CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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PART I: introduction to control theory

LECTURE 2: infinite-dimensional control systems

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Infinite-dimensional linear control

Infinite-dimensional control

Linear infinite-dimensional control problem

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{1}$$

- $A : \mathcal{D}(A) \subseteq H \to H$: linear operator generating a strongly continuous semi-group $S(t)_{t \geq 0}$.
- $B \in \mathcal{L}(U; \mathcal{D}(A))$: control operator.
- $(H, \langle \cdot, \cdot \rangle_H)$ and $(U, \langle \cdot, \cdot \rangle_U)$ Hilbert spaces.
- $x_0 \in H$.

Infinite-dimensional control

Well-posedness

Under the **admissibility condition**

$$\int_0^T \|B^*(t)S^*(t)z\|_U^2 \, dt \leq C_T \, \|z\|_H^2 \, , \quad \text{for all } z \in \mathcal{D}(A^*),$$

the Cauchy problem (1) is **well-posed in the sense of Hadamard**, i.e., for every $x_0 \in H$ and $u \in L^2(0,T;U)$ there exists a unique solution $x \in C([0,T];H)$ satisfying (1). Moreover,

$$\|x\|_{C([0,T];H)} \leq C\Big(\|x_0\|_H + \|u\|_{L^2(0,T;U)} \Big),$$

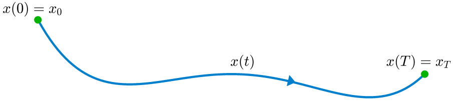
for a positive constant $C > 0$ depending on T, A and B.

Controllability notions

Exact controllability

System (1) is **exactly controllable** at time T if, for any $x_0, x_T \in H$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution x fulfills $x(T) = x_T$.

According to this definition the aim of exact controllability consists in driving the solution x of (1) from the initial state x_0 to the final one x_T in time T by acting on the system through the control u.

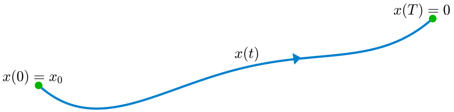


Controllability notions

Null controllability

System (1) is **null controllable** at time T if, for any $x_0 \in H$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution x fulfills $x(T) = 0$.

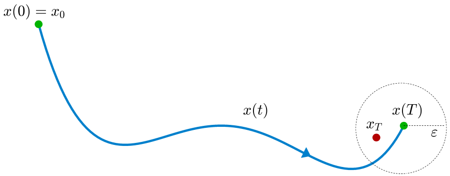
According to this definition the aim of null controllability consists in driving the solution x of (1) from the initial state x_0 to zero in time T by acting on the system through the control u.



Approximate controllability

System (1) is **approximately controllable** at time T if, for any $x_0, x_T \in H$ and any $\varepsilon > 0$, there exists $u \in L^2(0, T; U)$ such that the corresponding solution x fulfills $\|x(T) - x_T\|_H < \varepsilon$.

According to this definition the aim of approximate controllability consists in driving the solution x of (1) in time T from the initial state x_0 to a final one $x(T)$ which is ε-close to x_T by acting on the system through the control u.

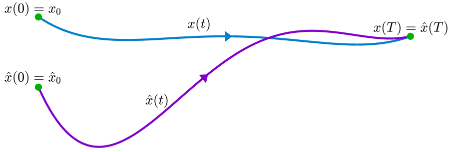


Controllability notions

Controllability to trajectories

System (1) is **exactly controllable to trajectory** at time T if, for any $x_0 \in H$ and any solution $\hat{x}$ of (1) with $\hat{x}(0) = \hat{x}_0 \in H$ and some given $\hat{u}$, there exists a control $u \in L^2(0,T;U)$ such that the corresponding solution x fulfills $x(T) = \hat{x}(T)$.

According to this definition the aim of controllability to trajectories consists in driving the solution x of (1) in time T from the initial state x_0 to match a particular solution $\hat{x}(T)$ by acting on the system through the control u.



The reachable set

Set of reachable states

$$R(T, x_0) = \left\{ x(T) \in H \ : \ x \text{ solution of (1) with } u \in (L^2(0,T); U) \right\}.$$

Remark

$R(T, x_0)$ is a convex subset of H.

The controllability notions previously introduced can be redefined through the reachable set

Exact controllability: $R(T, x_0) = H$ for any $x_0 \in H$.

Null controllability: $0 \in R(T, x_0)$ for any $x_0 \in H$.

Approximate controllability: $R(T, x_0)$ is dense in H for any $x_0 \in H$.

ATTENTION!

In infinite-dimensional control, exact controllability, null controllability and approximate controllability **are not equivalent in general**.

The observability property

The adjoint problem

Let A^* be the **adjoint** A, i.e. the operator such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \text{for all } x, y \in H.$$

Consider the following homogeneous **adjoint system** of (1):

$$\begin{cases} -p'(t) = A^* p(t), & t \in (0, T) \\ p(T) = p_T \end{cases} \tag{2}$$

We have the following equivalent condition for exact controllability.

Lemma

An initial datum $x_0 \in H$ of (1) is driven to zero in time T by using a control $u \in L^2(0, T; U)$ if and only if

$$\int_0^T \langle u, B^* p \rangle \, dt + \langle x_0, p(0) \rangle = 0, \qquad \text{for all } p_T \in H, \tag{3}$$

p being the corresponding solution of (2).

The adjoint problem

Identity (3) is in fact an optimality condition for the critical points of the quadratic functional $J : H \to \mathbb{R}$

$$J(p_T) = \frac{1}{2}\int_0^T |B^* p|^2 \, dt + \langle x_0, p(0) \rangle,$$

where p is the solution of the adjoint system (2) with initial datum p_T.

Lemma

Suppose that J has a minimizer $\widehat{p}_T \in H$ and let $\widehat{p}$ be the solution of the adjoint system (2) with initial datum $\widehat{\varphi}_T$. Then

$$u = B^* \widehat{p}$$

is a control of system (1) with initial datum x_0.

Remark

Minimizing the functional J requires of its coercivity, that is,

$$\lim_{|p_T|\to+\infty} J(p_T) = +\infty$$

Definition

System (2) is said to be **observable** in time $T > 0$ if there exists $c > 0$ such that

$$\int_0^T |B^* p|^2 \, dt \geq c|p(0)|^2, \tag{4}$$

for all $p_T \in H$, p being the corresponding solution of 2.

In the sequel (4) will be called the **observation** or **observability inequality**. It guarantees that the solution of the adjoint problem at $t = 0$ is uniquely determined by the observed quantity $B^* p(t)$ for $0 < t < T$.

The wave equation

Controllability of the wave equation

Let $N \geq 1$ and $T > 0$, Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary, $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$.

Controlled wave equation - interior control

$$\begin{cases} y_{tt} - \Delta y = u\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{5}$$

χ_ω denotes the **characteristic function** of the subset $\omega \subset \Omega$ where the control is active.

