CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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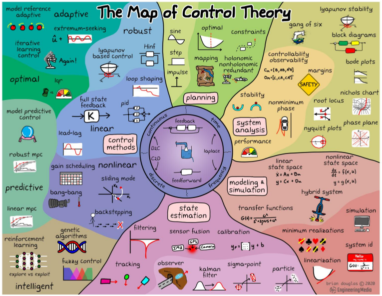
PART I: introduction to control theory

LECTURE 1: finite-dimensional control systems

Historical introduction

MATHEMATICAL CONTROL THEORY
or
CONTROL ENGINEERING
or simply
CONTROL THEORY?

An interdisciplinary field of research in between Mathematics and Engineering with strong connections with Scientific Computing, Technology, Communications,...



The origins

"... if every instrument could accomplish its own work, obeying or anticipating the will of others ... if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves."

Chapter 3, Book 1, of the monograph "Politics" by Aristotle (384-322 B. C.).

Main motivation

The need of automatizing processes to let the human being gain in liberty, freedom, and quality of life.

General principle

Modify the behavior of a dynamical system to drive its state to a desired final configuration, while possibly minimizing relevant aspects such as energy consumption, delay or overshoot, and ensuring a certain level of control stability.

Early examples of controlled systems

Ancient Mesopotamia: water regulation in irrigation systems.

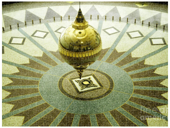
Roman aqueducts: system of water transportation endowed with devices of regulating valves for keeping the water level constant.

Ancient Egypt: the **harpenodaptai** (string stretchers), were specialized in stretching strings leading to long straight segments to help in large constructions. Two fundamental concepts were already well understood:

1. The shortest distance between two points is the straight line.
2. This is equivalent to the **dual property** that the straight line is the path generating the longest distance among all the paths of a given length between two points.

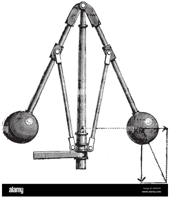
Modern examples of controlled systems

The pendulum: works of Huygens and Hooke (end of the XVII century), the goal being measuring in a precise way location and time, so precious in navigation.



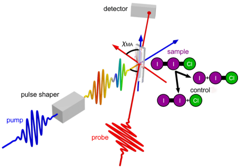
These studies inspired James Watt for the invention of his famous **steam engine**, marking the start of the industrial revolution.

When the velocity of the balls increases, one or several valves open to let the vapor escape and make the pressure diminish. When the pressure inside the boiler becomes weaker, the velocity begins to go down. The goal is to keep the velocity as close as possible to a constant.



Contemporary examples of controlled systems

Quantum control and computing: laser control in Quantum mechanical and molecular systems to design coherent vibrational states.



Aerospace industry:

- Control systems in flexible structures (satellites).
- Optimal shape design in aerodynamics.
- Control of rockets trajectories.

Mathematical formulation of control theory

Mathematical formulation

First mathematical analysis of control a control system: dated 1868 and due to Maxwell, who described some erratic behaviors in Watt's device and proposed control mechanisms to correct them. Until them it was not well understood why apparently more elaborated and perfect regulators could have a bad behavior. The reason is now referred to as the **overdamping phenomenon**.

Automatic control: the number of applications rapidly increased in the 1930's covering different areas like amplifiers in telecommunications, distribution systems in electrical plants, stabilization of aeroplanes, electrical mechanisms in paper production, petroleum and steel industry,...

In that period, two different approaches to control theory where defined:

- **State space approach**, based on modelling by means of Ordinary Differential equations (ODE).
- **Frequency domain approach**, based in the Fourier representation of signals.

IMPORTANT ADVANCES IN THE 1960's

By that time it was known that true systems are often nonlinear and nondeterministic, and this generated important new efforts in identifying novel efficient control techniques.

Three fundamental contributions

Kalman and his theory of filtering and algebraic approach to the control of systems.

Pontryagin and his maximum principle: a generalization of Lagrange multipliers.

