On problems of dynamic optimal nodal control for gas networks

Martin Gugat (Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
Chair in Applied Analysis (Alexander von Humboldt-Professorship)) joint work with
Jan Sokolowski (Systems Research Institute of Polish Academy of Sciences)
*Los Alamos National Laboratory's (LANL)

4th Grid Science Winter School and Conference

Outline

Dynamic Compressor Optimization

Model for the flow in a natural gas pipe

Problems of dynamic optimal nodal control for gas networks: $\mathbf{P_{dyn}}(T)$

Existence of a solution of $\mathbf{P_{dyn}}(T)$

The optimal controls approach the set-point

Inhalt

Dynamic Compressor Optimization in Natural Gas Pipeline Systems

is studied by TWK MAK , P V.HENTENRYCK , A ZLOTNIK, R BENT, INFORMS J Comp., 2019.

minimize compression cost

such that

the system state, whose evolution is governed by pdes, e.g.

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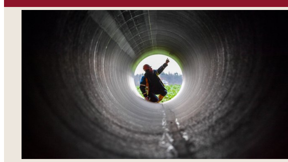

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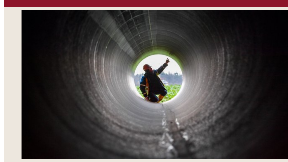

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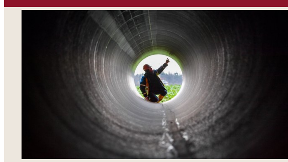

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Here we consider the hyperbolic model!

How to avoid shocks?

The static problem: Optimal steady gas flow

Static gas flows have been studied in depth, see e.g.

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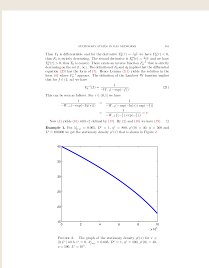

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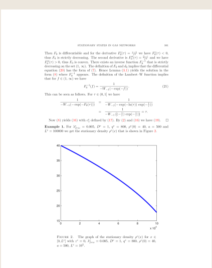

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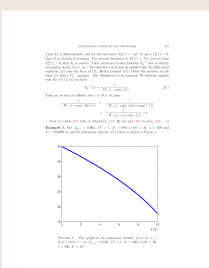

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For fixed(=steady) deliveries

many projects have determined optimal gas flows, that form **optimal stationary states on the networks**.

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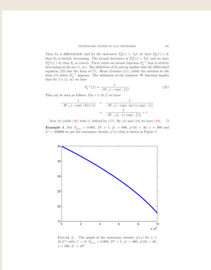

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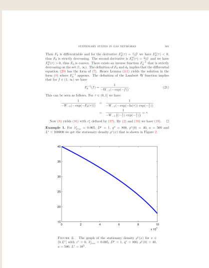

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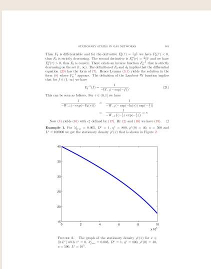

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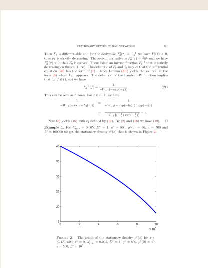

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In the hyperbolic transient model, in general shocks can occur!

We want controls that do **not** generate shocks!

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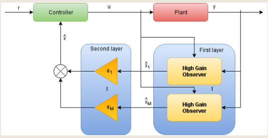

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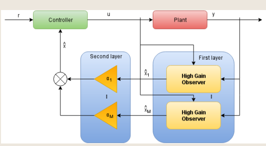

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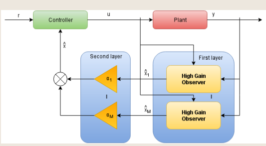

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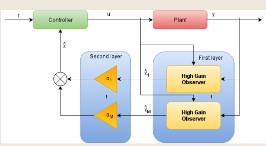

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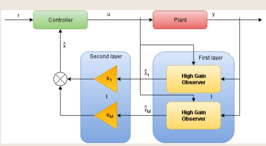

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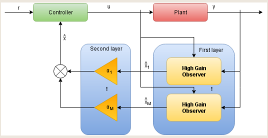

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The compressor cost is the essential obj.:

$$\int_0^T \sum_{v\in V_c} A_v \, q^v(t) \left[\left(\frac{p_{out,v}(t)}{p_{in,v}(t)} \right)^{R_v} - 1 \right] dt$$

(see Osiadac 2016).

