question is the existence of controls that insensitize the functional $\tilde{\Phi}_\tau$ given by

$$\tilde{\Phi}_\tau(y) = \frac{1}{2} \iint\limits_{\Gamma\times(0,T)} \left|\frac{\partial y}{\partial \nu}\right|^2 d\sigma\, dt,$$

where y is the solution of (3.50) associated with τ and v, and $\Gamma \subset \partial\Omega$ is a new open (non-empty) boundary such that $\gamma \cap \Gamma \neq \emptyset$, see [Pér04] for more information about this functional. The difficulty of this problem lies in analyzing the possibility of obtaining an appropriate Carleman inequality for a coupled adjoint system of parabolic equations with boundary control, and this is still an open question.

About Chapter 2:

Initially, note that (2.3) can be solved with the same techniques by taking $\bar{\nu}(\nabla\varsigma) := \nu_0 + \nu_1 \|\nabla \varsigma\|^2_{L^2}$. Furthermore, for our systems (2.1) and (2.3) it is also possible to obtain the local null controllability with control at the border $\Gamma_0 \times (0,T)$, where $\Gamma_0 \subset \partial\Omega$. Indeed, just construct a domain $\hat{\Omega}$ with boundary

$\partial\hat{\Omega}$ sufficiently regular via a subset U of $\mathbb{R}^N$ such that $\hat{\Omega} = \Omega \cup \mathrm{U}$ and $\bar{\mathrm{U}} \cap (\overline{\partial\Omega - \Gamma_0}) = \emptyset$. So, taking $\omega \subset \hat{\Omega} - \bar{\Omega}$ and keeping in mind the controllability result for distributed controls, the control at the boundary is obtained by considering the constraint trace in $\Omega \times (0,T)$ of the state of the distributed control system. That is, since $z(x,t)$ is the solution in $\hat{\Omega} \times (0,T)$ of the distributed control system then

$$u = \gamma(z\,|_{\Omega\times(0,T)}) = \begin{cases} \gamma(z) & \text{in} \quad \Gamma_0, \\ 0 & \text{in} \quad \partial\Omega - \Gamma_0 \end{cases}$$

is the control on the desired boundary, where $\gamma : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$.

Now, we comment on some open questions that arise naturally in the context of our results.

i) Is it possible the local exact controllability to the trajectories for the systems (2.1) and (2.3)? The main difficulty for this problem is finding a suitable Carleman estimate.

ii) Is it possible the local exact controllability to (2.1) when $N \geq 4$? This is a very difficult question, because in the proof of Lemma 2.6 we use the immersion $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and this is only valid when $N \leq 3$.

iii) Can we obtain controllability results for (2.3) when $2 < p < 3$? Note that in our case the fact that p is greater than the dimension $(p > 3)$ allowed us to use immersion $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

iv) Finally, can we deduce the null controllability of (2.1) and (2.3) in N dimensions, with $N-1$ controls?

About Chapter 3:

Firstly, note that it is possible to apply the same arguments here present in the Boussinesq system. More specifically, representing the temperature variable by θ, we have the well-known Boussinesq system

$$\begin{cases} u' - \nu\Delta u + (u\cdot\nabla)u = \theta e_N + f - \nabla p & \text{in} \quad \widehat{Q}, \\ \theta' - \Delta\theta + u\cdot\nabla\theta = f_1 & \text{in} \quad \widehat{Q}, \\ \nabla\cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0,\ \theta = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot,0) = u_0,\ \theta(.,0) = \theta_0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.51}$$

Hence, using (3.2) and (3.4), we obtain for $\epsilon > 0$ the following problem penalized

$$\begin{cases} u_\epsilon' - \nu\Delta u_\epsilon + (u_\epsilon\cdot\nabla)u_\epsilon + \frac{1}{\epsilon}\alpha(x,t)u_\epsilon' - \frac{1}{\epsilon}\beta(x,t)\Delta u_\epsilon' = {\theta_\epsilon}' e_N + \tilde{f} - \nabla p_\epsilon & \text{in} \quad Q, \\ {\theta_\epsilon}' - \Delta{\theta_\epsilon}' + {u_\epsilon}'\cdot\nabla{\theta_\epsilon}' + \frac{1}{\epsilon}\alpha(x,t){\theta_\epsilon}' - \frac{1}{\epsilon}\beta(x,t)\Delta{\theta_\epsilon}' = \tilde{f}_1 & \text{in} \quad Q, \\ \nabla\cdot u_\epsilon = 0 & \text{in} \quad Q, \\ u_\epsilon = 0,\ \theta_\epsilon = 0 & \text{on} \quad \partial\Omega\times[0,T), \\ u_\epsilon(\cdot,0) = \tilde{u}_0,\ \theta_\epsilon(.,0) = \tilde{\theta}_0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.52}$$

Therefore, it is feasible to prove a result for (3.52) similar to Lemma 3.1 and consequently obtain the existence and uniqueness of strong solutions for (3.51).

Now, we will indicate here some that, as far as we know, are open.

The Ladyzhenskaya-Smagorinsky kind differential turbulence model, where μ_0 and μ_1 are positive constants that represent the kinematic viscosity and turbulent viscosity, respectively.

$$\begin{cases} u' - \nabla \cdot ((\mu_0 + \mu_1 \displaystyle\int_\Omega |\nabla u|^2)\nabla u) + (u \cdot \nabla)u = f - \nabla p & \text{in} \quad \widehat{Q}, \\ \nabla \cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.53}$$

Certainly some specific difficulties, due to the occurrence of non-local nonlinear terms, will be encountered in this problem. For a broader perspective on this model, one can refer to [M C01]. In light of this, we can contemplate:

$$\begin{cases} u' - a(\displaystyle\int_\Omega |\nabla u|^2)\Delta u + (u \cdot \nabla)u = f - \nabla p & \text{in} \quad \widehat{Q}, \\ \nabla \cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.54}$$

where $a \in C^1(\mathbb{R})$ and $0 < m \leq a(r) \leq M$, for all $r \in \mathbb{R}$.

Furthermore, akin to [DG91], the subsequent model involving the gradient of u in $\mathbb{R}^N$ warrants consideration:

$$\begin{cases} u' - \nabla \cdot (a(|\nabla u|^2_{\mathbb{R}^N})\nabla u) + (u \cdot \nabla)u = f - \nabla p & \text{in} \quad \widehat{Q}, \\ \nabla \cdot u = 0 & \text{in} \quad \widehat{Q}, \\ u = 0 & \text{on} \quad \widehat{\Sigma}, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega_0. \end{cases} \tag{3.55}$$

with a satisfying the same conditions as before.

APPENDIX **A**

Appendix to Chapter 1

A.1 Regularity for the nonlinear cascade system (1.7)

Here we prove the existence and uniqueness of, solution for (1.7).

We know from [FLM15] and [HLC18] that, when $N = 2$, for any $y^0 \in \mathrm{V}$ and any $v \in L^2(\omega \times (0,T)),\ f \in L^2(Q)^N$, the system

$$\begin{cases} w_t - \nabla \cdot \big((\nu_0 + \nu_1 \|\nabla w\|^2) Dw\big) + (w \cdot \nabla)\, w + \nabla p^0 = f + v\chi_\omega, & \\ \nabla \cdot w = 0 & \text{in} \quad Q, \\ w = 0 & \text{on} \quad \Sigma, \\ w(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{A.1}$$

possesses exactly one strong solution (w, p^0), with

$$w \in L^2(0,T; D(A)) \cap C^0([0,T]; \mathrm{V}),\ w_t \in L^2(0,T; \mathrm{H}),$$

where $A : D(A) \to \mathrm{H}$, the Stokes operator. By definition, one has

$$D(A) = H^2(\Omega)^N \cap \mathrm{V},\ A(w) = P(-\Delta w)\ \forall w \in D(A),$$

with $P : L^2(\Omega)^N \to \mathrm{H}$ denoting the usual orthogonal projector. And, when $N = 3$, this is true if v and f are sufficiently small.