Bellman and his principle of dynamic programming: a trajectory is optimal if it is optimal at every time.

IMPORTANT FURTHER DEVELOPMENTS IN THE LAST DECADES

Modern advances

- **Nonlinear problems**, with the introduction of Lie brackets.
- **Stochastic models**: Human beings introduce more uncertainty in already uncertain systems.
- **Infinite dimensional systems = Partial Differential Equations (PDE)**, also referred to as Distributed Parameter Systems. When the number of degrees of freedom is too large one is obliged to deal with models in Continuum Mechanics.

Mathematical formulation

State equation

$$A(y) = f(v)$$

- y is the **state** to be controlled.
- v is the **control**. It belongs to the set of admissible controls $\mathcal{U}_{ad}$.

Goal

Roughly speaking the goal is to drive the state y close to a desired state y_d: $y \sim y_d$.

Mathematical formulation

In this general functional setting many different mathematical models feet:

- **Linear** or **nonlinear** problems.
- **Deterministic** or **stochastic** models.
- **Finite-dimensional** (ODE) or **infinite-dimensional** (PDE) models.

And, of course, when facing complex real life processes, often, **hybrid models** might also be needed.

Mathematical formulation

Several kinds of different control problems may also feet in this frame depending on how the control objective is formulated:

Optimal control (related with the Calculus of Variations)

$$\min_{v \in \mathcal{U}_{ad}} \|y - y_d\|^2$$

Controllability: drive exactly the state y to the prescribed one

$$y = y_d$$

Stabilization or **feedback control** (real time control)

$$v = F(y)$$
$$A(y) = f(F(y))$$

Feedback process

The one in which the state of the system determines the way the control has to be exerted in real time (cause-effect principle).

Finite-dimensional linear control

Finite-dimensional control

Let $N, M \in \mathbb{N}^*$ and $T > 0$. Consider the following finite-dimensional system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{1}$$

- $A \in \mathbb{R}^{N \times N}$ describes the **dynamics**.
- $B \in \mathbb{R}^{N \times M}$ defines the **control's action**.
- $x_0 \in \mathbb{R}^N$.
- $x : [0, T] \longrightarrow \mathbb{R}^N$ represents the **state**.
- $u : [0, T] \longrightarrow \mathbb{R}^M$ is the **control**.

Given an initial datum $x^0 \in \mathbb{R}^N$ and a vector function $u \in L^2(0, T; \mathbb{R}^M)$, system (1) has a unique solution $x \in H^1(0, T; \mathbb{R}^N)$ characterized by the variation of constants formula

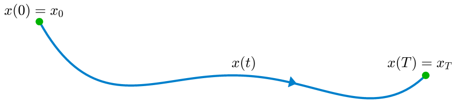
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)\, ds, \quad \forall t \in [0, T].$$

Controllability notions

Exact controllability

System (1) is **exactly controllable** at time T if, for any $x_0, x_T \in \mathbb{R}^N$, there exists $u \in L^2(0, T; \mathbb{R}^M)$ such that the corresponding solution x fulfills $x(T) = x_T$.

According to this definition the aim of exact controllability consists in driving the solution x of (1) from the initial state x_0 to the final one x_T in time T by acting on the system through the control u.

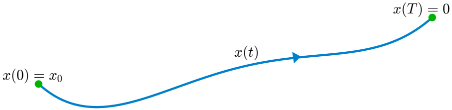


Controllability notions

Null controllability

System (1) is **null controllable** at time T if, for any $x_0 \in \mathbb{R}^N$, there exists $u \in L^2(0, T; \mathbb{R}^M)$ such that the corresponding solution x fulfills $x(T) = 0$.

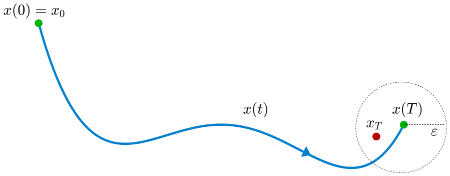
According to this definition the aim of null controllability consists in driving the solution x of (1) from the initial state x_0 to zero in time T by acting on the system through the control u.



