Sidewise control of 1-d waves

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Abstract We analyze the sidewise controllability for the variable coefficients one-dimensional wave equation. The control is acting on one extreme of the string with the aim that the solution tracks a given path at the other free end. This sidewise control problem is also often referred to as nodal profile or tracking control. First, the problem is reformulated as a dual observability property for the corresponding adjoint system. Using sidewise energy propagation arguments the sidewise observability is shown to hold, in a sufficiently large time, in the class of BV-coefficients. We also present a number of open problems and perspectives for further research.

Key words 1-d wave equations, sidewise controllability, nodal profile control, BV-coefficients, sidewise energy estimates.

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1 Introduction

Consider the following variable coefficient controlled 1-d wave equation:

$$\begin{cases} \rho(x)y_{tt} - (a(x)y_x)_x = 0, & 0 < x < L,\ 0 < t < T, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & 0 < x < L, \\ y(0,t) = u(t),\ y(L,t) = 0, & 0 < t < T. \end{cases} \tag{1.1}$$

In (1.1), $0 < T < \infty$ stands for the length of the time-horizon, L is the length of the string where waves propagate, $y = y(x,t)$ is the state and $u = u(t)$ is a control that acts on the system through the extreme $x = 0$.

We assume that the coefficients ρ and a are in BV and to be uniformly bounded above and below by positive constants, i. e.

$$0 < \rho_0 \leq \rho(x) \leq \rho_1, \quad 0 < a_0 \leq a(x) \leq a_1 \ \text{ a.e. in } \ (0,L) \tag{1.2}$$

and

$$\rho, a \in BV(0,L). \tag{1.3}$$

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The main goal of this paper is to analyze the sidewise boundary controllability of (1.1). More precisely, we want to solve the following problem: *Given a time-horizon $T > 0$, initial data $y_0(x)$, $y_1(x)$ and a target $p(t)$ for the flux at $x = L$, to find $u(t)$ such that the corresponding solution of the system* (1.1) *satisfies*

$$y_x(L,t) = p(t), \quad t \geq 0 \tag{1.4}$$

in a time-subinterval of $[0,T]$ to be made precise and under suitable conditions on T, according to the velocity of propagation of waves.

In other words, the string being fixed at the right extreme $x = L$ and the control $u = u(t)$ acting on the left-extreme $x = 0$, given a profile $p = p(t)$, we want to choose the control so that the tension of the string at $x = L$, namely $y_x(L,t)$, tracks the profile $p = p(t)$.

This is a non-standard controllability problem since, most often, controllability refers to the possibility of steering the solution to a target in the final time $t = T$. But here the aim is rather to assure that a given trace is achieved on the boundary.

There is an extensive literature on the controllability of wave-like equations (see, for instance, [10, 11, 17, 21, 24]). Most techniques to handle these problems rely on the dual observability problem for the adjoint uncontrolled wave equation, that is then addressed using methods such as multipliers, microlocal analysis or Carleman inequalities, among others. The one-dimensional case is particularly well understood and sharp results can be obtained using sidewise energy estimates in the class of BV-coefficients of bounded variation, [6]. In that context, BV is the minimal requirement on the regularity of the coefficients since counterexamples can be built in the class of Hölder continuous ones ([3]). In case coefficients are slightly less regular than BV one may obtain weaker controllability properties in the sense that initial data to be controlled need to be more smooth than expected according to the BV frame, [7].

Sidewise control problems have also been previously investigated. For instance, motivated by applications on gas-flow on networks, Gugat et al. [8] proposed the so-called nodal profile control problem, the goal being to assure that the state fits a given profile on one or some nodes of the network, after a waiting time, by means of boundary controls. In [12], [20], [13] and [22] this analysis was extended to 1-D quasilinear hyperbolic systems by a constructive method employing the method of characteristics.

In this paper we address this sidewise or nodal-controllability problem in the context of the 1-D wave equation above. Our two main contributions are as follows:

- The first one is of a methodological nature. We introduce the dual version of this problem, which leads to a non-standard observability inequality for the adjoint wave equation, involving a non-homogeneous boundary condition at $x = L$, that needs to be estimated out of measurements done at $x = 0$. This is inspired on the classical duality approach to exact controllability as introduced in [10] (see also [17]).