Therefore, we need regularity for the variable z. Defining, $\bar{z}(x,t) = z(x, T-t)$ in $(1.7)_2$ we get

$$\begin{cases} \bar{z}_t - (\nu_0 + \nu_1 \|\nabla w\|^2) \Delta \bar{z} + 2\nu_1 \left((\Delta w, \bar{z})_{L^2} \Delta w \right) + \left(\bar{z} \cdot \nabla^t \right) w & \\ -(w \cdot \nabla) \bar{z} + \nabla q = w\chi_{\mathcal{O}},\ \ \nabla \cdot \bar{z} = 0 & \text{in} \quad Q, \\ \bar{z} = 0 & \text{on} \quad \Sigma, \\ \bar{z}(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{A.2}$$

For simplicity we will do the calculations with z instead of $\bar{z}$.

We introduce the eigenfunctions of the Stokes operator, i.e. the solutions to

$$\begin{cases} \Delta k^j + \nabla \gamma^j = \lambda_j k^j & \text{in} \quad \Omega, \\ k^j = 0 & \text{on} \quad \partial\Omega, \\ \|k^j\| = 1,\ \lambda_j \to +\infty, & \end{cases}$$

Also, consider the spaces $V_m := span\{k^1, \ldots, k^m\}$ and the following associated Galerkin approximations

$$\begin{cases} (z_m', k) + (\nu_0 + \nu_1 \|\nabla w\|^2)(\nabla z_m, \nabla k) + 2\nu_1 \left((\Delta w, z_m)\Delta w, k\right) \\ + \big((z_m \cdot \nabla^t) w, k\big) - ((w \cdot \nabla) z_m, k) = (w\chi_{\mathcal{O}}, k), \quad \forall k \in V_m, \\ z_m : [0,T] \to V_m, \; z(0) = 0, \end{cases} \tag{A.3}$$

By the classical theory of ODE, we can state that the existence and uniqueness of solutions (local in time) for (A.3) is assured. The uniform estimates that we will obtain next will allow us to define such solutions for all time t.

Estimate I: Taking $k = z_m$, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|z_m\|^2 + (\nu_0 + \nu_1\|\nabla w\|^2)\|\nabla z_m\|^2 + 2\nu_1\left((\Delta w, z_m)\Delta w, z_m\right) \\ &+ \big((z_m \cdot \nabla^t) w, z_m\big) - ((w \cdot \nabla) z_m, z_m) = (w\chi_{\mathcal{O}}, z_m). \end{aligned} \tag{A.4}$$

Note that, by Lemma 6.1 in Chapter 1 of [Lio69],

$$\begin{array}{rcl} b : \mathrm{V} \times \mathrm{V} \times \mathrm{V} & \longrightarrow & \mathbb{R} \\ (u, v, w) & \longmapsto & b(u,v,w) = \displaystyle\sum_{i,k=1}^{N} \int_\Omega u_k \left(\frac{\partial v_i}{\partial x_k}\right) w_i \, dx \end{array}$$

defines a continuous trilinear form such that $b(u, v, v) = 0$. Thus, in (A.4), $((w \cdot \nabla) z_m, z_m) = b(w, z_m, z_m) = 0$.

Furthermore, since

$$\iint_Q (u \cdot \nabla) v \, w \, dxdt = \iint_Q (w \cdot \nabla^t) v \, u \, dxdt$$

then let's deal with $((z_m \cdot \nabla) w, z_m) = b(z_m, w, z_m)$ instead of $\big((z_m \cdot \nabla^t) w, z_m\big)$.

If $N = 2$, taking into account that $\|z_m\|_{L^4(\Omega)} \leq C\|z_m\|^{1/2}\|\nabla z_m\|^{1/2}$,

$$\begin{aligned} |b(z_m, w, z_m)| &\leq C\|z_m\|_{L^4}\|\nabla w\|\|z_m\|_{L^4} \\ &\leq C\|z_m\|^{1/2}\|\nabla z_m\|^{1/2}\|\nabla w\|\|z_m\|^{1/2}\|\nabla z_m\|^{1/2} \\ &\leq \frac{C}{2\nu_0}\|z_m\|^2\|\nabla w\|^2 + \frac{\nu_0}{2}\|\nabla z_m\|^2. \end{aligned}$$

If $N = 3$, since $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\|\nabla w\|_{L^3(\Omega)} \leq C\|\nabla w\|^{1/2}\|\Delta w\|^{1/2}$,

$$\begin{aligned} |b(z_m, w, z_m)| &\leq C\|z_m\|_{L^6(\Omega)}\|\nabla w\|_{L^3(\Omega)}\|z_m\| \\ &\leq C\|\nabla z_m\|\|\nabla w\|^{1/2}\|\Delta w\|^{1/2}\|z_m\| \\ &\leq C\|\nabla w\|\|\nabla z_m\|^2 + C\|\Delta w\|\|z_m\|^2. \end{aligned}$$

Also, for both $N = 2$ and $N = 3$,

$$\begin{aligned} |2\nu_1\left((\Delta w, z_m)\Delta w, z_m\right)| &\leq 2\nu_1\left(\int_\Omega \nabla w \nabla z_m \, dx\right)\left(\int_\Omega \Delta w z_m \, dx\right) \\ &\leq 2\nu_1\|\nabla w\|\|\nabla z_m\|\|\Delta w\|\|z_m\| \\ &\leq \frac{\nu_1}{2}\|\nabla w\|^2\|\nabla z_m\|^2 + C\|z_m\|^2\|\Delta w\|^2. \end{aligned}$$

Then, for $N = 2$ in (A.4),

$$\frac{1}{2}\frac{d}{dt}\|z_m\|^2 + (\nu_0 + \nu_1\|\nabla w\|^2)\|\nabla z_m\|^2 \leq \frac{C}{2\nu_0}\|\nabla w\|^2\|z_m\|^2 + \frac{\nu_0}{2}\|\nabla z_m\|^2$$
$$+\frac{\nu_1}{2}\|\nabla w\|^2\|\nabla z_m\|^2 + C\|z_m\|^2\|\Delta w\|^2 + C\|z_m\|^2 + C\int_{\mathcal{O}} |w|^2\, dx,$$

thus

$$\frac{1}{2}\frac{d}{dt}\|z_m\|_{\mathrm{H}}^2 + \frac{\nu_0}{2}\|z_m\|_{\mathrm{V}}^2 + \frac{\nu_1}{2}\|\nabla w\|^2\|z_m\|_{\mathrm{V}}^2$$
$$\leq C\int_{\mathcal{O}} |w|^2\, dx + \left(C + \frac{C}{2\nu_0}\|w\|_{\mathrm{V}}^2 + C\|\Delta w\|^2\right)\|z_m\|_{\mathrm{H}}^2.$$

Integrating from 0 to t and using the Gronwall's Lemma, since $w \in L^2(0,T; D(A)\cap \mathrm{V}) \cap C^0([0,T]; \mathrm{V})$ and

$$C + \frac{C}{2\nu_0}\|w\|_{\mathrm{v}}^2 + C\|\Delta w\|^2 \in L^1(0,T),$$

we get

$$|z_m|_{L^\infty(0,T;\mathrm{H})} + |z_m|_{L^2(0,T;\mathrm{V})} < +\infty. \tag{A.5}$$

And, if $N = 3$ in (A.4),

$$\frac{1}{2}\frac{d}{dt}\|z_m\|^2 + (\nu_0 + \nu_1\|\nabla w\|^2)\|\nabla z_m\|^2 \leq \frac{\nu_1}{2}\|\nabla w\|^2\|\nabla z_m\|^2 + C\|z_m\|^2\|\Delta w\|^2$$
$$+C\|\nabla w\|\|\nabla z_m\|^2 + C\|\Delta w\|\|z_m\|^2 + C\|z_m\|^2 + C\int_{\mathcal{O}} |w|^2\, dx.$$

Using the same arguments as above, we have

$$|z_m|_{L^\infty(0,T;\mathrm{H})} + |z_m|_{L^2(0,T;\mathrm{V})} < +\infty. \tag{A.6}$$

Estimate II: Noticing that $Az_m(t) \in V_m$ and taking $k = Az_m(t)$ in (A.3), we see that

$$\big((z_m \cdot \nabla^t)w, Az_m\big) = ((Az_m \cdot \nabla)w, z_m) = b(Az_m, w, z_m) = -b(Az_m, z_m, w),$$

since $b(u,v,w) = -b(u,w,v)$.