Approximate controllability

System (1) is **approximately controllable** at time T if, for any $x_0, x_T \in \mathbb{R}^N$ and any $\varepsilon > 0$, there exists $u \in L^2(0,T;\mathbb{R}^M)$ such that the corresponding solution x fulfills $\|x(T) - x_T\|_{\mathbb{R}^N} < \varepsilon$.

According to this definition the aim of approximate controllability consists in driving the solution x of (1) in time T from the initial state x_0 to a final one $x(T)$ which is ε-close to x_T by acting on the system through the control u.

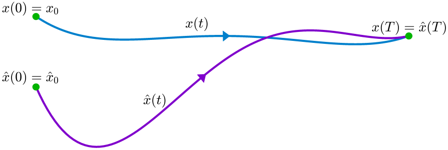


Controllability notions

Controllability to trajectories

System (1) is **exactly controllable to trajectory** at time T if, for any $x_0 \in \mathbb{R}^N$ and any solution $\hat{x}$ of (1) with $\hat{x}(0) = \hat{x}_0 \in \mathbb{R}^N$ and some given $\hat{u}$, there exists a control $u \in L^2(0, T; \mathbb{R}^M)$ such that the corresponding solution x fulfills $x(T) = \hat{x}(T)$.

According to this definition the aim of controllability to trajectories consists in driving the solution x of (1) in time T from the initial state x_0 to match a particular solution $\hat{x}(T)$ by acting on the system through the control u.



Controllability examples

Example 1: consider the case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then the system (1) can be written as

$$\begin{cases} x_1'(t) = x_1(t) + u(t) \\ x_2'(t) = x_2(t) \end{cases} \qquad \equiv \qquad \begin{cases} x_1'(t) = x_1(t) + u(t) \\ x_2(t) = x_{0,2}e^t \end{cases}$$

where $x_0 = (x_{0,1}, x_{0,2})$ are the initial data.

This system is not controllable since the control u does not act on the second component x_2 of the state which is completely determined by the initial data $x_{0,2}$.

Controllability examples

Example 2: by the contrary, the equation of the harmonic oscillator

$$x''(t) + x(t) = u(t)$$

is controllable.

In this case, the matrices are

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One can easily check the controllability by simply building a smooth curve $x = x(t)$ taking the initial values at $t = 0$ and the final ones at $t = T$, and then, computing a posteriori the control $u(t) = x''(t) + x(t)$.

The reachable set

Set of reachable states

$$R(T, x_0) = \left\{ x(T) \in \mathbb{R}^N \ : \ x \text{ solution of (1) with } u \in (L^2(0,T);\mathbb{R}^M) \right\}.$$

Remark

$R(T, x_0)$ is a convex subset of $\mathbb{R}^N$.

The controllability notions previously introduced can be redefined through the reachable set

Exact controllability: $R(T, x_0) = \mathbb{R}^N$ for any $x_0 \in \mathbb{R}^N$.

Null controllability: $0 \in R(T, x_0)$ for any $x_0 \in \mathbb{R}^N$.

Approximate controllability: $R(T, x_0)$ is dense in $\mathbb{R}^N$ for any $x_0 \in \mathbb{R}^N$.

Remark

In linear finite-dimensional control, **exact controllability**, **null controllability** and **approximate controllability** are equivalent.

Approximate controllability is equivalent to exact controllability, since the only convex and dense subset of $\mathbb{R}^N$ is $\mathbb{R}^N$ itself.

Null controllability is equivalent to exact controllability: indeed, if $x(T) = x_T \neq 0$, then the function

$$z(t) = x(t) - x_T e^{(t-T)A}$$

verifies

$$\begin{cases} \dot{z} = Az + Bu \\ z(0) = x_0 - x_T e^{-TA} \end{cases}$$

and we have $x(T) = x_T \iff z(T) = 0$.