  As we shall see, this duality method can be applied also in several space dimensions and for other models, such as heat or Schrödinger equations, also when defined on networks. Nodal controllability is therefore systematically reduced to proving suitable observability estimates.

- In the particular case under consideration (1.1), we prove these observability inequalities using sidewise energy estimates.

The combination of these two contributions allows for a rather complete understanding of the problem under consideration for 1-D waves. But the development of techniques allowing to handle the corresponding observability inequalities in the multi-dimensional context seems to be a challenging problem.

The parabolic companion of this analysis has been recently developed in [1] where the nodal controllability of the 1-D heat equation is solved using the same duality principle and flatness methods, [16]. In the parabolic setting, contrarily to the results on this paper, results holds in an arbitrary time horizon. But, in the context (1.1), because of the finite velocity of propagation, the sidewise control property can only hold when the time horizon is large enough.

This paper is organized as follows. In Section 2 we present the dual sidewise observability inequality problem for the adjoint system. We also present the main sidewise controllability result and show what the dual equivalent in terms of sidewise observability is. In Section 4 we present the main sidewise controllability result. Section 4 is devoted to prove the relevant sidewise controllability inequality. We conclude in Section 5 with some conclusions and some open problems for future research.

2 The dual sidewise observability problem

Let us consider system (1.1) with coefficients satisfying the assumptions (1.2) and (1.3).

For any given (y_0, y_1) with $y_0 \in L^2(0,L)$ and $\rho y_1 \in H^{-1}(0,L)$ and any $u \in L^2(0,T)$, system (1.1) admits a unique solution y, defined by transposition (see [6]), enjoying the regularity property

$$y \in C([0,T]; L^2(0,L)), \quad \rho y_t \in C([0,T]; H^{-1}(0,L)). \tag{2.1}$$

In this functional setting the sidewise controllability problem can be formulated more precisely as follows: *Given a finite time $T > 2L\beta$ and $p \in H^{-1}(L\beta, T-L\beta)$ we aim to determine $u \in L^2(0,T)$ such that the solution y of system* (1.1) *satisfies the condition*

$$y_x(L,t) = p(t) \quad \textit{for all} \quad t \in (L\beta, T-L\beta) \tag{2.2}$$

where

$$\beta = \operatorname{ess}\sup_{x\in[0,L]} \sqrt{\frac{\rho}{a}}. \tag{2.3}$$

Remark 2.1. *Several remarks are in order:*

- *Note that this problem makes sense since solutions of* (1.1) *in the above regularity class fulfill the added boundary regularity condition*

$$y_x(L,t) \in H^{-1}(0,T). \tag{2.4}$$

- *Note also that in the present formulation of the sidewise controllability problem the velocity of propagation plays an important role. On one hand the sidewise controllability property is only guaranteed when $T > 2L\beta$. On the other, the tracking condition is only assured in the time sub-interval $(L\beta, T-L\beta)$.*

- *Without loss of generality, using the principle of additive superposition of solutions of linear problems, the problem can be reduced to the particular case* $y_0(x) \equiv y_1(x) \equiv 0$.

Let us now consider the adjoint system:

$$\begin{cases} \rho(x)\psi_{tt} - (a(x)\psi_x)_x = 0, & 0 < x < L,\ 0 < t < T \\ \psi(x,T) = 0,\ \psi_t(x,T) = 0, & 0 < x < L \\ \psi(0,t) = 0,\ \psi(L,t) = s(t), & 0 < t < T \end{cases} \tag{2.5}$$

where the boundary data are of the form

$$s(t) = \begin{cases} 0, & T - L\beta \le t \le T \\ s_0(t), & L\beta \le t \le T - L\beta \\ 0, & 0 \le t \le L\beta \end{cases} \tag{2.6}$$

with $s_0 \in H_0^1(L\beta, T - L\beta)$.