When $N = 2$, one has

$$\begin{aligned} |b(Az_m, z_m, w)| &\leq C\|Az_m\|\|\nabla z_m\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)} \\ &\leq C\|\nabla z_m\|^2\|\nabla w\|^4 + \frac{\nu_0}{6}\|Az_m\|^2; \end{aligned}$$

$$\begin{aligned} |\,((w\cdot\nabla)z_m, Az_m)\,| = |b(w,, z_m, Az_m)| &\leq C\|Az_m\|\|\nabla z_m\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)} \\ &\leq C\|\nabla z_m\|^2\|\nabla w\|^4 + \frac{\nu_0}{6}\|Az_m\|^2. \end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\nabla z_m\|^2 + \frac{1}{2}(\nu_0 + \nu_1\|\nabla w\|^2)\|\Delta z_m\|^2 \leq C\left(\|\nabla w\|^4 + \|\Delta w\|^2\right)\|\nabla z_m\|^2 + C\int_{\mathcal{O}} |w|^2\, dx. \tag{A.7}$$

Consequently, by Gronwall's Lemma,

$$|z_m|_{L^\infty(0,T;\mathrm{V})} + |z_m|_{L^2(0,T;D(A))} < +\infty. \tag{A.8}$$

Now, when $N = 3$, the nonlinear term can be bounded as follows:

$$\begin{aligned}
|b(Az_m, z_m, w)| &\leq C\|w\|_{L^6(\Omega)}\|\nabla z_m\|_{L^3(\Omega)}\|Az_m\| \\
&\leq C\|\nabla w\|\|\nabla z_m\|^{1/2}\|\Delta z_m\|^{1/2}\|\Delta z_m\| \\
&= C\|\nabla w\|\|\nabla z_m\|^{1/2}\|\Delta z_m\|^{3/2} \\
&\leq C\|\nabla w\|^4\|\nabla z_m\|^2 + \frac{\nu_0}{6}\|\Delta z_m\|^2;
\end{aligned}$$

$$\begin{aligned}
|\,((w\cdot\nabla)z_m, Az_m)\,| = |b(w, z_m, Az_m)| &\leq C\|w\|_{L^6(\Omega)}\|\nabla z_m\|_{L^3(\Omega)}\|Az_m\| \\
&\leq C\|\nabla w\|^4\|\nabla z_m\|^2 + \frac{\nu_0}{6}\|Az_m\|^2.
\end{aligned}$$

Thus, we have an estimate analogous to (A.7) and therefore

$$|z_m|_{L^\infty(0,T;\mathrm{V})} + |z_m|_{L^2(0,T;D(A))} < +\infty. \tag{A.9}$$

Estimate III: Taking $k = z_m'$ in (A.3),

$$\begin{aligned}
&\|z_m'\|^2 = (\nu_0 + \nu_1\|\nabla w\|^2)(\Delta z_m, z_m') + (w\chi_{\mathcal{O}}, z_m') \\
&-2\nu_1\,((\Delta w, z_m)\Delta w, z_m') - b(z_m', w, z_m) + b(w, z_m, z_m').
\end{aligned} \tag{A.10}$$

Note that the third term on the right-hand side can be bounded as follows:

$$|2\nu_1\,((\Delta w, z_m)\Delta w, z_m')\,| \leq C\|\nabla w\|^2\|\nabla z_m\|^2\|\Delta w\|^2 + \frac{1}{10}\|z_m'\|^2.$$

Therefore, when $N = 2$, using the continuous embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, we can compute that

$$\begin{aligned}
|b(z_m', w, z_m)| &\leq C\|z_m'\|\|\nabla w\|_{L^4(\Omega)}\|z_m\|_{L^4(\Omega)} \\
&\leq C\|\Delta w\|^2\|\nabla z_m\|^2 + \frac{1}{10}\|z_m'\|^2;
\end{aligned}$$

and

$$\begin{aligned}
|b(w, z_m, z_m')| &\leq C\|w\|_{L^4(\Omega)}\|\nabla z_m\|_{L^4(\Omega)}\|z_m'\| \\
&\leq C\|\nabla w\|\|\nabla z_m\|^{1/2}\|\Delta z_m\|^{1/2}\|z_m'\| \\
&\leq C\|\nabla w\|^2\|\nabla z_m\|\|\Delta z_m\| + \frac{1}{10}\|z_m'\|^2 \\
&\leq C\|\nabla w\|^2\|\nabla z_m\|^2 + C\|\nabla w\|^2\|\Delta z_m\|^2 + \frac{1}{10}\|z_m'\|^2;
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2}\|z_m'\|^2 \quad\leq\quad &\left[(\nu_0 + \nu_1\|\nabla w\|^2)^2 + C\|\nabla w\|^2\right]\|\Delta z_m\|^2 + C\left(\|\nabla w\|^2\|\Delta w\|^2\right. \\
&\left. + \|\Delta w\|^2 + \|\nabla w\|^2\right)\|\nabla z_m\|^2 + \int_{\mathcal{O}} |w|^2\,dx
\end{aligned}$$

which, integrating from 0 to t and using the regularity of w and Estimate II, we obtain

$$|z_m'|_{L^2(0,T;L^2(\Omega)^2)} < +\infty. \tag{A.11}$$

Now, if $N = 3$, note that

$$\begin{aligned}
|b(z_m', w, z_m)| &\leq C\|z_m\|_{L^6(\Omega)}\|\nabla w\|_{L^3(\Omega)}\|z_m'\|_{L^2(\Omega)} \\
&\leq C\|\nabla z_m\|\|\nabla w\|^{1/2}\|\Delta w\|^{1/2}\|z_m'\| \\
&\leq C\|\nabla w\|\|\Delta w\|\|\nabla z_m\|^2 + \frac{1}{10}\|z_m'\|^2 \\
&\leq C\|w\|_{D(A)}^2\|\nabla z_m\|^2 + \frac{1}{10}\|z_m'\|^2;
\end{aligned}$$

and

$$
\begin{aligned}
|b(w, z_m, z_m')| &\leq C\|w\|_{L^6(\Omega)}\|\nabla z_m\|_{L^3(\Omega)}\|z_m'\|_{L^2(\Omega)} \\
&\leq C\|\nabla w\|\|\nabla z_m\|^{1/2}\|\Delta z_m\|^{1/2}\|z_m'\| \\
&\leq C\|\nabla w\|^2\|\nabla z_m\|\|\Delta z_m\| + \frac{1}{10}\|z_m'\|^2 \\
&\leq C\|\nabla w\|^2\|z_m\|_{D(A)}^2 + \frac{1}{10}\|z_m'\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\|z_m'\|^2 \quad &\leq \quad (\nu_0 + \nu_1\|\nabla w\|^2)^2\|\Delta z_m\|^2 + C(\|\nabla w\|^2\|\Delta w\|^2 + \|w\|_{D(A)}^2)\|\nabla z_m\|^2 \\
&\quad + C\|\nabla w\|^2\|z_m\|_{D(A)}^2.
\end{aligned}
$$

Whence can we conclude, by regularity of w and Estimate II, that

$$
|z_m'|_{L^2(0,T;L^2(\Omega)^3)} < +\infty. \tag{A.12}
$$

The uniform bounds Estimate I - Estimate III, allow us to take limits in (A.3), at least for a subsequence, as $m \to \infty$. In other words, we have

$$
\begin{cases}
z_m \longrightarrow z & \text{weak in } L^2(0,T; D(A)\cap \mathrm{V}), \\
z_m' \longrightarrow z & \text{weak in } L^2(Q)^N,
\end{cases} \tag{A.13}
$$

with (z, q) soution of (A.2). Indeed, let's look at some terms:

For simplicity we will omit the summation in

$$
b(w, z_m, k) = \sum_{i,j=1}^{N} \int_\Omega w_j \left(\frac{\partial z_{mi}}{\partial x_j}\right) k_i \, dx.
$$