The observability property

The adjoint problem

Let $A^\top$ be the **adjoint** (transpose) matrix of A, i.e. the matrix such that

$$\langle Ax, y\rangle = \langle x, A^\top y\rangle \quad \text{for all } x, y \in \mathbb{R}^N.$$

Consider the following homogeneous **adjoint system** of (1):

$$\begin{cases} -p'(t) = A^\top p(t), & t \in (0,T) \\ p(T) = p_T \end{cases} \tag{2}$$

We have the following equivalent condition for exact controllability.

Lemma

An initial datum $x_0 \in \mathbb{R}^N$ of (1) is driven to zero in time T by using a control $u \in L^2(0,T;\mathbb{R}^M)$ if and only if

$$\int_0^T \langle u, B^\top p\rangle \, dt + \langle x_0, p(0)\rangle = 0, \qquad \text{for all } p_T \in \mathbb{R}^N, \tag{3}$$

p being the corresponding solution of (2).

The adjoint problem

PROOF: let p_T be arbitrary in $\mathbb{R}^N$ and p be the corresponding solution of (2). By multiplying (1) by p and (2) by x, we deduce that

$$\begin{aligned} \langle x', p\rangle &= \langle Ax, p\rangle + \langle Bu, p\rangle = \langle x, A^\top p\rangle + \langle Bu, p\rangle \\ -\langle x, p'\rangle &= \langle A^\top p, x\rangle \end{aligned}$$

Hence,

$$\langle x', p\rangle + \langle x, p'\rangle = \frac{d}{dt}\langle x, p\rangle = \langle Bu, p\rangle$$

which, after integration in time, gives that

$$\langle x(T), p_T\rangle - \langle x^0, p(0)\rangle = \int_0^T \langle Bu, p\rangle\, dt = \int_0^T \langle u, B^\top p\rangle\, dt.$$

We obtain that $x(T) = 0$ if and only if (3) is verified for any $\varphi_T \in \mathbb{R}^N$.

The adjoint problem

Identity (3) is in fact an optimality condition for the critical points of the quadratic functional $J : \mathbb{R}^N \to \mathbb{R}$

$$J(p_T) = \frac{1}{2} \int_0^T |B^\top p|^2 \, dt + \langle x_0, p(0) \rangle,$$

where p is the solution of the adjoint system (2) with initial datum p_T.

Lemma

Suppose that J has a minimizer $\widehat{p}_T \in \mathbb{R}^N$ and let $\widehat{p}$ be the solution of the adjoint system (2) with initial datum $\widehat{\varphi}_T$. Then

$$u = B^\top \widehat{p}$$

is a control of system (1) with initial datum x_0.

The adjoint problem

PROOF: if $\widehat{p}_T$ is a point where J achieves its minimum value, then

$$\lim_{h\to 0} \frac{J\left(\widehat{p}_T + hp_T\right) - J\left(\widehat{p}_T\right)}{h} = 0, \qquad \text{for all } p_T \in \mathbb{R}^N.$$

This is equivalent to

$$\int_0^T \langle B^\top \widehat{p}, B^\top p\rangle \, dt + \langle x_0, p(0)\rangle = 0, \qquad \text{for all } p_T \in \mathbb{R}^N.$$

In view of the previous lemma, $u = B^\top \widehat{p}$ is a control for (1).

The observability inequality

Remark

Minimizing the functional J requires of its coercivity, that is,

$$\lim_{|p_T|\to+\infty} J(p_T) = +\infty$$

Definition

System (2) is said to be **observable** in time $T > 0$ if there exists $c > 0$ such that

$$\int_0^T |B^\top p|^2\,dt \geq c|p(0)|^2, \tag{4}$$

for all $p_T \in \mathbb{R}^N$, p being the corresponding solution of 2.

In the sequel (4) will be called the **observation** or **observability inequality**. It guarantees that the solution of the adjoint problem at $t = 0$ is uniquely determined by the observed quantity $B^\top p(t)$ for $0 < t < T$.

The observability inequality

The following result is very important in finite dimensional control.