This system admits an unique finite-energy solution ψ such that

$$(\psi, \frac{\partial \psi}{\partial t}) \in C([0,T], H^1(0,L) \times L^2(0,L))$$

and

$$\psi_x(0,.) \in L^2(0,L).$$

The sidewise observability inequality that, as we shall see, this adjoint system fulfills, and which is equivalent to the sidewise controllability problem under consideration is the following:

Proposition 2.1. *Let* $T > 2L\beta$ *(*β *being given as in* (2.3)*).*
Then, there exists $C_1 > 0$ *such that the observability inequality*

$$\|s_0(t)\|_{H^1(L\beta, T-L\beta)} \le C_1 \|\psi_x(0,t)\|_{L_2(0,T)} \tag{2.7}$$

is satisfied for every finite energy solution of (2.5)*, i. e. for all* $s_0 \in H_0^1(L\beta, T - L\beta)$.

Remark 2.2. *All terms in the inequality* (2.7) *make sense. In fact, the sidewise energy estimates in [6] allow showing that there exists a constant* $C_2 = C_2(\rho, a, T) > 0$ *such that the finite energy solution* ψ (2.5) *satisfies*

$$\|\psi_x(0,t)\|_{L_2(0,T)} \le C_2 \|s_0(t)\|_{H^1(0,T)} \tag{2.8}$$

for all $s_0 \in H_0^1(L\beta, T - L\beta)$.

This observability inequality (2.7) is equivalent to the sidewise controllability property. In fact, out of the observability property above, one can obtain the sidewise control of minimal $L^2(0,T)$-norm by a variational principle that we present now.

Let us consider the continuous and strictly convex quadratic functional

$$J : H_0^1(L\beta, T - L\beta) \longrightarrow \mathbb{R}$$

defined as

$$J(s_0) = \frac{1}{2}\int_0^T [(a\psi_x)(0,t)]^2 dt - a(L)\int_{L\beta}^{T-L\beta} s_0(t)p(t)dt \tag{2.9}$$

defined for $s_0 \in H_0^1(L\beta, T-L\beta)$, where ψ is the solution of the adjoint system (2.5) corresponding to the boundary datum $s \in H_0^1(0,T)$ given by (2.6).

Note that, in (2.9), the integral $\int_{L\beta}^{T-L\beta} p(t)s_0(t)dt$ represents the duality pairing between $p(t) \in H^{-1}(L\beta, T-L\beta)$ and $s_0(t) \in H_0^1(L\beta, T-L\beta)$.

The observability inequality above guarantees that the functional is also coercive. The Direct method of the Calculus of Variations then ensures that J has a unique minimizer.

The following lemma states that the minimum of the functional J provides the desired sidewise control.

Lemma 2.1. *Suppose that $\bar{s}_0 \in H_0^1(L\beta, T-L\beta)$ is the unique minimizer of J in the space $H_0^1(L\beta, T-L\beta)$. If $\bar{\psi}$ is the solution of the adjoint system (2.5) corresponding to the boundary datum $\bar{s}_0 \in H_0^1(L\beta, T-L\beta)$, then*

$$u(t) = -a(0)\bar{\psi}_x(0,t) \tag{2.10}$$

is a control such that

$$y_x(L,t) = p(t), \quad L\beta \le t \le T - L\beta, \tag{2.11}$$

when the initial data $y_0 \equiv y_1 \equiv 0$.

Remark 2.3. *As mentioned above, once the control is built for $y_0 \equiv y_1 \equiv 0$, using the linear superposition of solutions of the wave equation, the control for arbitrary initial data can be built. The functional J above can be also modified so to lead directly the control corresponding to non-trivial initial data.*

Proof. Since J achieves its minimum at $\bar{s}_0$, the Gateaux derivative of J vanishes at that point and, in other words,

$$\begin{aligned} 0 &= \lim_{h\to 0}\frac{1}{h}(J(\bar{s}_0 + hs_0) - J(\bar{s}_0)) \\ &= \int_0^T (a(0))^2 \bar{\psi}_x(0,t)\psi_x(0,t)dt - \int_{L\beta}^{T-L\beta} a(L)s_0(t)p(t)dt \end{aligned} \tag{2.12}$$

for any other $s_0 \in H_0^1(L\beta, T-L\beta)$ where ψ stands for the solution of adjoint system (2.5) with s_0 as boundary datum.