We assume that $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$ and $u \in L^2(Q)$ so that (5) admits a unique weak solution

$$(y, y_t) \in C\left([0,T]\,; H^1_0(\Omega) \times L^2(\Omega)\right).$$

given by the variation of constants formula

$$(y, y_t)(t) = S(t)(y_0, y_1) + \int_0^t S(t-s)(0, u(s)\chi_\omega)\, ds. \tag{6}$$

Remark

The wave equation is **reversible in time**. Hence, we may solve it for $t \in (0, T)$ by considering initial data (y_0, y_1) in $t = 0$ or final data $(y_{0,T}, y_{1,T})$ in $t = T$. In the former case the solution is given by (6) and in the latter one by

$$(y, y_t)(t) = S(T-t)(y_{0,T}, y_{1,T}) + \int_{T-t}^{T} S(s-T+t)(0, u(s)\chi_\omega)\, ds.$$

Controllability of the wave equation

Let $N \geq 1$ and $T > 0$, Ω be a bounded open set of $\mathbb{R}^N$ with smooth boundary $\Gamma := \partial\Omega$, and let Γ_0 be an open nonempty subset of Γ. Denote $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$.

Controlled wave equation - boundary control

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q \\ y = u\chi_{\Gamma_0} & \text{on } \Sigma \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{7}$$

χ_{Γ_0} denotes the **characteristic function** of the subset $\Gamma_0 \subset \Gamma$ where the control is active.

We assume that $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $u \in L^2(\Sigma)$ so that (7) admits a unique very weak solution **defined by transposition**

$$(y, y_t) \in C([0,T]; L^2(\Omega) \times H^{-1}(\Omega)).$$

Controllability of the wave equation

EXACT CONTROLLABILITY: to find a control function u such that $(y(\cdot,T), y_t(\cdot,T)) = (y_T, y'_T)$ in Ω.

NULL CONTROLLABILITY: to find a control function u such that $y(\cdot,T) = y_T(\cdot,T) = 0$ in Ω.

In view of the reversibility, **exact and null controllability are equivalent concepts in the context of the wave equation**.

Proposition

System (7) is exactly controllable if and only if it is null controllable.

PROOF: exact controllability of (7) implies null controllability since, clearly, $(0,0) \in L^2(\Omega) \times H^{-1}(\Omega)$.

In view of the reversibility, **exact and null controllability are equivalent concepts in the context of the wave equation**.

Proposition

System (7) is exactly controllable if and only if it is null controllable.

PROOF: suppose now that (7) is null controllable, i.e., $(0,0) \in R(T;(y_0,y_1))$ for any initial datum $(y_0,y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. It follows that any initial datum in $L^2(\Omega) \times H^{-1}(\Omega)$ can be driven to $(0,0)$ in time T by the control u.

Since the wave equation is time-reversible, we deduce that any state in $L^2(\Omega)\times H^{-1}(\Omega)$ can be reached in time T by starting from $(0,0)$. This means that $R(T,(0,0)) = L^2(\Omega) \times H^{-1}(\Omega)$.

Moreover, the linearity of (7) implies that

$$R(T;(y_0,y_1)) = R(T;(0,0)) + S(T)(y_0,y_1).$$

The exact controllability property holds from these two facts.

The adjoint problem

Adjoint system

$$\begin{cases} p_{tt} - \Delta p = 0, & \text{in } Q \\ p = 0, & \text{on } \Sigma \\ p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), & \text{in } \Omega. \end{cases} \tag{8}$$

Recall that, for any initial datum $(p_0, p_1) \in H_0^1(\Omega) \times L^2(\Omega)$, (8) admits a unique weak solution $(p, p_t) \in C([0,T]; H_0^1(\Omega) \times L^2(\Omega))$.

Moreover, the energy

$$E(t) = \frac{1}{2} \int_\Omega \left(|p_t(x,t)|^2 + |\nabla p(x,t)|^2 \right) dx$$

is conserved in time:

$$\frac{d}{dt} E(t) = 0$$

$$\downarrow$$

$$E(t) = E(0) = \frac{1}{2} \int_\Omega \left(|p_1|^2 + |\nabla p_0|^2 \right) dx \sim \|p_0\|_{H_0^1(\Omega)} + \|p_1\|_{L^2(\Omega)}.$$

The adjoint problem

For all $(p_0, p_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, we shall introduce the duality product

$$\langle (p_0, p_1), (y_0, y_1) \rangle := \int_\Omega p_1 y_0 \, dx - \langle y_1, p_0 \rangle_{1,-1},$$

where with $\langle \cdot, \cdot \rangle_{1-1}$ we indicate the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

We have the following equivalent condition for boundary controllability.

Lemma

An initial datum $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ in (7) is controllable to zero if and only if there exists $u \in L^2((0,T) \times \Gamma_0)$ such that

$$\int_0^T \int_{\Gamma_0} \frac{\partial p}{\partial \nu} u \, d\sigma dt + \langle (p_0, p_1), (y_0, y_1) \rangle = 0, \tag{9}$$

for all $(p_0, p_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where p is the solution of (8).

The adjoint problem

Identity (11) is in fact an optimality condition for the critical points of the quadratic functional $J : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$

$$J(p_0, p_1) = \frac{1}{2}\int_0^T \int_{\Gamma_0} \left(\frac{\partial p}{\partial \nu}\right)^2 d\sigma dt + \langle (p_0, p_1), (y_0, y_1) \rangle,$$

where p is the solution of the adjoint system (8) with initial datum (p_0, p_1).

Lemma

Suppose that J has a minimizer $(\widehat{p}_0, \widehat{p}_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and let $\widehat{p}$ be the solution of the adjoint system (8) with initial datum $(\widehat{p}_0, \widehat{p}_1)$. Then

$$u = \left.\frac{\partial \widehat{p}}{\partial \nu}\right|_{\Gamma_0}$$

is a control of system (7) with initial datum $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

The observability inequality

Minimizing the functional J requires of its coercivity, that is given by the observability inequality

$$E(0) \sim \|p_0\|^2_{H^1_0(\Omega)} + \|p_1\|^2_{L^2(\Omega)} \leq C \int_0^T \int_{\Gamma_0} \left(\frac{\partial p}{\partial \nu} \right)^2 d\sigma dt. \tag{10}$$

The adjoint problem

For the case of interior controllability we have a similar situation.

Lemma

An initial datum $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ in (5) is controllable to zero if and only if there exists $u \in L^2(\omega)$ such that

$$\int_0^T \int_\omega pu \, dxdt + \int_\Omega p_0 y_1 \, dx - \langle y_0, p_1 \rangle_{1,-1} = 0, \tag{11}$$

for all $(p_0, p_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and where p is the solution of (8).

Identity (11) is in fact an optimality condition for the critical points of the quadratic functional $J : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$

$$J(p_0, p_1) = \frac{1}{2} \int_0^T \int_\omega |p|^2 \, dxdt + \int_\Omega p_0 y_1 \, dx - \langle y_0, p_1 \rangle_{1,-1}$$

where p is the solution of the adjoint system (8) with initial datum (p_0, p_1).