Then, for the first convergence of (A.13), for all $k \in L^2(0,T;\mathrm{V})$

$$
\begin{aligned}
\int_0^T b(w, z_m, k) \quad &= \quad \int_0^T \int_\Omega w_j \left(\frac{\partial z_{mi}}{\partial x_j}\right) k_i \, dx \\
&= \quad -\int_0^T \int_\Omega \frac{\partial}{\partial x_j}(w_j k_i) z_{mi} \, dx \, dt \\
&= \quad -\int_0^T \int_\Omega (\nabla \cdot w \, k_i \, z_{mi} + w_j \frac{\partial k_i}{\partial x_j} z_{mi}) \, dx \, dt \\
&= \quad -\int_0^T \int_\Omega w_j \frac{\partial k_i}{\partial x_j} z_{mi} \, dx \, dt.
\end{aligned}
$$

So

$$
\bullet \int_0^T b(w, z_m, k) \longrightarrow -\int_0^T \int_\Omega w_j \frac{\partial k_i}{\partial x_j} z_i \, dx \, dt \quad = \quad -\int_0^T b(w, k, z)\, dt = \int_0^T b(w, z, k)\, dt,
$$

as $m \to \infty$. Analogously, we have

$$
\begin{aligned}
\bullet \int_0^T \big((z_m \cdot \nabla^t) w, k\big)\, dt \quad &= \quad \int_0^T ((k \cdot \nabla) w, z_m)\, dt \\
&= \quad \int_0^T b(k, w, z_m)\, dt \longrightarrow \int_0^T b(k, w, z)\, dt = \int_0^T \big((z \cdot \nabla^t) w, k\big)\; dt,
\end{aligned}
$$

as $m \to \infty$.

Now, notice the following:

$$\begin{aligned}
\int_0^T ((\Delta w, z_m)\Delta w, k)\, dt &= \int_0^T ((\nabla w, \nabla z_m)\nabla w, \nabla k)\ dt \\
&\leq \int_0^T \left(\int_\Omega \nabla w \nabla z_m\right) \int_\Omega \nabla w \nabla k\, dx\, dt \\
&\leq \int_0^T \|\nabla w\| \|\nabla z_m\| \|\nabla w\| \|\nabla k\| dt \\
&\leq C \int_0^T \|\nabla z_m\| \|\nabla k\|\, dt \\
&\leq C |z_m|_{L^\infty(0,T;\mathrm{V})} |k|_{L^2(0,T;\mathrm{V})} < +\infty.
\end{aligned} \tag{A.14}$$

We know from the Aubin-Lions Lemma (see, Theorem 5.1 Chap. 1 in [Lio69]) that the following immersion is compact

$$\mathcal{W} = \{u;\ u \in L^2(0,T;H_0^1(\Omega)),\ u_t \in L^2(0,T;H^{-1}(\Omega))\} \hookrightarrow L^2(0,T;L^2(\Omega)).$$

Therefore, since $|z_m|_{L^2(0,T;H_0^1(\Omega)^N)} \leq C$ and $|z'_m|_{L^2(0,T;H^{-1}(\Omega)^N)} \leq C$ we have, at least one subsequence, that

$$\begin{cases} z_m \longrightarrow z & \text{strong in } L^2(Q)^N. \\ z_m \longrightarrow z & \text{a.e in } Q. \end{cases} \tag{A.15}$$

Fixing $w \in L^2(0,T;\mathrm{V})$, it is easily obtained that the map

$$\begin{aligned}
G_w : L^2(0,T;\mathrm{V}) &\longrightarrow \mathbb{R} \\
z &\longmapsto G_w(z) = \int_0^T (\nabla w, \nabla z)\, dt
\end{aligned}$$

is continuous. Then, $(\nabla w, \nabla z_m) \longrightarrow (\nabla w, \nabla z)$ a.e. in Q and consequently

$$((\Delta w, z_m)\Delta w, k) \longrightarrow ((\Delta w, z)\Delta w, k) \quad \text{a.e in } Q. \tag{A.16}$$

Thus, from (A.14) and (A.16) we can apply the Lions Lemma (see, Lemma 1.3, Chap. 1 in [Lio69]),

$$((\Delta w, z_m)\Delta w, k) \longrightarrow ((\Delta w, z)\Delta w, k) \quad \text{weak in } L^2(Q)^N.$$

The other terms follow in a standard way. This shows us that (z, q) satisfies

$$\begin{aligned}
&\int_0^T \left[(z', k) + (\nu_0 + \nu_1 \|\nabla w\|^2)(\nabla z, \nabla k) + 2\nu_1 ((\Delta w, z)\Delta w, k) + b(k, w, z)\right. \\
&\qquad \left. - b(w, z, k)\right] dt = \int_0^T (w\chi_{\mathcal{O}}, k)\, dt,\ \ \forall\ k = \sum_{j=1}^N h_j k^i,\ h_j \in L^2(0,T)
\end{aligned} \tag{A.17}$$

which is dense in $L^2(0,T;\mathrm{V})$, i.e., (A.17) holds for all $k \in L^2(0,T;\mathrm{V})$ and consequently (z,q) is strong solution of (A.2), with

$$z \in L^2(0,T;D(A) \cap \mathrm{V}) \cap C^0([0,T];\mathrm{V}), \quad \text{and} \quad z_t \in L^2(0,T;\mathrm{H}).$$

This ends the existence of a solution for (1.7).

The uniqueness of the strong solution to (1.7) can be proved in a standard way. Indeed, we know that $(1.7)_1$ has a unique solution. Suppose then that (A.2) has two solutions, i.e., (z^1, q^1) and (z^2, q^2) are strong solutions and we set $z = z^1 - z^2$. Therefore, z satisfies

$$\begin{cases} (z_t, k) - (\nu_0 + \nu_1\|\nabla w\|^2)(\Delta z, k) + 2\nu_1 \left((\Delta w, z)\Delta w, k\right) + \big((z^1 \cdot \nabla^t)w, k\big) \\ - \big((z^2 \cdot \nabla^t)w, k\big) - \big((w \cdot \nabla)z^1, k\big) + \big((w \cdot \nabla)z^2, k\big) = 0, \text{ in } \mathcal{D}'(0,T), \, \forall k \in \mathrm{V}. \end{cases}$$

Taking $k = -\Delta z$ in the previous equality, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla z\|^2 + (\nu_0 + \nu_1\|\nabla w\|^2)\|\Delta z\|^2 \quad &= \quad -2\nu_1 \left((\Delta w, z)\Delta w, -\Delta z\right) \\ &\quad + \left((w \cdot \nabla)z, -\Delta z\right) - \big((z \cdot \nabla^t)w, -\Delta z\big) \, . \end{aligned}$$

Arguing in the same way as for the existence, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla z\|^2 + \frac{1}{2}(\nu_0 + \nu_1\|\nabla w\|^2)\|\Delta z\|^2 \leq C \left(\|w\|^2 + \|\Delta w\|^2\right) \|\nabla z\|^2.$$

Integrating from 0 to t, using Gronwall's Lemma and the fact that $z(0) = 0$ in Ω, we conclude that

$$\|z\|_{\mathrm{V}} = 0 \text{ implying } z^1 = z^2 \text{ in } \Omega,$$

proving the uniqueness.

APPENDIX **B**

Appendix to Chapter 2

B.1 Existence and uniqueness of solution for (2.1)

The following theorem will show the existence and uniqueness of strong solutions for (2.1). In sequel, unless otherwise specified, the symbol C represents a generic positive constant.