On the other hand, multiplying the equation (1.1) by the solution $\psi(x,t)$ of the adjoint problem and integrating on $(0,L)\times(0,T)$ we get (recall that, without loss of generality, we have assumed that $y_0(x) \equiv y_1(x) \equiv 0$)

$$\int_0^T a(0)u(t)\psi_x(0,t)dt + \int_0^T a(L)s(t)y_x(L,t)dt = 0. \tag{2.13}$$

Comparing (2.12) and (2.13) we get that $u(t) = -a(0)\bar{\psi}_x(0,t)$ is a control which leads $y_x(x,t)$ to $p(t)$ on $x = L$ for all $L\beta < t < T - L\beta$. □

Remark 2.4. *Using classical arguments it can be seen that the control obtained by this minimization method is the one of minimal $L^2(0,T)$-norm (see [17]).*

3 The main sidewise controllability result

Summarizing, the main sidewise controllability result is as follows:

Theorem 3.1. *Let us consider system* (1.1) *with coefficients satisfying the assumptions* (1.2) *and* (1.3).

Let $T > 2L\beta$ (β being given as in (2.3)*).*

Then, for any $p \in H^{-1}(L\beta, T - L\beta)$ there exists a control $u \in L^2(0,T)$ such that the solution of (1.1) *satisfies* (2.2).

4 Proof of the sidewise observability inequality

Let us now proceed to the proof of Proposition 2.1.

In order to prove the observability inequality (2.7), we will use the sidewise energy estimates (see [6, 23]) (see also [4], [18] for other applications of this technique).

Assume that $T > 2L\beta$ and set

$$F(x) = \frac{1}{2}\int_{\beta x}^{T-\beta x} [\rho(x)(\psi_t(x,t))^2 + a(x)(\psi_x(x,t))^2]dt, \qquad \forall x \in [0,L].$$

Because of the boundary condition $\psi(0,t) = 0$, we have

$$F(0) = \frac{a(0)}{2}\int_0^T (\psi_x(0,t))^2 dt$$

Let us compute the derivative of F with respect to x:

$$\begin{aligned}\frac{dF(x)}{dx} = &-\frac{\beta}{2}[\rho\psi_t^2 + a\psi_x^2]\Big|_{t=T-\beta x} - \frac{\beta}{2}[\rho\psi_t^2 + a\psi_x^2]\Big|_{t=\beta x} \\ &+ \int_{\beta x}^{T-\beta x}\left(\rho(x)\psi_t\psi_{tx} + a(x)\psi_x\psi_{xx} + \frac{\rho'(x)}{2}\psi_t^2 + \frac{a'(x)}{2}\psi_x^2\right)dt.\end{aligned} \tag{4.1}$$

Integrating by parts and using the equation (2.5) we have

$$\begin{aligned}&\int_{\beta x}^{T-\beta x} (\rho(x)\psi_t\psi_{tx} + a(x)\psi_x\psi_{xx})\,dt \\ &\qquad = -\int_{\beta x}^{T-\beta x} a'(x)\psi_x^2 dt + \rho(x)\psi_t\psi_x\Big|_{t=T-\beta x} - \rho(x)\psi_t\psi_x\Big|_{t=\beta x}.\end{aligned}$$

Combining this identity and (4.1) and using the elementary inequality

$$|\rho(x)\psi_t\psi_x| \le \frac{\beta}{2}[\rho(x)\psi_t^2 + a(x)\psi_x^2],$$

we obtain

$$\begin{aligned}\frac{dF(x)}{dx} &\le \frac{1}{2}\int_{\beta x}^{T-\beta x} \left(\rho'(x)\psi_t^2 - a'(x)\psi_x^2\right)dt \\ &\le \frac{1}{2}\max\left\{\frac{|\rho'|}{\rho}, \frac{|a'|}{a}\right\}\int_{\beta x}^{T-\beta x} \left(\rho(x)\psi_t^2 + a(x)\psi_x^2\right)dt \\ &= \max\left\{\frac{|\rho'|}{\rho}, \frac{|a'|}{a}\right\}F(x).\end{aligned}$$