Lemma

Suppose that J has a minimizer $(\widehat{p}_0, \widehat{p}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and let $\widehat{p}$ be the solution of the adjoint system (8) with initial datum $(\widehat{p}_0, \widehat{p}_1)$. Then

$$u = p|_\omega$$

is a control of system (5) with initial datum $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

The observability inequality

Minimizing the functional J requires of its coercivity, that is given by the observability inequality

$$\|p_0\|^2_{L^2(\Omega)} + \|p_1\|^2_{H^{-1}(\Omega)} \leq C \int_0^T \int_\omega |p|^2 \, dxdt. \tag{12}$$

The observability inequality

There are several ways of proving the observability inequalities (10) and (12). The most classical ones are the following.

Space-dimension $N = 1$

- ▷ Ingham's inequalities

Space-dimension $N \geq 1$

- ▷ Multiplier method

Ingham's inequalities

Ingham's inequalities

Theorem

Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that $\lambda_{k+1} - \lambda_k \geq \gamma > 0$, for all $k \in \mathbb{Z}$. Then, for any real $T > \pi/\gamma$, there exists a positive constant $\mathcal{C} = \mathcal{C}(T,\gamma) > 0$ such that, for any finite sequence $(a_k)_{k\in\mathbb{Z}}$,

$$\mathcal{C}\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_{-T}^{T} \left|\sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t}\right|^2 dt. \tag{13}$$

PROOF: first of all, notice that we can reduce the problem to the case $T = \pi$ and $\gamma > 1$ since, if $T\gamma > \pi$, then the change of variables $s = (T/\pi)t$ yields

$$\int_{-T}^{T} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t} \right|^2 dt = \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k s} \right|^2 ds,$$

where

$$\mu_k := \frac{T\lambda_k}{\pi}$$

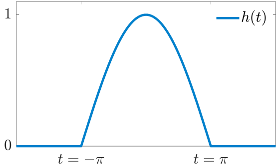
$$\mu_{k+1} - \mu_k = \frac{T}{\pi}(\lambda_{k+1} - \lambda_k) \geq \gamma_1 := \frac{T\gamma}{\pi} > 1.$$

Hence, it will be sufficient to prove the existence of another positive constant (still denoted by $\mathcal{C}$) such that

$$\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \mathcal{C} \int_{-\pi}^{\pi} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt.$$

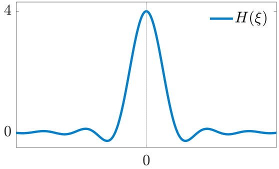
PROOF: consider now the function $h : \mathbb{R} \to \mathbb{R}$ defined as

$$h(t) = \begin{cases} \cos\left(\frac{t}{2}\right), & \text{if } |t| \leq \pi \\ 0, & \text{if } |t| > \pi \end{cases}$$



whose Fourier transform is given by

$$H(\xi) = \int_{-\infty}^{\infty} h(t)e^{i\xi t}\, dt = \frac{4\cos(\pi\xi)}{1 - 4\xi^2}$$



PROOF: since $0 \leq h(t) \leq 1$ for all $t \in [-\pi, \pi]$ and $H(\xi)$ is an even function, we have that

$$\int_{-\pi}^{\pi} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \geq \int_{-\pi}^{\pi} h(t) \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt = \int_{-\pi}^{\pi} h(t) \sum_{k,\ell\in\mathbb{Z}} a_k \bar{a}_\ell e^{i(\mu_k-\mu_\ell)t}\, dt$$

$$= \sum_{k,\ell\in\mathbb{Z}} a_k \bar{a}_\ell H(\mu_k-\mu_\ell) = H(0)\sum_{k\in\mathbb{Z}} |a_k|^2 + \sum_{k\neq\ell} a_k\bar{a}_\ell H(\mu_k-\mu_\ell)$$

$$\geq 4\sum_{k\in\mathbb{Z}} |a_k|^2 - \frac{1}{2}\sum_{k\neq\ell}(|a_k|^2+|a_\ell|^2)|H(\mu_k-\mu_\ell)|$$

$$= 4\sum_{k\in\mathbb{Z}} |a_k|^2 - \sum_{k\in\mathbb{Z}} |a_k|^2 \sum_{k\neq\ell} |H(\mu_k-\mu_\ell)|.$$

PROOF: on the other hand,

$$\sum_{k\neq\ell}|H(\mu_k-\mu_\ell)| \leq \sum_{k\neq\ell}\frac{4}{4|\mu_k-\mu_\ell|^2-1} \leq \sum_{k\neq\ell}\frac{4}{4\gamma_1^2|k-\ell|^2-1}$$
$$= \sum_{r\geq 1}\frac{8}{4\gamma_1^2 r^2-1} \leq \frac{8}{\gamma_1^2}\sum_{r\geq 1}\frac{1}{4r^2-1} = \frac{4}{\gamma_1^2}.$$

Therefore,

$$\int_{-\pi}^{\pi}\left|\sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\right|^2 dt \geq \left(4-\frac{4}{\gamma_1^2}\right)\sum_{k\in\mathbb{Z}}|a_k|^2$$

and the proof is concluded by taking

$$\mathcal{C} = \frac{T}{\pi}\left(4-\frac{4}{\gamma_1^2}\right) = \frac{4\pi}{T}\left(T^2-\frac{\pi^2}{\gamma^2}\right).$$

Notice that the assumption $T > \pi/\gamma$ is necessary for the positivity of $\mathcal{C}$.

Ingham's inequalities

Theorem

Let $(\lambda_k)_{k\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that $\lambda_{k+1} - \lambda_k \geq \gamma > 0$, for all $k \in \mathbb{Z}$. Then, for any real $T > 0$, there exists a positive constant $\mathcal{C} = \mathcal{C}(T,\gamma) > 0$ such that, for any finite sequence $(a_k)_{k\in\mathbb{Z}}$,

$$\int_{-T}^{T} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t} \right|^2 dt \leq \mathcal{C} \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{14}$$

PROOF: let us first consider the case $T\gamma \geq \pi/2$, and notice that, as in the proof of the previous theorem, we can reduce the problem to $T = \pi/2$ and $\gamma \geq 1$. Indeed

$$\int_{-T}^{T} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t} \right|^2 dt = \frac{2T}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k s} \right|^2 ds,$$

where

$$\mu_k := \frac{2T\lambda_k}{\pi}$$

$$\mu_{k+1} - \mu_k = \frac{2T}{\pi}(\lambda_{k+1} - \lambda_k) \geq \gamma_1 := \frac{2T\gamma}{\pi} > 1.$$

PROOF: let $h(t)$ be the function introduced in the previous theorem. Since $\sqrt{2}/2 \leq h(t) \leq 1$ for all $t \in [-\pi/2, \pi/2]$, we obtain that