Proposition

Inequality (4) is equivalent to the following **unique continuation principle**:

$$B^\top p(t) = 0 \quad \text{for all } t \in [0, T] \Rightarrow p_T = 0. \tag{5}$$

General principle

UNIQUE CONTINUATION
$\downarrow$
OBSERVABILITY INEQUALITY
$\downarrow$
CONTROLLABILITY

with a constructive procedure to build controls by minimizing a coercive functional.

The Kalman rank condition

What about the observability property? Are there algebraic conditions on the state matrix A and the control one B for it to be true?

The following classical result is due to Kalman and gives a complete answer to the problem of exact controllability of finite dimensional linear systems.

Theorem

System (1) is exactly controllable in some time T if and only if

$$\operatorname{rank}\left[B|AB|A^2B|\cdots|A^{N-1}B\right] = N. \tag{6}$$

The Kalman rank condition

PROOF: for simplifying the notation, from now on we will indicate

$$C := \Big[B|AB|A^2B|\cdots|A^{N-1}B\Big].$$

From the variation of constants formula, we know that (1) is controllable at time T if and only if it holds the identity

$$e^{TA}x_0 + \int_0^T e^{(T-s)A}Bu(s)\,ds = x_T.$$

- Without losing generality, we can assume $x_0 = 0$ by eventually changing the target x_T with $y_T := x_T - e^{TA}x_0$.
- $x_0 = 0 \quad \rightarrow \quad x(T) = \displaystyle\int_0^T e^{(T-s)A}Bu(s)\,ds = x_T = y_T.$

  Hence $x_T = y_T \neq$ if and only if $B \neq 0$.

The Kalman rank condition

PROOF: we introduce the matrices

$$\mathcal{F}(s) := e^{(T-s)A}B \in \mathbb{R}^{N\times M} \quad \text{and} \quad \underbrace{\mathcal{G} := \int_0^T \mathcal{F}(s)\mathcal{F}(s)^\top \, ds \in \mathbb{R}^{N\times N}}_{\textit{Controllability Grammian}}.$$

We can easily see that (1) is controllable if and only if $\mathcal{G}$ is invertible.

- If $\mathcal{G}$ is invertible we can define the control $u^\star(s) := B^\top e^{(T-s)A^\top}\mathcal{G}^{-1}y_T$ and we have

$$\begin{aligned}\int_0^T e^{(T-s)A}Bu^\star(s)\, ds &= \int_0^T e^{(T-s)A}BB^\top e^{(T-s)A^\top}\mathcal{G}^{-1}y_T\, ds \\ &= \int_0^T \mathcal{F}(s)\mathcal{F}(s)^\top \mathcal{G}^{-1}y_T\, ds = \mathcal{G}\mathcal{G}^{-1}y_T = y_T.\end{aligned}$$

- If (1) is controllable ($B \neq 0$)

$$\mathbf{det}(\mathcal{G}) = \int_0^T \mathbf{det}\left(e^{(T-s)A}\right)^2 \mathbf{det}(B)^2\, ds \neq 0.$$

The Kalman rank condition

PROOF: we have that $det(\mathcal{G}) \neq 0$ if and only if the rank condition (6) holds.

- If $\mathbf{det}(\mathcal{G}) = 0$, there exists $0 \neq v \in \mathbb{R}^N$ such that $v^\top \mathcal{G} = 0$. Thus

  $$v^\top \mathcal{G} v = \int_0^T v^\top \mathcal{F}(s)\mathcal{F}(s)^\top v \, ds = 0,$$

  and this is possible if and only if $v^\top \mathcal{F}(s)\mathcal{F}(s)^\top v = 0$ for all $s \in [0, T]$, since $v \mapsto v^\top \mathcal{F}(s)\mathcal{F}(s)^\top v$ is a positive definite quadratic form. In particular, $v^\top \mathcal{F}(s) = 0$ for all $s \in [0, T]$. Therefore,

  $$0 = v^\top \mathcal{F}(s) = v^\top e^{(T-s)A} B = v^\top \left(I + \sum_{k \geq 1} \frac{(T-s)^k}{k!} A^k \right) B,$$

  from which we conclude that $v^\top A^k B = 0$ for all $k \geq 0$.