Theorem B.1. *There exists* $R > 0$ *such that if,*

$$\|v\|^2_{L^2(\omega\times(0,T))^N} + \|v_0\|^2_{L^2(\omega\times(0,T))} + \|y^0\|_V + \|\theta^0\|_{W_0^{1,3/2}(\Omega)} < R.$$

then there exists a unique (y, p, θ) *strong solution of (2.1) in the class*

$$\begin{cases} y \in L^2(0,T; H^2(\Omega)^N \cap V) \cap C(0,T;V), \;\; y_t \in L^2(0,T;H) \\ \theta \in L^2(0,T; W^{2,3/2}(\Omega)), \;\; \theta_t \in L^2(0,T; L^{3/2}(\Omega)). \end{cases}$$

Proof. ***Existence:*** We will apply Faedo-Galerkin method to obtain the proof, be orthonormal eigenfunctions of the Stokes operator, i.e, the solutions to

$$\begin{cases} -\Delta u_m + \nabla p_m = \lambda_m u_m & \text{in} \quad \Omega, \\ u_m = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and $\{w_m\}_{m\in\mathrm{N}}$ the basis formed by the eigenfunctions of the Dirichlet Laplacian in Ω. Consider, for $m \in \mathbb{N}$, $U_m = span\{u_1, u_2, \ldots, u_m\}$ and $V_m = span\{w_1, w_2, \ldots, w_m\}$. Let us introduce the finite dimensional Galerkin approximations as follows: find y_m, θ_m, with $y_m(t) \in U_m$ and $\theta_m(t) \in V_m$ for all t, associated with the initial data (y^0, θ^0), such that

$$\begin{cases} (y_m', u) + ((\nu_0 + \nu_1\|\nabla y_m\|^2)\nabla y_m, \nabla u) + ((y_m\cdot\nabla)y_m, u) = (\nu_0\theta_m e_N, u) \\ +(v1_\omega, u), \forall\, u \in U_m, \\ (\theta_m', w) + ((\nu_0 + \nu_1\|\nabla y_m\|^2)\nabla\theta_m, \nabla w) + (y_m\cdot\nabla\theta_m, w) = (v_0 1_\omega, w) \\ +((\nu_0 + \nu_1\|\nabla y_m\|^2)Dy_m : \nabla y_m, w), \forall\, w \in V_m, \\ y_m(0) = y_m^0 \to y^0 \;\text{ in }\; V, \;\; \theta_m(0) = \theta_m^0 \to \theta^0 \;\text{ in }\; L^2(\Omega). \end{cases} \tag{B.1}$$

The classical ODE theory gives us the existence and uniqueness of a solution for (B.1), in local time. By means of the uniform estimates that we will obtain below, we will be able to define such solutions for all time t.

Estimate I: Multiplying the first row of (B.1) by λ_1, taking $u = -\Delta y_m(t)$ and $w = \theta_m(t)$ in the first and second equation of (B.1) and knowing that $\|.\|_{L^3} \leq C\|.\|^{1/2}\|.\|_{H^1}^{1/2}$, we have

- $\lambda_1(y_m', -\Delta y_m) = \dfrac{\lambda_1}{2}\dfrac{d}{dt}\|\nabla y_m\|^2;$
- $(\theta_m', \theta_m) = \dfrac{1}{2}\dfrac{d}{dt}\|\theta_m\|^2;$
- $((\nu_0 + \nu_1\|\nabla y_m\|^2)\nabla\theta_m, \nabla\theta_m) = (\nu_0 + \nu_1\|\nabla y_m\|^2)\|\nabla\theta_m\|^2;$
- $$\begin{aligned}\lambda_1((\nu_0 + \nu_1\|\nabla y_m\|^2)\nabla y_m, \nabla(-\Delta y_m)) &= \lambda_1(\nu_0 + \nu_1\|\nabla y_m\|^2)\|\Delta y_m\|^2\\ &= \lambda_1\nu_0\|\Delta y_m\|^2 + \lambda_1\nu_1\|\nabla y_m\|^2\|\Delta y_m\|^2;\end{aligned}$$
- $$\begin{aligned}\lambda_1|(\nu_0\theta_m e_N, -\Delta y_m)| &\leq \lambda_1\nu_0\|\theta_m e_N\|\|\Delta y_m\|\\ &\leq \frac{\lambda_1\nu_0}{2}\left(\|\theta_m\|^2 + \|\Delta y_m\|^2\right)\\ &\leq \frac{\nu_0}{2}\|\nabla\theta_m\|^2 + \frac{\lambda_1\nu_0}{2}\|\Delta y_m\|^2; \; (\|.\|^2 \leq 1/\lambda_1\|\nabla.\|^2)\end{aligned}$$
- $\lambda_1|(v1_\omega, -\Delta y_m)| \leq \lambda_1\|v1_\omega\|\|\Delta y_m\| \leq \hat{C}_1(\nu_0, \lambda_1)\|v\|^2_{L^2(\omega)^N} + \dfrac{\lambda_1\nu_0}{4}\|\Delta y_m\|^2; \; (\epsilon = \nu_0/4)$
- $$\begin{aligned}|(v_0 1_\omega, \theta_m)| &\leq \|v_0 1_\omega\|\|\theta_m\| \leq C\|v_0 1_\omega\|\|\nabla\theta_m\|\\ &\leq \hat{C}_2(\Omega, \nu_0)\|v_0\|^2_{L^2(\omega)} + \frac{\nu_0}{8}\|\nabla\theta_m\|^2; \; ab < \epsilon a^2 + (1/4\epsilon)b^2; \; (\epsilon = \nu_0/8)\end{aligned}$$
- $$\begin{aligned}\lambda_1|((y_m\cdot\nabla)y_m, -\Delta y_m)| &\leq \lambda_1 C\|y_m\|_{L_6}\|\nabla y_m\|_{L^3}\|\Delta y_m\|\\ &\leq \lambda_1 C\|\nabla y_m\|\|\nabla y_m\|^{1/2}\|\Delta y_m\|^{1/2}\|\Delta y_m\|\\ &\leq \lambda_1 C\|\nabla y_m\|^{3/2}\|\Delta y_m\|^{3/2}(\|\nabla y_m\| \leq C\|\Delta y_m\|)\\ &\leq \hat{C}_3(\Omega, \lambda_1)\|\nabla y_m\|\|\Delta y_m\|^2;\end{aligned}$$

and

- $\left|\displaystyle\int_\Omega (\nu_0 + \nu_1\|\nabla y_m\|^2)|\nabla y_m|^2|\theta_m|dx\right|$

$$\begin{aligned}&\leq (\nu_0 + \nu_1\|\nabla y_m\|^2)\left(\int_\Omega (|\nabla y_m|^2)^{3/2}dx\right)^{2/3}\left(\int_\Omega |\theta_m|^3 dx\right)^{1/3} \quad (2/3 + 1/3 = 1)\\ &= (\nu_0 + \nu_1\|\nabla y_m\|^2)\|\nabla y_m\|^2_{L^3}\|\theta_m\|_{L^3}\\ &\leq C(\nu_0 + \nu_1\|\nabla y_m\|^2)\|\nabla y_m\|\|\Delta y_m\|\|\theta_m\|^{1/2}\|\nabla\theta_m\|^{1/2}(\|\theta_m\| \leq 1/\sqrt{\lambda_1}\|\nabla\theta_m\|)\\ &\leq C(\nu_0 + \nu_1\|\nabla y_m\|^2)\|\nabla y_m\|\|\Delta y_m\| \; \|\nabla\theta_m\|\\ &\leq C\nu_0\|\nabla y_m\|\|\Delta y_m\| \; \|\nabla\theta_m\| + C\nu_1\|\nabla y_m\|^2\|\nabla y_m\|\|\Delta y_m\| \; \|\nabla\theta_m\|\\ &\leq \hat{C}_4(\Omega, \nu_0, \lambda_1)\|\nabla y_m\|^2\|\Delta y_m\|^2 + \frac{\nu_0}{16}\|\nabla\theta_m\|^2 + \hat{C}_5(\Omega, \nu_0, \nu_1, \lambda_1)\|\nabla y_m\|^6\|\Delta y_m\|^2 + \frac{\nu_0}{16}\|\nabla\theta_m\|^2.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\lambda_1\|\nabla y_m\|^2 + \|\theta_m\|^2) + \frac{\nu_0}{4}\|\nabla\theta_m\|^2 + \nu_1\|\nabla y_m\|^2\|\nabla\theta_m\|^2\\ &+\lambda_1\nu_1\|\nabla y_m\|^2\|\Delta y_m\|^2 + \frac{\lambda_1\nu_0}{8}\|\Delta y_m\|^2 + \left[\frac{\lambda_1\nu_0}{8} - \hat{C}_3(\Omega, \lambda_1)\|\nabla y_m\|\right.\\ &\left.-\hat{C}_4(\Omega, \nu_0, \lambda_1)\|\nabla y_m\|^2 - \hat{C}_5(\Omega, \nu_0, \nu_1, \lambda_1)\|\nabla y_m\|^6\right]\|\Delta y_m\|^2\\ &\leq \hat{C}_2(\Omega, \nu_0)\|v_0\|^2_{L^2(\omega)} + \hat{C}_1(\nu_0, \lambda_1)\|v\|^2_{L^2(\omega)^N},\end{aligned} \tag{B.2}$$