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt &\leq 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(t) \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq 2 \int_{-\pi}^{\pi} h(t) \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \\
&= 2 \sum_{k,\ell\in\mathbb{Z}} a_k \bar{a}_\ell H(\mu_k - \mu_\ell) = 8 \sum_{k\in\mathbb{Z}} |a_k|^2 + 2 \sum_{k\neq\ell} a_k \bar{a}_\ell H(\mu_k - \mu_\ell) \\
&\leq 8 \sum_{k\in\mathbb{Z}} |a_k|^2 + \sum_{k\neq\ell} (|a_k|^2 + |a_\ell|^2) |H(\mu_k - \mu_\ell)| \\
&\leq 8 \left(1 + \frac{1}{\gamma_1^2} \right) \sum_{k\in\mathbb{Z}} |a_k|^2,
\end{aligned}
$$

where we used the fact that

$$\sum_{k\neq\ell} |H(\mu_k - \mu_\ell)| < \frac{4}{\gamma_1^2}.$$

Then, (14) follows immediately with

$$\mathcal{C} = 8 \left(\frac{4T^2}{\pi^2} + \frac{1}{\gamma^2} \right).$$

PROOF: when $T\gamma < \pi/2$, instead, we have that

$$\int_{-T}^{T} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t} \right|^2 dt = \frac{1}{\gamma} \int_{-T\gamma}^{T\gamma} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\frac{\lambda_k}{\gamma}s} \right|^2 ds \leq \frac{1}{\gamma} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\frac{\lambda_k}{\gamma}s} \right|^2 ds.$$

Moreover, since $(\lambda_{k+1} - \lambda_k)/\gamma \geq 1$ from the analysis of the previous case we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\frac{\lambda_k}{\gamma}s} \right|^2 ds \leq 16 \sum_{k\in\mathbb{Z}} |a_k|^2,$$

and (14) follows with

$$\mathcal{C} = \frac{16}{\gamma}.$$

Joining the two cases, we finally obtain that (14) holds for all $T > 0$ with

$$\mathcal{C} = 8\max\left\{ \frac{4T^2}{\pi^2} + \frac{1}{\gamma^2}, \frac{2}{\gamma} \right\}.$$

Remark

Notice that (14) holds for all $T > 0$ while, instead, (13) requires the length T of the time interval to be sufficiently large, depending on the gap γ between two consecutive exponents λ_k. In view of that, when the gap becomes small the value of T must increase proportionally.

Remark

The constant $\mathcal{C}$ in (13) blows-up when T goes to π/γ. In this critical case, the inequality may hold or not, depending on the particular family of exponential functions.

Remark

The length T of the time interval in (13) does not depend on the smallest distance between two consecutive exponents but on the asymptotic gap defined by

$$\gamma_\infty = \liminf_{|k|\to+\infty} |\lambda_{k+1} - \lambda_k|. \tag{15}$$

An induction argument due to A. Haraux allows to give an Ingham-type inequality in which condition the gap condition for γ is replaced by a similar one for γ_∞.

Remarks on Ingham's inequalities

Theorem

Let $(\lambda_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers such that $\lambda_{k+1} - \lambda_k \geq \gamma > 0$ for any $k \in \mathbb{Z}$, and let $\gamma_\infty > 0$ be given by (15). Then, for any real $T > \pi/\gamma_\infty$ there exist two positive constants $\mathcal{C}_1, \mathcal{C}_2 > 0$ such that, for any finite sequence $(a_k)_{k\in\mathbb{Z}}$,

$$\mathcal{C}_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_{-T}^{T} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\lambda_k t} \right|^2 dt \leq \mathcal{C}_2 \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{16}$$

Spectral analysis for the wave operator

We give here a **Fourier expansion** for the solutions of the one-dimensional wave equation

$$\begin{cases} p_{tt} - p_{xx} = 0, & (x,t) \in (0,1) \times (0,T) \\ p(0,t) = p(1,t) = 0, & t \in (0,T) \\ p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x), & x \in (0,1) \end{cases} \tag{17}$$

as a preliminary tool for obtaining observability properties by means of the Ingham's inequalities previously presented.

Let us firstly remark that (17) can be rewritten as an abstract Cauchy problem

$$\begin{cases} \Phi_t + A\Phi = 0, & (x,t) \in (0,1) \times (0,T) \\ \Phi(x,0) = \Phi_0(x), & x \in (0,1) \end{cases} \tag{18}$$

where $\Phi = (p, p_t)^\top$, $\Phi_0 = (p_0, p_1)^\top$ and A is the unbounded operator in $H := L^2(0,1) \times H^{-1}(0,1)$, $A : \mathcal{D}(A) \subset H \to H$, defined by

$$\mathcal{D}(A) = H_0^1(0,1) \times L^2(0,1), \quad A = \begin{pmatrix} 0 & -I \\ -\partial_x^2 & 0 \end{pmatrix}.$$

A is an isomorphism from $H_0^1(0,1) \times L^2(0,1)$ to $L^2(0,1) \times H^{-1}(0,1)$.

Lemma

The eigenvalues of A are $\lambda_k = ik\pi$, $k \in \mathbb{Z}^*$. The corresponding eigenfunctions are given by

$$\Phi_k = \left(\frac{1}{\lambda_k}, -1\right)^\top \sin(k\pi x), \quad k \in \mathbb{Z}^*, \tag{19}$$

and form an orthonormal basis in $H_0^1(0,1) \times L^2(0,1)$.

Spectral analysis for the wave operator

Since $(\Phi_k)_{k\in\mathbb{Z}^*}$ is an orthonormal basis in $H_0^1(0,1)\times L^2(0,1)$ and A is an isomorphism from $H_0^1(0,1)\times L^2(0,1)$ to $L^2(0,1)\times H^{-1}(0,1)$, we have that also $(A(\Phi_k))_{k\in\mathbb{Z}^*}$ is an orthonormal basis in $L^2(0,1)\times H^{-1}(0,1)$. Moreover $(\lambda_k\Phi_k)_{k\in\mathbb{Z}^*}$ is an orthonormal basis in $L^2(0,1)\times H^{-1}(0,1)$, and we have that

- $\Phi=\sum_{k\in\mathbb{Z}^*} a_k\Phi_k \in H_0^1(0,1)\times L^2(0,1)$ if and only if $\sum_{k\in\mathbb{Z}^*}|a_k|^2<+\infty$.
- $\Phi=\sum_{k\in\mathbb{Z}^*} a_k\Phi_k \in L^2(0,1)\times H^{-1}(0,1)$ if and only if $\sum_{k\in\mathbb{Z}^*}\frac{|a_k|^2}{|\lambda_k|^2}<+\infty$.

In addition, the solution of the Cauchy problem (18) corresponding to an initial datum

$$\Phi_0=\sum_{k\in\mathbb{Z}^*} a_k\Phi_k \in L^2(0,1)\times H^{-1}(0,1)$$

is given by

$$\Phi(t)=\sum_{k\in\mathbb{Z}^*} a_k e^{\lambda_k t}\Phi_k.$$

Boundary observability inequality

By means of Ingham's inequality we can prove the following.