Uncertainty managemnt for time-dependent constr.

Often constraints have to hold for all $t \in [0, T]$ (like pressure bounds):

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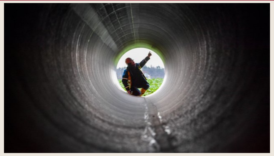
Probabilistic robustness is less costly than classical robustness!

Inhalt

Model for the flow in a single pipe

The isothermal Euler equations for ideal gas



For a horizontal pipe, we have

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(c^2\,\rho + \frac{q^2}{\rho}\right)_x = -\frac{1}{2}\,\theta\,\frac{q\,|q|}{\rho} \end{cases}$$

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- q: flow rate
- c: sound speed
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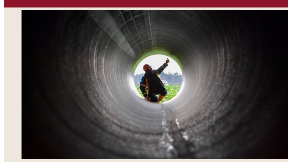

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Slow flow ($|q/\rho| << c$)

See the results of DFG CRC 154-2

TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

For subsonic flow, at each boundary point one boundary condition is set.

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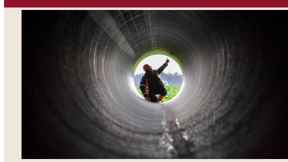

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$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(c^2\,\rho + \frac{q^2}{\rho}\right)_x = -\frac{1}{2}\,\theta\,\frac{q\,|q|}{\rho} \end{cases}$$

- ρ: density
- q: flow rate
- c: sound speed
- $\theta = \frac{f_g}{\delta}$
- f_g: friction, δ: diameter

Slow flow ($|q/\rho| << c$)

See the results of DFG CRC 154-2

TRR 154 | Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks

For subsonic flow, at each boundary point one boundary condition is set.

Slow flow: Boundary conditions

In terms of the Riemann invariants, we can state the boundary conditions as

$$R_+(t, 0) = g_0(t),$$
$$R_-(t, L) = g_L(t).$$

Inhalt

Dynamic optimal nodal control for gas networks

Let a stationary reference solution $p_{ref}(x)^e$, $q_{ref}(x)^e$ $(e \in E)$ with constant controls u_{ref}^v, $(v \in V_c =$ set of compressor nodes) be given.

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$q^e(0, x) = q^e_0(x),\ x \in (0, L^e),\ e \in E,$ INITIAL CONDITIONS

$\rho^e(0, x) = \rho^e_0(x),\ x \in (0, L^e),\ e \in E,$

$q^e_{out}(t, x^e(v)) = q^e_{ref},\ t \in (0, T),\ v \in V,\ e \in E_0(v), \text{if } |E_0(v)| = 1,\ v \neq v^*,$ B.COND.

$p^e_{in}(t, x^e(v)) = p^e_{ref},\ t \in (0, T),\ v \in V,\ e \in E_0(v), \text{if } |E_0(v)| = 1,\ v = v^*,$

$\sum_{e \in E_0(v)} \mathfrak{s}(v, e)\,(D^e)^2\, q^e(t, x^e(v)) = 0,\ t \in (0, T),\ \text{if } |E_0(v)| \geq 2,$ FLOW BALANCE

$p(\rho^e(t, x^e(v))) = p(\rho^f(t, x^f(v))),\ t \in (0, T),\ \text{if } |E_0(v)| \geq 2,\ e, f \in E_0(v),$ PRESS. CONT.