For simplicity of notation, we will omit the dependencies of the constants already known. Hence, the following statement is valid:

Affirmation 1.

$$\bar{A}(t) = \hat{C}_3\|\nabla y_m(t)\| + \hat{C}_4\|\nabla y_m(t)\|^2 + \hat{C}_5\|\nabla y_m(t)\|^6 < \frac{\lambda_1\nu_0}{8}, \ \forall\, t \in [0, T_m]. \tag{B.3}$$

Indeed, assuming by contradiction that (B.3) is false then there exist t_{1m} such that

$$\bar{A}(t) < \frac{\lambda_1\nu_0}{8}, \ \forall\ 0 \leq t < t_{1m}$$

and

$$\bar{A}(t_{1m}) = \frac{\lambda_1\nu_0}{8}. \tag{B.4}$$

By hypothesis, there is $\tilde{\rho}_0 > 0$ such that

$$\|v\|^2_{L^2(\omega\times(0,T))^N} + \|v_0\|^2_{L^2(\omega\times(0,T))} + \|y^0\|_V + \|\theta^0\|_{W_0^{1,3/2}(\Omega)} < \tilde{\rho}_0.$$

Then, we have

$$\begin{cases} \hat{C}_3\|\nabla y^0\| + \hat{C}_4\|\nabla y^0\|^2 + \hat{C}_5\|\nabla y^0\|^6 < \dfrac{\lambda_1\nu_0}{8}, \\ \hat{C}_1\|v\|^2_{L^2(\omega\times(0,T))^N} + \hat{C}_2\|v_0\|^2_{L^2(\omega\times(0,T))} + \dfrac{1}{2}(\lambda_1\|\nabla y^0\|^2 + \|\theta^0\|^2) \\ < \min\left\{\dfrac{{\lambda_1}^2\nu_0}{48\hat{C}_4}, {\lambda_1}^3\left(\dfrac{\nu_0}{24\sqrt{2}\hat{C}_3}\right)^2, {\lambda_1}^{4/3}\left(\dfrac{\nu_0}{192\hat{C}_5}\right)^{1/3}\right\}. \end{cases} \tag{B.5}$$

Integrating (B.2) from 0 to t_{1m}, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\lambda_1\|\nabla y_m(t_{1m})\|^2 + \|\theta_m(t_{1m})\|^2\right) + \frac{\nu_0}{4}\int_0^{t_{1m}} \|\nabla\theta_m\|^2 dt + \frac{\lambda_1\nu_0}{8}\int_0^{t_{1m}} \|\Delta y_m\|^2 dt \\ &\leq \hat{C}_1\|v\|^2_{L^2(\omega\times(0,T))^N} + \hat{C}_2\|v_0\|^2_{L^2(\omega\times(0,T))} + \frac{1}{2}(\lambda_1\|\nabla y^0\|^2 + \|\theta^0\|^2). \end{aligned} \tag{B.6}$$

Then from (B.5) and (B.6) we arrive at $\bar{A}(t_{1m}) < \lambda_1\nu_0/8$, which contradicts (B.4). Therefore, (B.3) holds and we obtain that

$$\begin{aligned} &\frac{1}{2}\left(\lambda_1\|\nabla y_m(t)\|^2 + \|\theta_m(t)\|^2\right) + \frac{\nu_0}{4}\int_0^t \|\nabla\theta_m(s)\|^2 ds + \frac{\lambda_1\nu_0}{8}\int_0^t \|\Delta y_m(s)\|^2 ds \\ &\leq \hat{C}_1\|v\|^2_{L^2(\omega\times(0,T))^N} + \hat{C}_2\|v_0\|^2_{L^2(\omega\times(0,T))} + \frac{1}{2}(\lambda_1\|\nabla y^0\|^2 + \|\theta^0\|^2)\ \forall\, t \in [0, T_m]. \end{aligned} \tag{B.7}$$

As the term on the right side of (B.7) is independent of m, we can extend the solution (y_m, θ_m) to the entire interval $[0, T]$ and in the same way we can estimate (B.7) for $t \in [0, T]$. More precisely,

$$\begin{aligned} &\|y_m\|^2_{L^\infty(0,T;V)} + \|\theta_m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_m\|^2_{L^2(0,T;H^2(\Omega)^N\cap V)} + \|\theta_m\|^2_{L^2(0,T;H_0^1(\Omega))} \\ &\leq C(\|y^0\|^2_V + \|\theta^0\|^2 + \|v\|^2_{L^2(\omega\times(0,T))^N} + \|v_0\|^2_{L^2(\omega\times(0,T))}). \end{aligned} \tag{B.8}$$

Estimate II: Taking $u = y'_m$ in the first equation of (B.1), we obtain after some calculations

$$\begin{aligned} &\frac{1}{2}\int_0^t \|y_{t,m}(s)\|^2 ds + \frac{\nu_0}{2}\|\nabla y_m(t)\|^2 + \frac{\nu_1}{4}\|\nabla y_m(t)\|^4 \\ &\leq C\int_0^t \|\Delta y_m(s)\|^2\|\nabla y_m(s)\|^2 ds + C\int_0^t \|\theta_m(s)\|^2 ds + \|v\|^2_{L^2(\omega\times(0,T))^N} \\ &\quad + \frac{\nu_0}{2}\|\nabla y^0\|^2 + \frac{\nu_1}{4}\|\nabla y^0\|^4, \end{aligned}$$

therefore, using Estimate I and the Gronwall's Lemma, we arrive at

$$
\begin{aligned}
&\|y_{t,m}\|^2_{L^2(0,T;H)} + \|y_m\|^2_{L^\infty(0,T;V)} \\
&\leq C(\|y^0\|^2_V + \|y^0\|^4_V + \|\theta^0\|^2 + \|v\|^2_{L^2(\omega\times(0,T))^N} + \|v_0\|^2_{L^2(\omega\times(0,T))}).
\end{aligned}
\tag{B.9}
$$

Estimate III: Since the θ_m are the eigenfunctions of $-\Delta$ in $H^1_0(\Omega)$, we have from Estimate I,

$$
\|\theta_{t,m}\|^2_{L^2(0,T;H^{-1}(\Omega))} \leq C(\|y^0\|^2_V + \|\theta^0\|^2 + \|v\|^2_{L^2(\omega\times(0,T))^N} + \|v_0\|^2_{L^2(\omega\times(0,T))}). \tag{B.10}
$$

From estimates (B.8), (B.9) and (B.10) we can extract subsequences of $\{y_m\}$ and $\{\theta_m\}$ denoted equal, so that taking the limit $m \to \infty$ in the equation (B.1), y_m and θ_m converge to a solution (weak) of (2.1). Indeed, to obtain the a.e. convergence of nonlocal terms, just use the fact that the sequence y_m is pre-compact in $L^2(0,T;V)$.

This solution must satisfy

$$
\begin{cases}
y \in L^2(0,T;H^2(\Omega)^N \cap V) \cap C(0,T;V), \ y_t \in L^2(0,T;H) \\
\theta \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \ \theta_t \in L^2(0,T;H^{-1}(\Omega))
\end{cases}
$$

where y strong solution in first equation in (2.1) and θ weak solution in second equation in (2.1).