Theorem

Let $T \geq 2$. There exists a positive constant $\mathcal{C} > 0$ such that, for any initial datum $(p_0, p_1) \in H_0^1(0,1) \times L^2(0,1)$, the corresponding solution p of (17) satisfies

$$\mathcal{C} \,\|(p_0, p_1)\|^2_{H_0^1(0,1)\times L^2(0,1)} \leq \int_0^T |p_x(1,t)|^2 \, dt. \tag{20}$$

Boundary observability inequality

PROOF: if $(p_0, p_1) = \sum_{k\in\mathbb{Z}^*} a_k \Phi_k$ then, using the orthonormality of the eigenfunctions on $H_0^1(0,1) \times L^2(0,1)$ we have

$$\|(p_0, p_1)\|^2_{H_0^1(0,1)\times L^2(0,1)} = \sum_{k\in\mathbb{Z}^*} |a_k|^2.$$

On the other hand,

$$\int_0^T |p_x(1,t)|^2 \, dx = \int_0^T \left| \sum_{k\in\mathbb{Z}^*} (-1)^k a_k e^{ik\pi t} \right|^2 dt.$$

Hence, (20) reduces to the following inequality

$$\mathcal{C} \sum_{k\in\mathbb{Z}^*} |a_k|^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}^*} (-1)^k a_k e^{ik\pi t} \right|^2 dt. \qquad (21)$$

PROOF: notice that (21) is in the form of an Ingham's inequality, in which $\lambda_k = k\pi$ and the family $(\lambda_k)_{k\in\mathbb{Z}^*}$ satisfies the gap condition with $\gamma = \pi$. Hence, since the time integration in the interval $[0, T]$ is equivalent, up to a change of variables, to considering $t \in [-T/2, T/2]$, from (13) we have that (21) holds for any $T > 2\pi/\gamma = 2$.

Finally, when $T = 2$, by using the orthogonality in $L(0, 2)$ of the exponentials $(e^{ik\pi t})_{k\in\mathbb{Z}^*}$, we immediately get that

$$\int_0^2 \left| \sum_{k\in\mathbb{Z}^*} (-1)^k a_k e^{ik\pi t} \right|^2 dt = \sum_{k\in\mathbb{Z}^*} |a_k|^2.$$

Hence (20) is actually an identity.

Interior observability inequality

By means of Ingham's inequality we can prove the following.

Theorem

Let $T \geq 2$. There exists a positive constant $\mathcal{C} > 0$ such that, for any initial datum $(p_0, p_1) \in L^2(0,1) \times H^{-1}(0,1)$, the corresponding solution p of (17) satisfies

$$\mathcal{C}\, \|(p_0, p_1)\|^2_{L^2(0,1)\times H^{-1}(0,1)} \leq \int_0^T \int_\omega |p|^2\, dxdt. \tag{22}$$

Remark

The need of a large time horizon ($T > 2$ in this case) for observability is not only a consequence of Ingham's inequality. It is actually an intrinsic property of hyperbolic systems, related to the finite velocity of propagation of their solutions.

Interior observability inequality

PROOF: since A is an isomorphism form $H_0^1(0,1)\times L^2(0,1)$ into $L^2(0,1)\times H^{-1}(0,1)$, and $A^{-1}\Phi_k=\lambda_k^{-1}\Phi_k$, we have

$$\|(p_0,p_1)\|^2_{L^2(0,1)\times H^{-1}(0,1)}=\left\|A^{-1}(p_0,p_1)\right\|^2_{H_0^1(0,1)\times L^2(0,1)}=\sum_{k\in\mathbb{Z}^*}\frac{|a_k|^2}{k^2\pi^2}.$$

On the other hand, we obtain from Fubini's Theorem that

$$\int_0^T\int_\omega |p|^2\,dxdt=\int_\omega\int_0^T\left|\sum_{k\in\mathbb{Z}^*}\frac{a_k}{k\pi}e^{i\lambda_k t}\sin(k\pi x)\right|^2\,dtdx.$$

Hence, (20) is equivalent to the following inequality

$$\mathcal{C}\sum_{k\in\mathbb{Z}^*}\frac{|a_k|^2}{k^2\pi^2}b_k\le\int_\omega\int_0^T\left|\sum_{k\in\mathbb{Z}^*}\frac{a_k}{k\pi}e^{i\lambda_k t}\sin(k\pi x)\right|^2\,dtdx,\tag{23}$$

where we denoted

$$b_k:=\int_\omega \sin^2(k\pi x)\,dx.$$

Interior observability inequality

PROOF: notice that, for $T > 2$, (23) holds true by applying (13) with $\lambda_k = k\pi$ and by replacing the family $(a_k)_{k\in\mathbb{Z}^*}$ with $(\frac{a_k}{\lambda_k}\sqrt{b_k})_{k\in\mathbb{Z}^*}$.

Moreover, when $T = 2$, by using again the orthogonality in $L(0,2)$ of the exponentials $(e^{ik\pi t})_{k\in\mathbb{Z}^*}$, we immediately get that

$$\int_\omega \int_0^2 \left| \sum_{k\in\mathbb{Z}^*} \frac{a_k}{k\pi} e^{i\lambda_k t} \sin(k\pi x) \right|^2 dtdx = \sum_{k\in\mathbb{Z}^*} \frac{|a_k|^2}{k^2\pi^2} b_k.$$

Hence, we can conclude that (23) holds true for all $T \geq 2$.

Interior observability inequality

PROOF: from (23), we can now derive (22). To this end, let us firstly notice that, from the definition of b_k we have

$$b_k = \int_\omega \sin^2(k\pi x)\,dx = \frac{|\omega|}{2} - \frac{1}{2}\int_\omega \cos(2k\pi x)\,dx \geq \frac{|\omega|}{2} - \frac{1}{2|k|\pi}.$$

Since $1/(2|k|\pi) \to 0$ when $k \to \infty$, there exists $k_0 > 0$ such that

$$b_k \geq \frac{|\omega|}{2} - \frac{1}{2|k|\pi} \geq \frac{|\omega|}{2} > 0, \qquad \text{if } |k| > k_0.$$

Hence, $\inf_{|k|>k_0} b_k > 0$ and, since $b_k > 0$ for all $k \in \mathbb{Z}^*$, we conclude that

$$B := \inf_{k\in\mathbb{Z}^*} b_k > 0.$$

Therefore,

$$B\sum_{k\in\mathbb{Z}^*}\frac{|a_k|^2}{k^2\pi^2} \leq \int_\omega\int_0^T\left|\sum_{k\in\mathbb{Z}^*}\frac{a_k}{k\pi}e^{i\lambda_k t}\sin(k\pi x)\right|^2 dtdx$$

or, equivalently,

$$B\left(\|p_0\|^2_{L^2(0,1)} + \|p_1\|_{H^{-1}(0,1)^2}\right) \leq \int_0^T\int_\omega |p|^2\,dxdt.$$

THE MULTIPLIER METHOD

The multiplier method

$$\Gamma_0 = \Gamma_0(x_0) := \left\{ x \in \partial\Omega \,:\, (x - x_0)\cdot\nu \geq 0,\ \exists x_0 \in \mathbb{R}^N \right\}$$

$$\Gamma_1 = \Gamma \setminus \Gamma_0$$

For simplifying the notation, let us define

- $m(x) := x - x_0$
- $X := \left. (p_t, m\cdot\nabla p)\right|_0^T$
- $Y := \left. (p_t, p)\right|_0^T$
- $\Sigma_i := \Gamma_i \times (0,T),\ i = 0,1$