  Since we are assuming $v \neq 0$, the above identities immediately imply that $v^\top C = 0$, i.e. $\text{rank}(C) < N$.

The Kalman rank condition

PROOF: we have that $det(\mathcal{G}) \neq 0$ if and only if the rank condition (6) holds.

- Finally, let us assume that $\text{rank}(C) < N$. Then, there exists $0 \neq v \in \mathbb{R}^N$ such that $v^\top C = 0$. By definition of C, this implies that

$$v^\top A^k B = 0 \text{ for all } k = 0, \ldots, N-1.$$

Hamilton-Cayley Theorem: $A^N = c_1 A^{N-1} + c_2 A^{N-2} + \ldots + c_N I$. Hence,

$$\begin{aligned} v^\top A^N B &= v^\top \left(c_1 A^{N-1} + c_2 A^{N-2} + \ldots + c_N I \right) B \\ &= c_1 v^\top A^{N-1} B + c_2 c^\top A^{N-2} B + \ldots + c_N v^\top B = 0. \end{aligned}$$

Therefore, $v^\top A^k B = 0 \text{ for all } k \geq 0$, which is equivalent to $v^\top \mathcal{G} = 0$. Hence $\mathcal{G}$ is singular.

Bang-bang control

Bang-bang control

Let us consider the particular case $B \in \mathbb{R}^{N\times 1}$, i. e. $M = 1$, in which only one control $u : [0, T] \to \mathbb{R}$ is available and B is a column vector.

L^1 quadratic functional

$$J_{bb}(p_T) = \frac{1}{2}\left[\int_0^T |B^\top p| \, dt\right]^2 + \langle x_0, p(0)\rangle \tag{7}$$

The same argument as above shows that J_{bb} is also continuous and coercive. It follows that J_{bb} attains a minimum in some point $\widehat{p}_T \in \mathbb{R}^N$.

Bang-bang control

Optimality condition (Euler-Lagrange equation)

$$\int_0^T \left[\operatorname{sgn}(B^\top \widehat{p}) \int_0^T |B^\top \widehat{p}| \, dt\right] B^\top p \, dt + \langle x_0, p(0)\rangle = 0$$

for all $p_T \in \mathbb{R}^N$, where p is the corresponding solution of (2).

This gives the control

$$u_{bb} = \operatorname{sgn}(B^\top \widehat{p})\Lambda_{T,bb} \quad \text{with} \quad \Lambda_{T,bb} = \int_0^T |B^\top \widehat{p}| \, dt,$$

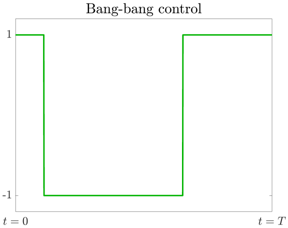
that is of **bang-bang** form, and takes only two values $\pm\Lambda_{T,bb}$ switching finitely many times when the function $B^\top \widehat{p}$ changes sign. It has minimal $L^\infty(0,T)$ norm.

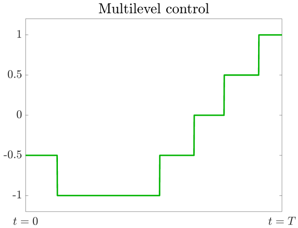
Multilevel control

Multilevel control

Multilevel controls

Generalization of the concept of bang-bang control. They are **piece-wise constant functions** with a finite number o jumps, taking values in a finite-dimensional set.





U. Biccari and E. Zuazua, Multilevel control by duality, 2022

Multilevel control

Multilevel control functional

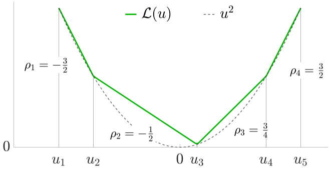
$$J_{ml}(p_T) = \int_0^T \mathcal{L}(B^\top p)\,dt + \langle x_0, p(0)\rangle$$

$\mathcal{L}$: piece-wise affine interpolation of $\mathcal{P}(u) = u^2$.