Furthermore, since $(\nu_0 + \nu_1\|\nabla y\|^2)Dy : \nabla y + v_0 1_\omega \in L^2(0,T;L^{3/2}(\Omega))$(see (2.47)) and $\theta^0 \in W^{1,3/2}_0(\Omega)$, from $L^p - L^q$ regularity for parabolic equation (see, [DHP07]), we have θ solution of

$$
\begin{cases}
\theta_t - \nabla\cdot(\nu(\nabla y)D\theta) + y\cdot\nabla\theta = v_0 1_\omega + \nu(\nabla y)Dy : \nabla y & \text{in} \quad Q, \\
\theta(x,t) = 0 & \text{on} \quad \Sigma, \\
\theta(x,0) = \theta^0(x) & \text{in} \quad \Omega.
\end{cases}
\tag{B.11}
$$

in class $\theta \in L^2(0,T;W^{2,3/2}(\Omega)), \ \theta_t \in L^2(0,T;L^{3/2}(\Omega))$.

This yields (2.5).

Uniqueness: Let $(u,q,w) = (y^1,p^1,\theta^1) - (y^2,p^2,\theta^2)$, where (y^1,p^1,θ^1) and (y^2,p^2,θ^2) are solutions of problem (2.1). Then, we got

$$
\begin{cases}
u_t - \nu_0\Delta u - \nu_1\|\nabla y^1\|^2\Delta y^1 + \nu_1\|\nabla y^2\|^2\Delta y^2 + (u\cdot\nabla)y^1 + (y^2\cdot\nabla)u & \\
+\nabla q = \nu_0 w e_N, \ \nabla\cdot u = 0 & \text{in} \quad Q, \\
w_t - \nu_0\Delta w - \nu_1(\|\nabla y^1\|^2)\Delta\theta^1 - \|\nabla y^2\|^2\Delta\theta^2) + u\cdot\nabla\theta^1 + y^2\cdot\nabla w & \\
= \nu_0 Dy^1 : \nabla y^1 - \nu_0 Dy^2 : \nabla y^2 + \nu_1\|\nabla y^1\|^2 Dy^1 : \nabla y^1 - \nu_1\|\nabla y^2\|^2 Dy^2 : \nabla y^2 & \text{in} \quad Q, \\
u(x,t) = 0, w(x,t) = 0 & \text{on} \quad \Sigma, \\
u(x,0) = 0, \ w(x,0) = 0 & \text{in} \quad \Omega.
\end{cases}
$$

Which we can rewrite as follows

$$
\begin{cases}
u_t - \nu_0\Delta u - \nu_1[\|\nabla y^1\|^2\Delta u + (\|\nabla y^1\| + \|\nabla y^2\|)(\|\nabla y^1\| - \|\nabla y^2\|)\Delta y^2] & \\
+(u\cdot\nabla y^1) + (y^2\cdot\nabla)u + \nabla q = \nu_0 w e_N, \ \nabla\cdot u = 0 & \text{in} \quad Q, \\
w_t - \nu_0\Delta w - \nu_1[\|\nabla y^1\|^2\Delta w + (\|\nabla y^1\| + \|\nabla y^2\|)(\|\nabla y^1\| - \|\nabla y^2\|)\Delta\theta^2] & \\
+u\cdot\nabla\theta^1 + y^2\cdot\nabla w = \nu_0(Du : \nabla y^1 + Dy^2 : \nabla u) + \nu_1[\|\nabla y^1\|^2 Du : \nabla y^1 & \\
+(\|\nabla y^1\| + \|\nabla y^2\|)(\|\nabla y^1\| - \|\nabla y^2\|)Dy^2 : \nabla u] & \text{in} \quad Q, \\
u(x,t) = 0, w(x,t) = 0 & \text{on} \quad \Sigma, \\
u(x,0) = 0, \ w(x,0) = 0 & \text{in} \quad \Omega.
\end{cases}
\tag{B.12}
$$

Multiplying by $-\Delta u$ and w in first and second line of (B.12), respectively, and integrating in Ω, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2+\|w\|^2)+(\nu_0+\nu_1\|\nabla y^1\|^2)(\|\Delta u\|^2+\|\nabla w\|^2)\\
&=\int\limits_\Omega(\|\nabla y^1\|+\|\nabla y^2\|)(\|\nabla y^1\|-\|\nabla y^2\|)\Delta y^2\Delta u+\int\limits_\Omega[(u\cdot\nabla y^1)\Delta u\\
&+(y^2\cdot\nabla u)\Delta u]-\int_\Omega we_3\Delta u+\int\limits_\Omega(\|\nabla y^1\|+\|\nabla y^2\|)(\|\nabla y^1\|-\|\nabla y^2\|)\nabla\theta^2\nabla w\\
&-\int\limits_\Omega u\cdot\nabla\theta^1 w+\int\limits_\Omega\nu_0[(Du:\nabla y^1)w+(Dy^2:\nabla u)w]\\
&+\int\limits_\Omega(\|\nabla y^1\|+\|\nabla y^2\|)(\|\nabla y^1\|-\|\nabla y^2\|)Dy^2:\nabla u\,w=\sum_{i=1}^{7}L_i.
\end{aligned}
$$

Notice that,

$$
\begin{aligned}
|L_1|&\leq\int\limits_\Omega C(\|\nabla y^1\|+\|\nabla y^2\|)\|\nabla u\||\Delta y^2||\Delta u\|\\
&\leq C(\|\nabla y^1\|+\|\nabla y^2\|)^2\|\nabla u\|^2\|\Delta y^2\|^2+\frac{1}{\epsilon}\|\Delta u\|^2;
\end{aligned}
$$

Since $N\leq 3$, by continuous embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, $H^1(\Omega)\hookrightarrow L^4(\Omega)$ and inequality $\|f\|_{H^2(\Omega)}\leq C\|\Delta f\|$ to any $f\in H^2(\Omega)\cap H^1_0(\Omega)$, we achieved

$$
\begin{aligned}
|L_2|&\leq C\|u\|_{L^4(\Omega)}\|\nabla y^1\|_{L^4(\Omega)}\|\Delta u\|+C\|y^2\|_\infty\|\nabla u\|\|\Delta u\|\\
&\leq C\|\nabla u\|\|\Delta y^1\|\|\Delta u\|+C\|\Delta y^2\|\|\nabla u\|\|\Delta u\|\\
&\leq\frac{1}{\epsilon}\|\Delta u\|^2+C(\|\Delta y^1\|^2+\|\Delta y^2\|^2)\|\nabla u\|^2;
\end{aligned}
$$

$$
|L_3|\leq\frac{1}{\epsilon}\|\Delta u\|^2+C\|w\|^2;
$$

$$
|L_4|\leq C(\|\nabla y^1\|+\|\nabla y^2\|)^2\|\nabla u\|^2+C\|\nabla\theta^2\|^2\|\nabla u\|^2;
$$

$$
\begin{aligned}
|L_5|&\leq C\|u\|_\infty\|\nabla\theta^1\|\|w\|\\
&\leq C\|\Delta u\|\|\nabla\theta^1\|\|w\|\\
&\leq\frac{1}{\epsilon}\|\Delta u\|^2+C\|\nabla\theta^1\|^2\|w\|^2;
\end{aligned}
$$

$$
\begin{aligned}
|L_6|&\leq C\|\nabla u\|_{L^4(\Omega)}\|\nabla y^1\|_{L^4(\Omega)}\|w\|+C\|\nabla y^2\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|w\|\\
&\leq\frac{1}{\epsilon}\|\Delta u\|^2+C(\|\Delta y^1\|^2+\|\Delta y^2\|^2)\|w\|^2;
\end{aligned}
$$