The technique consists in multiplying the adjoint equation by $m\cdot\nabla p$ and integrate by parts over Q. In this way, we obtain

$$\begin{aligned} 0 &= \int_Q (p_{tt} - \Delta p)(m\cdot\nabla p)\, dxdt \\ &= X - \int_Q p_t m\cdot\nabla p_t\, dxdt - \int_\Sigma (m\cdot\nabla p)\frac{\partial p}{\partial\nu}\, d\sigma dt + \int_Q \nabla p\cdot\nabla(m\cdot\nabla p)\, dxdt \end{aligned}$$

J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev., 1988

The multiplier method

Through some simple algebraic computation, the last term on the right hand side of this previous identity may be developed as

$$\int_Q \nabla p \cdot \nabla (m \cdot \nabla p)\, dxdt = \int_Q |\nabla p|^2\, dxdt + \frac{1}{2} \sum_{k=1}^{N} \int_Q m_k \frac{\partial}{\partial x_k} |\nabla p|^2\, dxdt.$$

Hence, we get

$$X - \frac{1}{2} \int_Q m \cdot \nabla (p_t^2)\, dxdt + \frac{1}{2} \sum_{k=1}^{N} \int_Q m_k \frac{\partial}{\partial x_k} |\nabla p|^2\, dxdt + \int_Q |\nabla p|^2\, dxdt - \int_\Sigma (m \cdot \nabla p) \frac{\partial p}{\partial \nu}\, d\sigma dt = 0.$$

Then, a further integration by parts yields

$$X + \frac{N}{2} \int_Q p_t^2\, dxdt + \left(1 - \frac{N}{2}\right) \int_Q |\nabla p|^2\, dxdt - \sum_{k=1}^{N} \int_\Sigma \left[\frac{\partial p}{\partial \nu} \left(m_k \frac{\partial p}{\partial x_k} \right) - \frac{1}{2} m_k \left(\frac{\partial p}{\partial x_k} \right)^2 \nu_k \right] d\sigma dt = 0.$$

Since $p = 0$ on Γ, we have $\frac{\partial p}{\partial x_k} = \nu_k \frac{\partial p}{\partial \nu}$ and

$$
\begin{aligned}
0 &= X + \frac{N}{2}\int_Q p_t^2\,dxdt + \left(1 - \frac{N}{2}\right)\int_Q |\nabla p|^2\,dxdt - \frac{1}{2}\int_\Sigma \left(\frac{\partial p}{\partial \nu}\right)^2 (m\cdot\nu)\,d\sigma dt \\
&= X + \frac{N-1}{2}\int_Q \left(p_t^2 - |\nabla p|^2\right)\,dxdt + \frac{1}{2}\int_Q \left(p_t^2 + |\nabla p|^2\right)\,dxdt \\
&\quad - \frac{1}{2}\int_\Sigma \left(\frac{\partial p}{\partial \nu}\right)^2 (m\cdot\nu)\,d\sigma dt.
\end{aligned}
$$

Moreover, we have

$$
\int_Q \left(p_t^2 - |\nabla p|^2\right)\,dxdt = Y - \int_Q (p_{tt} - \Delta p)p\,dxdt = Y,
$$

and we then obtain

$$
X + \frac{N-1}{2}Y + \frac{1}{2}\int_Q \left(p_t^2 + |\nabla p|^2\right)\,dxdt - \frac{1}{2}\int_\Sigma \left(\frac{\partial p}{\partial \nu}\right)^2 (m\cdot\nu)\,d\sigma dt = 0,
$$

that is,

$$
X + \frac{N-1}{2}Y + E(0) - \frac{1}{2}\int_\Sigma \left(\frac{\partial p}{\partial \nu}\right)^2 (m\cdot\nu)\,d\sigma dt = 0.
$$

We then get

$$TE_0 - \frac{1}{2}\int_{\Sigma_1}\left(\frac{\partial p}{\partial \nu}\right)^2 (m\cdot\nu)\, d\sigma dt = -X - \frac{N-1}{2}Y + \frac{1}{2}\int_{\Sigma_0}\left(\frac{\partial p}{\partial \nu}\right)^2 (m\cdot\nu)\, d\sigma dt.$$

Since $m\cdot\nu \leq 0$ on Γ_1, the second term on the left hand side of the previous identity is positive. In view of that, we have

$$TE_0 \leq \left|X + \frac{N-1}{2}Y\right| + \frac{R(x_0)}{2}\int_{\Sigma_0}\left(\frac{\partial p}{\partial \nu}\right)^2 d\sigma dt,$$

where $R(x_0) := \sup_\Gamma (m\cdot\nu)$.

If we define

$$\xi(t) := X + \frac{N-1}{2}Y = \left(p_t, m\cdot\nabla p + \frac{N-1}{2}p\right)\Big|_0^T,$$

we have

$$\left|X + \frac{N-1}{2}Y\right| \leq |\xi(T)| + |\xi(0)|.$$

On the other hand, Young's inequality yields

$$|\xi(t)| \leq \frac{R(x_0)}{2}|p_t|^2 + \frac{1}{2R(x_0)}\left|m\cdot\nabla p + \frac{N-1}{2}p\right|^2, \quad \text{for all } t\in[0,T].$$

Moreover, we can estimate

$$\left|m\cdot\nabla p + \frac{N-1}{2}p\right|^2 \leq |m\cdot\nabla p|^2 \leq R(x_0)^2|\nabla p|^2,$$

so to obtain

$$|\xi(t)| \leq \frac{R(x_0)}{2}(|p_t|^2|\nabla p|^2) = R(x_0)E(0).$$

The multiplier method

Consequently,

$$\left| X + \frac{N-1}{2} Y \right| \leq 2R(x_0)E(0),$$

and we finally obtain

$$TE(0) \leq 2R(x_0)E(0) + \frac{R(x_0)}{2} \int_{\Sigma_0} \left(\frac{\partial p}{\partial \nu} \right)^2 d\sigma dt.$$

Therefore, assuming $T > 2R(x_0)$, (10) follows immediately with

$$\mathcal{C} := \frac{R(x_0)}{2(T - 2R(x_0))}.$$

The heat equation

Let $N \geq 1$ and $T > 0$, Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$.

Controlled heat equation

$$\begin{cases} y_t - \Delta y = u\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{24}$$

χ_ω denotes the **characteristic function** of the subset $\omega \subset \Omega$ where the control is active.

We assume that $y_0 \in L^2(\Omega)$ and $u \in L^2(Q)$ so that (24) admits an unique solution

$$y \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

Approximate controllability

For all initial data y_0, all final data $y_T \in L^2(\Omega)$ and all $\varepsilon > 0$ there exists a control u_ε such that the solution satisfies

$$\|y(\cdot, T) - y_T\|_{L^2(\Omega)} \leq \varepsilon.$$

Unique continuation

Approximate controllability holds **if and only if** the following unique continuation property (UCP) is true

$$p = 0 \text{ in } \omega \times (0, T) \implies p \equiv 0, \text{ i.e. } \varphi_0 \equiv 0, \tag{25}$$

where p is the unique solution of the adjoint system

$$\begin{cases} -p_t - \Delta p = 0 & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ p(x, T) = p_T(x) & \text{in } \Omega \end{cases} \tag{26}$$

This UCP is a consequence of Holmgren's uniqueness Theorem and holds for all ω and all $T > 0$.