$u^v(t) + u^v_{ref} = \left(\frac{p_{out,v}(t)}{p_{in,v}(t)}\right)^{R_v},\ t \in (0, T),\ \text{if } |E_0(v)| = 2, v \in V_c,$

$q^e(t, x^e(v))) = q^f(t, x^f(v))),\ t \in (0, T),\ \text{if } |E_0(v)| = 2, v \in V_c; e, f \in E_0(v),$

$$\begin{pmatrix} \rho^e \\ q^e \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ a^2 - \frac{(q^e)^2}{(\rho^e)^2} & 2\,\frac{q^e}{\rho^e} \end{pmatrix}_x \begin{pmatrix} \rho^e, \\ q^e \end{pmatrix}_x = \begin{pmatrix} 0 \\ -\frac{1}{2}\theta^e \frac{q^e\,|q^e|}{\rho^e} \end{pmatrix} \text{ on } [0, T] \times [0, L^e],\ e \in E.$$

Semi-global solutions, Ta-Tsien Li

- The theory of *semi-global solutions* asserts that for any given time horizon $T_0 > 0$ there exists a number $\varepsilon(T_0) > 0$ such that for all initial states with

$$\|q_0^e - q_{ref}\|_{C^1([0,\, L^e])} \leq \varepsilon(T_0) \quad \text{and} \quad \|\rho_0^e - \rho_{ref}\|_{C^1([0,\, L^e])} \leq \varepsilon(T_0) \tag{1}$$

and all controls that satisfy

$$\|u^v\|_{C^1([0,\, T_0])} \leq \varepsilon(T_0) \tag{2}$$

and are C^1**–compatible** with the initial state there exists a classical solution of (**S**) on $[0,\ T_0]$ that satisfies an a priori estimate for the corresponding C^1-norm.

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- There exists a constant $C_c > 0$ such that if (1), and (2) hold for two controls u_1 and u_2, we have

$$\max_{e\in E} \|p_1^e(t,\,x) - p_2^e(t,\,x)\|_{C([0,\,T_0]\times[0,\,L^e]} \leq C_c \max\{\max_{e\in E} \|u_1^e - u_2^e\|_{C([0,\,T_0]},\ \max_{v\in V_c} \|u_1^v - u_2^v\|_{C([0,\,T_0]}\}$$

and

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If $T_0 > 0$ is chosen sufficiently small, $\varepsilon(T_0)$ can be quite large!

Control and state constraints

- The control action in the **compressor** is bounded. With given minimum *compressor ratio* $\varepsilon_v^{\min}$ and maximum comp. ratio $\varepsilon_v^{\max}$ that satisfy
$$1 \leq \varepsilon_v^{\min} \leq \varepsilon_v^{\max}$$
we have the control constraint constraints
$$\varepsilon_v^{\min} \leq u^v(t) + u_{ref} \leq \varepsilon_v^{\max} \tag{3}$$
that gives bounds on the compression ratio. In practice we admit that $\varepsilon_v^{\min} > 1$. In this case, we assume that the compressor is switched on.

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- In the operation of gas networks, the pressure should remain between given bounds
$$p_{\min} < p_{\max}.$$
In our optimal control problem this will be taken into account in the cost functional by a **penalty term** in the form
$$\eta_p \sum_{e \in E} \|(p_{\min} - p^e(t,x))_+\|_{C([0,T]\times[0,L^e])} + \|(p^e(t,x) - p_{\max})_+\|_{C([0,T]\times[0,L^e])}$$
that penalizes a violation of the pressure bounds.
Here $\eta_p > 0$ is a penalty parameter and $(r)_+ = \max\{r,\, 0\}$.

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Here q^{ref} and p^{ref} denote a stationary reference state that is a classical steady state of the pde that is time-independent and is compatible with the boundary conditions, the node conditions and a feasible stationary compressor control u_{ref}. Moreover, we assume that q^{ref} and p^{ref} satisfy the *state constraints* and are compatible with the initial conditions, so that the feasible set is non-empty.

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The state constraint (5) *allows to make the time horizon* T *arbitrarily large!*

The optimal control problem

Let a Banach space $X(T) \subset \left(C^1([0,\ T])\right)^{|V_c|}$ be given.

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- First, we define the set $U(T)$ of **feasible controls**. The set $U(T)$ contains the control functions $u(t) \in X(T)$ such that the control constraints (3) and (4), and for the corresponding system state generated by **(S)** the state constraints (5) with $\varepsilon = \varepsilon(T_0)$ are satisfied.