Integrating this differential inequality with respesct to x, we get

$$F(x) \leq \exp\left(\int_0^x \max\left\{\frac{|\rho'(s)|}{\rho(s)}, \frac{|a'(s)|}{a(s)}\right\} ds\right) F(0)$$

From

$$\int_0^x \max\left\{\frac{|\rho'(s)|}{\rho(s)}, \frac{|a'(s)|}{a(s)}\right\} ds \leq \frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0},$$

we get

$$F(x) \leq \frac{a(0)}{2} \exp\left(\frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0}\right) \int_0^T (\psi_x(0,t))^2 dt, \quad \forall x \in [0, L]. \tag{4.2}$$

Integrating with respect to x in $(0, L)$, we have

$$\begin{aligned} &\frac{1}{2}\int_{\beta L}^{T-\beta L} \int_0^L [\rho(x)(\psi_t(x,t))^2 + a(x)(\psi_x(x,t))^2] dx dt \\ &\qquad \leq \frac{a(0)L}{2} \exp\left(\frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0}\right) \int_0^T (\psi_x(0,t))^2 dt. \end{aligned} \tag{4.3}$$

Because of the boundary condition $\psi(0,t) = 0$, we can write that

$$\begin{aligned} \int_{\beta L}^{T-\beta L} (\psi(L,t))^2 dt &= \int_{\beta L}^{T-\beta L} \left\{\int_0^L \psi_x(x,t) dx\right\}^2 dt \\ &\leq L \int_{\beta L}^{T-\beta L} \int_0^L \psi_x(x,t)^2 dx dt \\ &\leq \frac{L}{\min\{\rho_0, a_0\}} \int_{\beta L}^{T-\beta L} \int_0^L [\rho(x)(\psi_t(x,t))^2 + a(x)(\psi_x(x,t))^2] dx dt. \end{aligned}$$

Combining this inequality and (4.3), we get

$$\int_{\beta L}^{T-\beta L} (\psi(L,t))^2 dt \leq \frac{a(0)L^2}{\min\{\rho_0, a_0\}} \exp\left(\frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0}\right) \int_0^T (\psi_x(0,t))^2 dt. \tag{4.4}$$

On the other hand, from (4.2) we have

$$F(L) \leq \frac{a(0)}{2} \exp\left(\frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0}\right) \int_0^T (\psi_x(0,t))^2 dt.$$

Thenrefore we can write

$$\frac{\rho(L)}{2} \int_{\beta L}^{T-\beta L} (\psi_t(L,t))^2 dt \leq \frac{a(0)}{2} \exp\left(\frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0}\right) \int_0^T (\psi_x(0,t))^2 dt \tag{4.5}$$

where

$$\psi_t(L,t) = \begin{cases} 0, & T - L\beta \leq t \leq T \\ s_0'(t), & L\beta \leq t \leq T - L\beta \\ 0, & 0 \leq t \leq L\beta \end{cases}$$

Combining (4.4) and (4.5) we get the desired observability inequality (2.7) with

$$C_1^2 = \left(\frac{L^2}{\min\{\rho_0, a_0\}} + \frac{1}{\rho(L)}\right) a(0) \exp\left(\frac{TV(\rho)}{\rho_0} + \frac{TV(a)}{a_0}\right).$$

Thus, the proof of Proposition 2.1 is done.

5 Conclusions and open problems

In this paper we have proved the sidewise controllability of the 1-D wave equation with BV coefficients. This was done superposition, on one hand, a dual formulation of the problem, which leads to a novel sidewise observability inequality that, on the other hand, we prove by sidewise energy estimates that have been previously developed and implemented in the context of control of 1-D wave equations.