Finally,

$$
\begin{aligned}
|L_7|&\leq C\,(\|\nabla y^1\|+\|\nabla y^2\|)\|\nabla y^1\|\|\nabla y^2\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|w\|\\
&\quad+C\,(\|\nabla y^1\|+\|\nabla y^2\|)\|\nabla y^2\|\|\nabla y^2\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|w\|\\
&\leq\frac{1}{\epsilon}\|\Delta u\|^2+C\left[(\|\nabla y^1\|+\|\nabla y^2\|)^2(\|\nabla y^1\|^2+\|\nabla y^2\|^2)\|\Delta y^2\|^2\right]\|w\|^2.
\end{aligned}
$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2+\|w\|^2)+(\nu_0+\nu_1\|\nabla y^1\|^2)(\|\Delta u\|^2+\|\nabla w\|^2)\\
&\leq \frac{6}{\epsilon}\|\Delta u\|^2+C\left[(\|\nabla y^1\|+\|\nabla y^2\|)^2(\|\Delta y^2\|^2+1)+\|\nabla\theta^2\|^2\right.\\
&\left.+\|\Delta y^1\|^2+\|\Delta y^2\|^2\right]\|\nabla u\|^2+C\left[1+\|\nabla\theta^1\|^2+\|\Delta y^1\|^2+\|\Delta y^2\|^2\right.\\
&\left.+(\|\nabla y^1\|+\|\nabla y^2\|)^2(\|\nabla y^1\|^2+\|\nabla y^2\|^2)\|\Delta y^2\|^2\right]\|w\|^2\\
&=\frac{6}{\epsilon}\|\Delta u\|^2+CL_8\|\nabla u\|^2+CL_9\|w\|^2,
\end{aligned} \tag{B.13}$$

where

$$L_8=(\|\nabla y^1\|+\|\nabla y^2\|)^2(\|\Delta y^2\|^2+1)+\|\nabla\theta^2\|^2+\|\Delta y^1\|^2+\|\Delta y^2\|^2$$

and

$$L_9=1+\|\nabla\theta^1\|^2+\|\Delta y^1\|^2+\|\Delta y^2\|^2+(\|\nabla y^1\|+\|\nabla y^2\|)^2(\|\nabla y^1\|^2+\|\nabla y^2\|^2)\|\Delta y^2\|^2.$$

Taking $\epsilon=12/\nu_0$ and integrating (B.13) from 0 to t,

$$\begin{aligned}
&\|\nabla u(t)\|^2+\|w(t)\|^2+\int_0^t(\nu_0+\nu_1\|\nabla y^1\|^2)(\|\Delta u(s)\|^2+\|\nabla w(s)\|^2)ds\\
&\leq 0+\int_0^t C(L_8+L_9)(\|\nabla u(s)\|^2+\|w(s)\|^2)ds.
\end{aligned}$$

Hence, applying Gronwall's Lemma, we obtain $u(t)=0$ and $w(t)=0$, for all $t\in[0,T]$. Consequently, $(y^1,p^1,\theta^1)=(y^2,p^2,\theta^2)$ confirming the uniqueness of the solution. □

Proof of Lemma 2.12 Continuing as in the proof of the existence of solution, we will make an additional estimate for the temperature term. More accurately, taking $w=-\Delta\theta_m(t)$ in the second equation of (B.1), using the inequalities $\|.\|_{L^3}\leq C\|.\|^{1/2}\|.\|_{H^1}^{1/2}$, $\|.\|_{L^4}\leq C\|.\|^{1/2}\|.\|_{H^1}^{1/2}$ and (B.2) we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\lambda_1\|\nabla y_m\|^2+\|\theta_m\|^2+\|\nabla\theta_m\|^2)+\frac{\nu_0}{4}(\|\nabla\theta_m\|^2+\|\Delta\theta_m\|^2)\\
&+\nu_1\|\nabla y_m\|^2\|\Delta\theta_m\|^2+\lambda_1\nu_1\|\nabla y_m\|^2\|\Delta y_m\|^2+\frac{\lambda_1\nu_0}{8}\|\Delta y_m\|^2\\
&+\left(\nu_1-\hat{C}_6\|\nabla y_m\|^2\right)\|\nabla y_m\|^2\|\nabla\theta_m\|^2+\left[\frac{\lambda_1\nu_0}{8}-\hat{C}_3\|\nabla y_m\|\right.\\
&\left.-\hat{C}_9\|\nabla y_m\|^2-\hat{C}_{10}\|\nabla y_m\|^6\right]\|\Delta y_m\|^2\leq 0,
\end{aligned} \tag{B.14}$$

where $\hat{C}_9=\max\{\hat{C}_4,\hat{C}_7\}$ and $\hat{C}_{10}=\max\{\hat{C}_5,\hat{C}_8\}$ with $\hat{C}_7$ and $\hat{C}_8$ constants coming from the estimate of $-\Delta\theta_m(t)$. Remembering that $\hat{C}_i$, $i=\{1,\ldots,10\}$ are constants that may depend on Ω,ν_0,ν_1 and λ_1.

In a similar way to what was done for (B.2) we can obtain that all terms on the left side of (B.14) are positive. Just as was done for (B.3), we can obtain that the last term on the left side of (B.14) is positive. Therefore, we just need to prove the following statement:

Affirmation 2.

$$B(t)=\hat{C}_6\|\nabla y_m(t)\|^2<\nu_1,\ \forall\, t\in[0,T_m]. \tag{B.15}$$

Indeed, suppose by contradiction that (B.15) is false then there exist t_m^* such that

$$B(t) < \nu_1, \ \forall\ 0 \leq t < t_m^*$$

and

$$B(t_m^*) = \nu_1. \tag{B.16}$$

Since there is $r > 0$ such that

$$\|y^0\|_V + \|\theta^0\|_{H_0^1(\Omega)} < r$$

we have,

$$\|\nabla y^0\|^2 < \frac{\lambda_1 \nu_1}{2\hat{C}_6}. \tag{B.17}$$

Integrating (B.14) from 0 to t_m^*, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\lambda_1\|\nabla y_m(t_m^*)\|^2 + \|\theta_m(t_m^*)\|^2 + \|\nabla\theta_m(t_m^*)\|^2\right) + \frac{\nu_0}{4}\int_0^{t_m^*}\|\nabla\theta_m\|^2 dt \\
&+\frac{\lambda_1\nu_0}{8}\int_0^{t_m^*}\|\Delta y_m\|^2 dt + \frac{\nu_0}{4}\int_0^{t_m^*}\|\Delta\theta_m\|^2 dt \\
&\leq \frac{1}{2}(\lambda_1\|\nabla y^0\|^2 + \|\theta^0\|^2 + \|\nabla\theta^0\|^2)
\end{aligned} \tag{B.18}$$

and consequently $B(t_m^*) < \nu_1$, contradicting (B.16).

Therefore, (B.15) holds and we conclude that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\lambda_1\|\nabla y_m(t)\|^2 + \|\theta_m(t)\|^2 + \|\nabla\theta_m(t)\|^2) + \frac{\nu_0}{4}\|\nabla\theta_m(t)\|^2 \\
+\frac{\lambda_1\nu_0}{8}\|\Delta y_m(t)\|^2 \leq 0, \forall t \in [0,T]
\end{aligned}$$

which we will rewrite in the form

$$\frac{d}{dt}\Phi_m(t) + \frac{\nu_0}{2}\|\nabla\theta_m\|^2 + \frac{\lambda_1\nu_0}{4}\|\Delta y_m\|^2 \leq 0, \tag{B.19}$$

where $\Phi_m(t) = \lambda_1\|\nabla y_m\|^2 + \|\theta_m\|^2 + \|\nabla\theta_m\|^2$. Note that,

$$\Phi_m(t) \leq \frac{\tilde{C}_1\nu_0}{2}2\|\nabla\theta_m\|^2 + \frac{\tilde{C}_2\lambda_1\nu_0}{4}\|\Delta y_m\|^2 \leq \hat{C}\left(\frac{\nu_0}{2}\|\nabla\theta_m\|^2 + \frac{\lambda_1\nu_0}{4}\|\Delta y_m\|^2\right)$$

where $\tilde{C}_1, \tilde{C}_2 > 0$ and $\hat{C} = \max\{\tilde{C}_1, \tilde{C}_2\}$. Thus, from (B.19),

$$\frac{d}{dt}\Phi_m(t) + \frac{1}{\hat{C}}\Phi_m(t) \leq 0$$

which results in

$$\Phi_m(t) \leq e^{(-1/\hat{C})t}\Phi_m(0).$$

Then

$$\begin{aligned}
\lambda_1\|\nabla y(t,.)\|^2 + \|\theta(t,.)\|^2 + \|\nabla\theta(t,.)\|^2 &\leq \liminf_{m\to\infty}\Phi_m(t) \\
&\leq e^{(-1/\hat{C})t}(\lambda_1\|\nabla y^0\|^2 + \|\theta^0\|^2 + \|\nabla\theta^0\|^2),
\end{aligned}$$

and consequently one can deduce (2.71). □

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