Recall that (24) is approximately controllable in time T if, for every initial datum $y_0 \in L^2(\Omega)$, the set $\mathcal{R}(T; y_0)$ is dense in $L^2(\Omega)$, i.e.

$$\overline{R(T, y_0)}^{L^2(\Omega)} = L^2(\Omega).$$

Moreover, the linearity of (24) implies that

$$R(T; y_0) = R(T; 0) + S(T)y_0.$$

Hence, the problem of approximate controllability for (24) may be reduced to the case $y_0 = 0$.

Hahn-Banach Theorem: the set $\mathcal{R}(T; y_0)$ is dense in $L^2(\Omega)$ if the following property holds

> There is no $p_T \in L^2(\Omega)$, $p_T \neq 0$, such that $\langle y(\cdot, T), p_T\rangle = 0$ for all y solution of (24) with $u \in L^2(\omega \times (0, T))$.

Hence, the proof can be reduced to showing that, if $p_T \in L^2(\Omega)$ is such that

$$\langle y(\cdot, T), p_T\rangle = \int_\Omega y(x, T) p_T(x)\, dx = 0, \tag{27}$$

then, necessarily, $p_T = 0$.

Multiplying (24) by p and integrating by parts on Q taking into account that $y_0 = 0$ and $y = p = 0$ on Σ, we obtain

$$\begin{aligned}\int_0^T \int_\omega up\,dxdt &= \int_0^T \int_\Omega \Big(y_t - \Delta y\Big) p\,dxdt \\ &= \int_\Omega yp\,dx\,\Big|_{t=0}^{t=T} + \int_0^T \int_\Omega \Big(-p_t + \Delta p\Big) y\,dxdt \\ &= \int_\Omega y(x,T) p_T(x)\,dx\end{aligned}$$

Hence, (27) holds if and only if

$$\int_0^T \int_\omega up\,dxdt = 0 \quad \text{for all } u \in L^2(\omega \times (0,T)),$$

from where we deduce that $p = 0$ a.e. in $\omega \times (0,T)$.

Thanks to the unique continuation property (UCP), this implies that $p = 0$ in Q. Consequently $p_T = 0$.

Approximate controllability functional

$$J_\varepsilon(p_T) = \frac{1}{2}\int_0^T \int_\omega |p|^2 \, dxdt + \varepsilon \, \|p_T\|_{L^2(\Omega)} - \int_\Omega p_T y_T \, dx + \int_\Omega p(x,0) y_0 \, dx.$$

$J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ is continuous, and convex. Moreover, UCP implies coercivity:

$$\lim_{\|p_T\|_{L^2(\Omega)} \to +\infty} \frac{J_\varepsilon(p_T)}{\|p_T\|_{L^2(\Omega)}} \geq \varepsilon.$$

Accordingly, the minimizer $\hat{p}_T$ exists and the control

$$u_\varepsilon = \hat{p},$$

where $\hat{p}$ is the solution of the adjoint system corresponding to the minimizer, is such that

$$\|y(\cdot, T) - y_T\|_{L^2(\Omega)} \leq \varepsilon.$$

C. Fabre, J.-P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sec. A Math., 1995

Null controllability

For achieving $y(\cdot, T) = 0$ we have to consider the case in which $y_T = 0$ and $\varepsilon = 0$. Thus, we are led to considering the functional

Null controllability functional

$$J_0(p_T) = \frac{1}{2}\int_0^T \int_\omega |p|^2 \, dxdt + \int_\Omega p(x,0) y_0 \, dx.$$

Obviously, J_0 is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$. For coercivity, it is needed the observability inequality

$$\|p(x,0)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |p|^2 \, dxdt, \quad \text{for all } \varphi_T \in L^2(\Omega). \tag{28}$$

Null controllability

Remark

The observability inequality (28) **is very likely to hold**.

Because of the very strong regularizing effect of the heat equation, the norm of $p(x, 0)$ is a very weak measure of the total size of solutions. Indeed, in a Fourier series representation, this norm presents weights of the order of

$$\exp(-\lambda_j T),$$

$\lambda_j \to +\infty$ being the eigenvalues of the Dirichlet Laplacian $-\Delta$ on Ω.

Remark

Due to the **irreversibility** of the system, (28) is not easy to prove.

There are several ways of proving the observability inequality (28). The most classical ones are the following.

Space-dimension $N = 1$

- Parabolic Ingham's inequalities
- Moments method

Space-dimension $N \geq 1$

- Carleman estimates
- Lebeau-Robbiano strategy

Parabolic Ingham's inequalities

A spectral estimate

Theorem

Let $\{\lambda_k\}_{k\geq 1}$ be a sequence of real numbers satisfying the following conditions:

1. There exists $\gamma > 0$ such that $\lambda_{k+1} - \lambda_k \geq \gamma$ for all $k \geq 1$. (29a)
2. $\displaystyle\sum_{k\geq 1} \frac{1}{\lambda_k} < +\infty.$ (29b)

Then, for any $T > 0$, there is a constant $\mathcal{C}(T) > 0$ (depending only on T) such that, for any sequence $\{c_k\}_{k\geq 1}$ it holds the inequality

$$\sum_{k\geq 1} |c_k| e^{-\lambda_k T} \leq \mathcal{C}(T) \left\| \sum_{k\geq 1} c_k e^{-\lambda_k t} \right\|_{L^2(0,T)}. \tag{30}$$

Moreover, the function $\mathcal{C}(T)$ is uniformly bounded away from $T = 0$ and blows-up exponentially as $T \downarrow 0^+$.

Proof of the observability inequality

Let $\{\lambda_k, \phi_k\}_{k\geq 1}$ be the eigenvalues and eigenfunctions of the one-dimensional Laplacian on $\Omega = (0,1)$. Then

$$p(x,t) = \sum_{k\geq 1} p_k e^{-\lambda_k(T-t)}\phi_k(x) \quad \text{with } p_k = \int_0^1 p_t(x)\phi_k(x)\,dx$$

an d the observability inequality can be written as

$$\sum_{k\geq 1} |p_k|^2 e^{-2\lambda_k T} \leq \mathcal{C}\int_0^T\int_\omega \left|\sum_{k\geq 1} p_k e^{-\lambda_k t}\phi_k\right|^2 dxdt. \tag{31}$$

Since the eigenvalues $(\lambda_k)_{k\geq 1}$ satisfy (29a) and (29b), if we take $c_k := p_k\phi_k(x)$ for any $x \in (-1,1)$ fixed in (31), we obtain the estimate

$$\sum_{k\geq 1} |p_k\phi_k(x)| e^{-\lambda_k T} \leq \mathcal{C}(T)\left\|\sum_{k\geq 1} p_k e^{-\lambda_k t}\phi_k(x)\right\|_{L^2(0,T)}.$$

Hence

$$\sum_{k\geq 1}|p_k\phi_k(x)|^2 e^{-2\lambda_k T} \leq \left(\sum_{k\geq 1}|p_k\phi_k(x)|e^{-\lambda_k T}\right)^2 \tag{32}$$

$$\leq \mathcal{C}(T)^2\int_0^T\left|\sum_{k\geq 1}p_k e^{-\lambda_k t}\phi_k(x)\right|^2\,dt. \tag{33}$$