The methods and results in this paper lead top some interesting open problems and could be extended in various directions that we briefly describe now:

1. **Other boundary conditions.** For the sake of simplicity in this paper the case of Dirichlet boundary conditions has been addressed. But similar problems are relevant with other boundary conditions. Our techniques apply in those cases too.

   One could for instance consider the same model with Neumann boundary conditions and control:

   $$\begin{cases} \rho(x)y_{tt} - (a(x)y_x)_x = 0, & 0 < x < L,\ 0 < t < T, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & 0 < x < L, \\ y_x(0,t) = u(t),\ y_x(L,t) = 0, & 0 < t < T. \end{cases} \tag{5.1}$$

   In this case the problem consists on, given a time-dependent function $p = p(t)$, to find a control $u = u(t)$ such that the corresponding solution fulfills:

   $$y(L,t) = p(t), \quad t \geq 0 \tag{5.2}$$

   Our methods apply in this case too, leading to similar results with minor changes.

2. **Less regular coefficients.** The methods developed in this paper combined with those of [7] allow to consider coefficients with slightly weaker regularity and obtain sidewise controllability results for more smooth Sobolev targets. Note, however, that the results in [3] can also be adapted to show that in the class of Hölder continuous coefficients one cannot expect such results for targets in Sobolev classes.

3. **Non-harmonic Fourier series.**The results presented in this paper could be also obtained using other genuinely 1-D methods such as the D'Alembert formula or Fourier series representation methods.

   When dealing with Fourier series the sidewise observability inequality proved above leads to new variants of the classical Ingham inequalities (see [17]). It would be interesting to see if they can be obtained directly by non-harmonic Fourier series methods.

4. **Nonlinear problems.** The results in this paper, combined with those in [23], allow to extend our sidewise controllability result for semilinear wave equations of the form

   $$\rho(x)y_{tt} - (a(x)y_x)_x + f(y) = 0$$

   with f a locally Lipschitz nonlinearity such that

   $$\lim_{|s|\to\infty} \frac{f(s)}{s\log^2(s)}$$

   is small enough (see also [2]).

5. **Transmutation.** As explained in [6], transmutation techniques can be used to transfer controllability properties of wave equations into heat equations. It would be interesting to analyse whether this can be done in the context of the sidewise controllability/observability of the heat equation.

   Recall, as mentioned above, that the sidewise controllability of the 1-D heat equation has been directly addressed in [1] using flatness methods in Gevrey classes.

6. **Networks.** The techniques in this paper can be applied for 1-D wave propagation on networks. It could for instance allow to handle the case of a tree-shaped network with active controls on all but one free ends ([9]). But adapting them to more general networks, or to the case of fewer controls, would require substantial further developments in combination with graph and diophantine theory ([5]).

7. **Numerical analysis.** Most of the methods developed for the numerical controllability ([23]) of the wave equation can also be applied in the context of sidewise controllability. But this would require a careful adaptation since, most often, the needed numerical results are achieved using Fourier series techniques, and not the sidewise energy estimates presented here, that do not hold in the discrete setting.

8. **Multi-dimensional problems.** As explained in the introduction, the controllability theory of wave equations has been also developed in the multi-dimensional context. Duality arguments reduce the problem to boundary observability inequalities that can be obtained by different methods including multipliers, Carleman inequalities and microlocal analysis.

   The sidewise control problem discussed in this paper can also be easily reformulated in the multi-dimensional frame. The duality method described here can also be applied, reducing the problem to the obtention of new sidewise observability inequalities.

   However, adapting the existing techniques for the observability of waves to prove those sidewise observability inequalities for multi-dimensional wave equations seems to be a challenging problem.

   Let us present this interesting and challenging issue with some more detail.