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The **optimal control problem** is to find a control function $u \in U(T)$ such that

$$J(u) = \int_0^T \sum_{v \in V_c} A_v\, q^v(t)\,[u^v(t) + u^v_{ref} - 1]\; dt \tag{6}$$

$$+\eta_p \sum_{e \in E} \|(p_{\min} - p^e(t,x))_+\|_{C([0,T]\times[0,L^e])} + \|(p^e(t,x) - p_{\max})_+\|_{C([0,T]\times[0,L^e])}$$

$$+\gamma\, \|u\|_{X(T)}$$

is minimized. Here $\eta_p > 0$ and $\gamma > 0$ are penalty parameters. Notation: $\mathbf{P_{dyn}}(T)$.

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- For the operation of gas networks it is important to remain within the scenario of **classical solutions** in order to avoid damages in the system caused by shocks.

Inhalt

Existence of a solution of $\mathbf{P_{dyn}}(T)$

Assumption on $X(T)$ and a seminorm $\|\cdot\|_{X(T)}$:

*Every sequence of controls in the admissible set $U(T)$ that is bounded w.r.t. $\|\cdot\|_{X(T)}$ contains a subsequence that converges **strongly** in $\left(C^1([0,\ T])\right)^{|V_c|}$.*

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Also for $X(T) = \left(W^{2,\infty}(0,\ T)\right)^{|V_c|}$. For the latter space, we introduce a *special norm* that depends on a parameter $n \in \{1, 2, 3, ...\}$. It is defined as

$$\|u\|_n = \sum_{j=1}^{n} \sum_{v \in V_c} \|(u_{opt}^v)''\|_{L^\infty\left(\frac{j-1}{n}T, \frac{j}{n}T\right)} + \|u_{opt}^v\|_{L^\infty\left(\frac{j-1}{n}T, \frac{j}{n}T\right)}. \tag{7}$$

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Theorem (Let $T > T_0 > 0$ be given.)

A solution of the dynamic optimal control problem $\mathbf{P}_{\mathbf{dyn}}(T)$ with $\varepsilon = \varepsilon(T_0)$ in (4) and (5) does exist.

Existence of a solution of $\mathbf{P}_{\mathbf{dyn}}(T)$

Due to the **second order regularization term** the **existence of optimal controls** that generate a **smooth flow** can be shown! (Despite of the product in $J(u)$.)

Existence of a solution of $\mathbf{P_{dyn}}(T)$

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The proof for the existence of optimal controls

can be adapted to problems with **time-periodic** controls and states as considered in
Dynamic Compressor Optimization in Natural Gas Pipeline Systems
by *TWK Mak , P v.Hentenryck , A Zlotnik, R Bent,* INFORMS J. Comp., 2019,
but with a hyperbolic pde!

Inhalt

Dynamic Compressor Optimization

Model for the flow in a natural gas pipe

Problems of dynamic optimal nodal control for gas networks: $\mathbf{P}_{\mathbf{dyn}}(T)$

Existence of a solution of $\mathbf{P}_{\mathbf{dyn}}(T)$

The optimal controls approach the set-point

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***We say that the optimal control $u_{opt} \neq 0$ satisfies* Property A**

if there exist an integer $j_s \in \{1, 2, ..., n-1\}$ and a point $t_0 \in I^* = I^{j_s}$ such that $\frac{j_s - 1}{n} T \in [T - T_0, T)$ and the following conditions hold with

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1. $\|u''_{opt}\|_{L^\infty(I_0)} < \|u''_{opt}\|_{L^\infty(I^*)}$,
2. $\|u'_{opt}\|_{L^\infty(I_0)} < \|u'_{opt}\|_{L^\infty(I^*)}$,
3. $\|u_{opt}\|_{L^\infty(\frac{j_s-1}{n} T, \frac{t_0+t_s}{2})} < \|u_{opt}\|_{L^\infty(I^*)}$.

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Note that the inequalities with $\leq$ always hold.
If all components of u_{opt} are decreasing, 3. is violated.

The optimal controls approach the set-point

Now we state our result about the structure of the optimal controls.

Theorem

Let $T > T_0$ be given. Let $n \in \{1, 2, ...\}$ be given such that $n > \frac{T}{T_0}$.

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Let $X(T) = \left(W^{2,\infty}(0,\ T)\right)^{|V_c|}$ with the norm $\|\cdot\|_{X(T)} = \|\cdot\|_n$ as defined in (7).