For conservative or dissipative dynamics, J_{ml} has a minimizer $\widehat{p}_{T,ml}$ provided that the time horizon T **is large enough**.

Control

$$\widehat{u}_{ml} \in \partial\Big(\mathcal{L}(B^\top \widehat{p}_{T,ml})\Big)$$



Multilevel control

Multilevel control functional

$$\mathcal{J}_{ml}(p_T) = \frac{1}{2}\left(\int_0^T \mathcal{L}(B^\top p)\,dt\right)^2 + \langle x_0, p(0)\rangle$$

$\mathcal{L}$: piece-wise affine interpolation of $\mathcal{P}(u) = u^2$.

For general dynamics that satisfy the Kalman rank condition, $\mathcal{J}_{ml}$ has a minimizer $\widehat{p}_{T,ml}$ **for any** $T > 0$.

Control

$$u^*_{ml} \in \Lambda_{T,ml}\partial\Big(\mathcal{L}(B^\top \widehat{p}_{ml})\Big) \quad \text{with} \quad \Lambda_{T,ml} := \int_0^T \mathcal{L}(B^\top \widehat{p}_{ml}(t))\,dt$$

Remark

Bang-bang controls are obtained when $\mathcal{L}(u) = |u|$, which interpolates $\mathcal{P}(u) = u^2$ on the points $\{-1, 0, 1\}$.

Stabilization

The controls we have obtained so far are the so called **open loop** controls. In practice, it is interesting to get **closed loop** or **feedback** controls, so that its value is related in real time with the state itself.

We assume that A is **skew-adjoint**, i. e. $A^\top = -A$. In this case, $\langle Ax, x\rangle = 0$.

Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{8}$$

When $u \equiv 0$, the energy of the solution of (8) is conserved. Indeed, by multiplying (8) by x, if $u \equiv 0$, one obtains

$$\frac{d}{dt}|x(t)|^2 = 2\langle x'(t), x(t)\rangle = 2\langle Ax(t), x(t)\rangle = 0 \quad \to \quad |x(t)| = |x_0|, \quad \text{for all } t \geq 0.$$

Stabilization

Stabilization problem

Suppose that the pair (A, B) is controllable. We then look for a matrix L such that the solution of system (8) with the **feedback control law**

$$u(t) = Lx(t) \tag{9}$$

has a **uniform exponential decay**, i.e. there exist $c > 0$ and $\omega > 0$ such that

$$|x(t)| \leq ce^{-\omega t}|x_0| \tag{10}$$

for any solution.

Note that, according to the law (9), the control u is obtained in real time from the state x.

Stabilization

In other words, we are looking for matrices L such that the solution of the system

$$x' = (A + BL)x = Dx \tag{11}$$

has an uniform exponential decay rate.

Theorem

If A is skew-adjoint and the pair (A, B) is controllable then $L = -B^\top$ stabilizes the system, i.e. the solution of

$$\begin{cases} x' = Ax - BB^\top x \\ x(0) = x_0 \end{cases} \tag{12}$$

has a uniform exponential decay (10).

PROOF: with $L = -B^\top$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = -\langle BB^\top x(t), x(t)\rangle = -|B^\top x(t)|^2 \leq 0.$$

Hence, the norm of the solution decreases in time. Moreover,

$$|x(T)|^2 - |x_0|^2 = -2\int_0^T |B^\top x|^2 dt. \tag{13}$$

To prove the uniform exponential decay it is sufficient to show that there exist $T > 0$ and $c > 0$ such that

$$|x_0|^2 \leq c\int_0^T |B^\top x|^2\, dt \tag{14}$$

for any solution x of (12).