   Let Ω be a bounded smooth domain of $\mathbb{R}^n$, in dimension $n \geq 2$, and consider the wave equation:

$$\begin{cases} y_{tt} - \Delta y = 0, & \text{in } \Omega \times (0,T) \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & \text{in } \Omega \\ y = u, & \text{on } \Gamma_c \times (0,T) \\ y = 0, & \text{on } \Gamma_0 \times (0,T). \end{cases} \tag{5.3}$$

   Here Γ_0 and Γ_c stand for a partition of the boundary, Γ_0 being the fix part of the boundary and Γ_c the one under control. While the control u acts on Γ_c, the subset of the boundary Γ_0 remains fixed, thanks to the homogeneous Dirichlet boundary conditions.

   Given a smooth enough target $p : \Gamma_0 \times (0,T) \to \mathbb{R}$ to be tracked, the question is then to find a control u in, say, $L^2(\Gamma_c \times (0,T))$, such that the solution y of (5.3) satisfies the condition

$$\partial y/\partial \nu = p \quad \text{on } \Gamma_0 \times (0,T).$$

Here and in what follows ν denotes the outward unit normal vector and $\partial \cdot /\partial\nu$ the normal derivative.

This is the natural multi-d version of the sidewise controllability problem discussed above. In fact the same formulation can be easily adapted to consider other models such as multi-dimensional heat, Schrödinger equations or the system of thermoelasticity (see [25]).

The sidewise controllability problem above can be reduced, by duality, to a new class of sidewise observability inequalities for the adjoint system

$$\begin{cases} \psi_{tt} - \Delta\psi = 0, & \text{in } \Omega \times (0,T) \\ \psi(x,T) = \psi_t(x,T) = 0, & \text{in } \Omega \\ \psi = 0, & \text{on } \Gamma_c \times (0,T) \\ \psi = s, & \text{on } \Gamma_0 \times (0,T). \end{cases} \tag{5.4}$$

Here $s = s(x,t)$ is a smooth boundary condition given on $\Gamma_0 \times (0,T)$. The question is then whether one can prove the existence of an observability constant $C > 0$ such that

$$||s||_*^2 \leq C||\partial\psi/\partial||^2_{L^2(\Gamma_c\times(0,T))} \tag{5.5}$$

for every solution of this adjoint system.

Observe that here the norm $||\cdot||_*$ in this inequality is to be identified both in what corresponds the Sobolev regularity and the support within $\Gamma_0 \times (0,T)$.

As we mentioned above the existing techniques do not seem to yield this kind of inequalities in a direct manner.

However, as described in [11], using the Holmgren's uniqueness Theorem, a unique continuation property can be easily proved. This constitutes a weaker and non-quantitative version of this kind of inequality.

Indeed, it can be easily proved that, as soon as $\partial\psi/\partial\nu \equiv 0$ in $\Gamma_c \times (0,T)$ and T is large enough, one can guarantee that $s \equiv 0$ provided its support is localized in a subset of the boundary of the form $\gamma \times (\tau, T-\tau)$, with γ a suitable open subset of Γ_0 and $0 < \tau < T/2$. Both γ and τ can be easily characterized it terms of the cones of influence and dependence of solutions of the wave equation. Essentially, γ is constituted by the points for which the geodesic distance (within Ω) to Γ_c is less than τ.

Obviously, for this result to be active in some effective subset $\gamma \times (\tau, T-\tau)$, one needs T to be large enough, in particular $T > 2\delta$, where δ is the minimal geodesic distance from Γ_c to Γ_0.

This unique continuation result assures, in the corresponding geometric setting, that the wave equation enjoys the property of sidewise approximate controllability: i. e. that given any $p \in L^2(\gamma \times (\tau, T-\tau))$ and any $\epsilon > 0$ there exists a control $u \in L^2(\Gamma_c \times (0,T))$ (depending on ϵ) such that the corresponding solution y satisfies

$$||\partial y/\partial\nu - p||_{L^2(\gamma\times(\tau,T-\tau))} \leq \epsilon.$$

This result can be viewed as a partial extension of the 1-D results in this paper to the multi-dimensional case. Note however that these arguments, based purely on Holmgren uniqueness, do not yield any quantitative estimates.

The systematic analysis of these problems in the multi-dimensional context for the wave equation and other relevant models constitutes a very rich source of interesting open problems.

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