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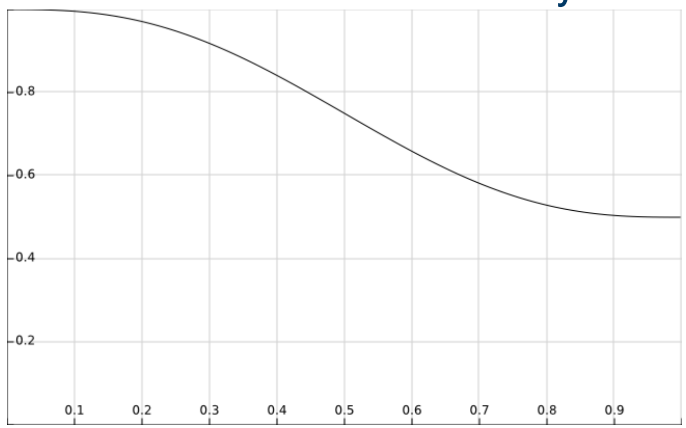
Let $X(T) = \left(W^{2,\infty}(0,\ T)\right)^{|V_c|}$ with the norm $\|\cdot\|_{X(T)} = \|\cdot\|_n$ as defined in (7).
If the penalty parameter $\gamma > 0$ is sufficiently large,
for any optimal control u_{opt} that solves the dynamic optimal control problem $\mathbf{P_{dyn}}(T)$ and satisfies ***Property A***
there exists a number $t_* \in (0,\ T)$ such that for all $t \in [t_*,\ T]$ we have

$$u_{opt}(t) = 0.$$

Terminal slide

In the proof, a $W^{2,\infty}$ variation of the control is used. It is obtained by multiplying the control u_{opt} with a function of this form: 

Terminal slide

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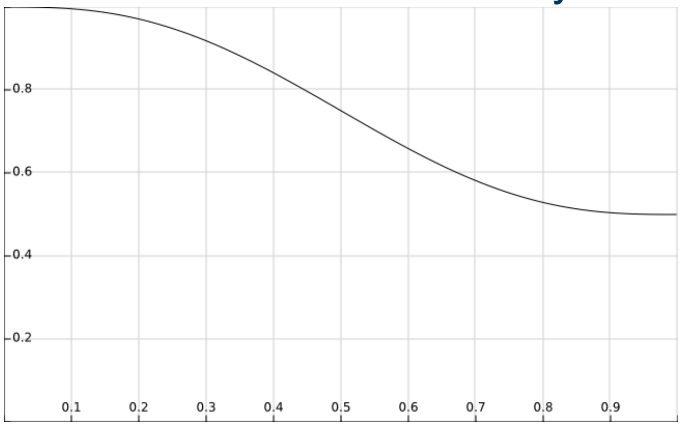

control u_{opt} with a function of this form:

- Future work along the same path:
  1. Numerical results; collab?
  2. Include probabilistic constraints? (see M. GUGAT, A turnpike result for convex hyperbolic optimal boundary control problems, *Pure and Applied Functional Analysis* 4, 849-866, 2019.)

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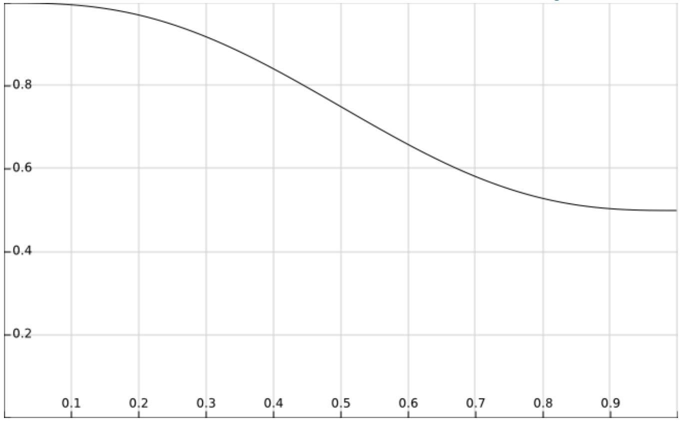

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- Thank you for your attention!
  Next time biking in Pipeline Rd Los Alamos, NM 87544, USA!