Indeed, from (13) and (14) we would obtain that

$$|x(T)|^2 - |x_0|^2 \leq -\frac{2}{c}|x(0)|^2 \quad \to \quad |x(T)|^2 \leq \gamma |x_0|^2 \quad \text{with } \gamma = 1 - \frac{2}{c} < 1.$$

Hence, since $\gamma < 1$

$$|x(kT)|^2 \leq \gamma^k |x_0|^2 = e^{k\ln(\gamma)}|x_0|^2 = e^{-k|\ln(\gamma)|}|x_0|^2, \quad \text{for all } k \in \mathbb{N}.$$

Now, given any $t > 0$ we write it in the form $t = kT + \delta$, with $\delta \in [0, T)$ and $k \in \mathbb{N}$ and we obtain that

$$\begin{aligned} |x(t)|^2 &\leq |x(kT)|^2 \leq e^{-k|\ln(\gamma)|}|x_0|^2 \\ &= e^{-\left(\frac{t}{T}\right)|\ln(\gamma)|} e^{\frac{\delta}{T}|\ln(\gamma)|}|x_0|^2 \leq \frac{1}{\gamma} e^{-\frac{|\ln(\gamma)|}{T}t}|x_0|^2. \end{aligned}$$

We have obtained the desired decay result (10) with

$$c = \frac{1}{\gamma} \quad \text{and} \quad \omega = \frac{|\ln(\gamma)|}{T}.$$

Stabilization

To prove (14) we decompose the solution x of (12) as $x = \varphi + y$ with φ and y solutions of the following systems:

$$\begin{cases} \varphi' = A\varphi \\ \varphi(0) = x_0 \end{cases} \tag{15}$$

and

$$\begin{cases} y' = Ay - BB^\top x \\ y(0) = 0. \end{cases} \tag{16}$$

Remark

Since A is skew-adjoint, (15) is exactly the adjoint system (2) except for the fact that the initial datum is taken at $t = 0$.

The pair (A, B) being controllable, the following observability inequality holds for system (15):

$$|x_0|^2 \leq C \int_0^T |B^\top \varphi|^2 \, dt. \tag{17}$$

Since $\varphi = x - y$ we deduce that

$$|x_0|^2 \leq 2C \left[\int_0^T |B^\top x|^2 \, dt + \int_0^T |B^\top y|^2 \, dt \right].$$

On the other hand, it is easy to show that the solution y of (16) satisfies

$$\frac{1}{2}\frac{d}{dt}|y(t)|^2 = -\langle B^\top x(t), B^\top y(t)\rangle$$
$$\leq |B^\top x(t)|\,|B^\top|\,|y(t)| \leq \frac{1}{2}\left(|y(t)|^2 + |B^\top|^2|B^\top x(t)|^2\right).$$

From **Gronwall's inequality** we deduce that

$$|y(t)|^2 \leq |B^\top|^2 \int_0^t e^{t-s}|B^\top x(s)|^2\,ds \leq |B^\top|^2 e^T \int_0^T |B^\top x(s)|^2\,ds$$

and consequently

$$\int_0^T |B^\top y(t)|^2\,dt \leq |B|^2 \int_0^T |y(t)|^2\,dt \leq T|B|^4 e^T \int_0^T |B^\top x(t)|^2\,dt.$$

Finally, we obtain that

$$|x_0|^2 \leq 2C \int_0^T |B^\top x(t)|^2\,dt + C|B^\top|^4 T e^T \int_0^T |B^\top x(t)|^2\,dt \leq C \int_0^T |B^\top x(t)|^2\,dt.$$

Example

Damped harmonic oscillator

$$mx'' + Rx + kx' = 0, \tag{18}$$

where m, k and R are positive constants.

Note that (18) may be written in the equivalent form

$$mx'' + Rx = -kx'$$

which indicates that an applied force, proportional to the velocity of the point-mass and of opposite sign, is acting on the oscillator.

Example

It is easy to see that the solutions of this equation have an exponential decay property. Indeed, it is sufficient to remark that the two characteristic roots have negative real part.

Damped harmonic oscillator

$$mr^2 + R + kr = 0 \quad \Leftrightarrow \quad r_\pm = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\Re(r_\pm) = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \leq 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \geq 4mR \end{cases}$$

We observe here the classical **overdamping phenomenon**. Contradicting a first intuition it is not true that the decay rate increases when the value of the damping parameter k increases.

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