Finally, since the eigenfunctions of the fractional Laplacian satisfy the estimate

$$\|\phi_k\|_{L^2(\omega)}\geq \beta|\omega|^{-1},\quad \text{for all } k\geq 1 \text{ and } \omega\subset(-1,1),$$

integrating over ω, we obtain that

$$\beta|\omega|^{-1}\sum_{k\geq 1}|p_k|^2 e^{-2\lambda_k T} \leq \int_\omega\sum_{k\geq 1}|p_k\phi_k(x)|^2 e^{-2\lambda_k T}\,dx$$

$$\leq \mathcal{C}(T)^2\int_0^T\int_\omega\left|\sum_{k\geq 1}p_k e^{-\lambda_k t}\phi_k(x)\right|^2\,dxdt.$$

Moment method

The moment method

Recall that the controllability of the heat equation is equivalent to the identity

$$\int_0^T \int_\omega up\,dt + \int_0^1 y_0 p(0)\,dx = 0, \quad \text{for all } p_T \in L^2(0,1). \tag{34}$$

Since $\{\phi_k\}_{k\geq 1}$ is an orthonormal basis of $L^2(0,1)$, it is sufficient that (34) holds for each eigenfunction. Writing

$$y_0(x) = \sum_{k\geq 1} y_k \phi_k(x) \quad \text{with} \quad y_k = \int_0^1 y_0(x)\phi_k(x)\,dx \ \text{ for all } k \geq 1,$$

and $p_T = \phi_k$ in (34), we obtain the new identity (equivalent to null controllability)

$$\int_0^T \int_\omega u\phi_k e^{\lambda_k t}\,dt = -y_k, \quad \text{for all } k \geq 1. \tag{35}$$

Identity (35) is known as a **problem of moments**.

H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, ARMA, 1971

The moment method

Biorthogonal sequence

$$\{\sigma_k(t)\}_{k\geq 1} \subset L^2(0,T) \quad \text{such that } \int_0^T \sigma_k(t) e^{\lambda_\ell t}\, dt = \delta_{k,\ell}.$$

The problem of moments (35) is satisfied by the control function

$$u(x,t) = \sum_{k\geq 1} -y_k \sigma_k(t) \frac{\phi_k(x)}{\|\phi_k\|_{L^2(\omega)}} \tag{36}$$

The existence of such a control u is guaranteed by two facts.

1. The biorthogonal sequence exists.

   Consequence of **Münz's Theorem** and the fact that $\sum_{k\geq 1} \lambda_k^{-1} < +\infty$.
2. The sum (36) converges

   Consequence of suitable bounds for $\|\sigma_k\|_{L^2(0,T)}$ that can be obtained under the gap condition $\lambda_{k+1} - \lambda_k \geq \gamma > 0$ for all $k \geq 1$.

H. O. Fattorini and D. L. Russell, *Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations*, Quart. Appl. Math., 1974

Carleman estimates

They take their name form the Swedish mathematician Torsten Carleman (1892-1949), who firstly introduced them in the mathematical literature in 1939 as a powerful tool to prove unique continuation result for elliptic partial differential equations with smooth coefficients

T. Carleman, *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat. Astr. Fys., 1939.

Firstly applied in control theory by Fursikov and Imanuvilov in 1996.

A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, Lecture notes, 1996.

Carleman estimates

Lemma

Let $\omega \subset\subset \Omega$ be a nonempty open set. Then, there exists $\eta^0 \in C^2(\overline{\Omega})$ such that

- $\eta^0 > 0$ in Ω.
- $\eta^0 = 0$ on $\partial\Omega$.
- $|\nabla\eta^0| > 0$ in $\overline{\Omega \setminus \omega}$.

In some particular cases, for instance when Ω is star-shaped with respect to a point in ω, η^0 can be built explicitly without difficulty. But the existence of this function is less obvious in general, when the domain has holes or its boundary oscillates, for instance.

For a parameter $\lambda > 0$, we define

$$\sigma(x) := e^{4\lambda\left\|\eta^0\right\|_\infty} - e^{\lambda\left(2\left\|\eta^0\right\|_\infty + \eta^0(x)\right)},$$

and we introduce the weight functions

$$\alpha(x,t) := \frac{\sigma(x)}{t(T-t)}, \quad \xi(x,t) := \frac{e^{\lambda\left(2\left\|\eta^0\right\|_\infty + \eta^0(x)\right)}}{t(T-t)}. \tag{37}$$

Proposition

There exist positive constants $\mathcal{C}$ and s_1 such that, for all $s \geq s_1$, $\lambda \geq \mathcal{C}$ and $p_T \in L^2(\Omega)$, the solution p to the adjoint equation (26) satisfies

$$s\lambda^2 \int_Q e^{-2s\alpha}\xi|\nabla p|^2\,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\alpha}\xi^3|p|^2\,dx\,dt \\ \leq \mathcal{C}s^3\lambda^4 \int_0^T \int_\omega e^{-2s\alpha}\xi^3|p|^2\,dxdt.$$

E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SICON, 2006.

From Carleman to observability

From the Carleman estimate we have

$$s^3 \int_Q e^{-2s\alpha}\xi^3 |p|^2 \, dxdt \leq \mathcal{C} s^3 \int_0^T \int_\omega e^{-2s\alpha}\xi^3 |p|^2 \, dx \, dt.$$

Moreover, due to the definition of the weight function α, if we choose $s \geq \mathcal{C}T^2$ we have the following two estimates:

1. $s^3 e^{-2s\alpha}\xi^3 \leq \mathcal{C} s^3 T^{-6} e^{-\frac{\mathcal{C}s}{T^2}} \leq \mathcal{C}(T)$.
2. $s^3 e^{-2s\alpha}\xi^3 \geq \mathcal{C} e^{-\frac{\mathcal{C}s}{T^2}}$, if $t \in [T/4, 3T/4]$.

Therefore, we obtain

$$\int_{\frac{T}{4}}^{\frac{3}{4}T} \int_\Omega |p|^2 \, dxdt \leq \mathcal{C} e^{\frac{\mathcal{C}s}{T^2}} \int_0^T \int_\omega |p|^2 \, dxdt. \tag{38}$$

Finally, classical energy estimates yield that $t \mapsto \|p(t)\|_{L^2(\Omega)}$ is an increasing function. Hence,

$$\begin{aligned} \frac{T}{2} \|p(x,0)\|^2_{L^2(\Omega)} &= \int_{\frac{T}{4}}^{\frac{3}{4}T} \int_\Omega |p(x,0)|^2 \, dxdt \\ &\leq \int_{\frac{T}{4}}^{\frac{3}{4}T} \int_\Omega |p(x,t)|^2 \, dxdt \leq \mathcal{C} e^{\frac{\mathcal{C}s}{T^2}} \int_0^T \int_\omega |p|^2 \, dxdt. \end{aligned}$$

Lebeau-Robbiano strategy

The Lebeau-Robbiano strategy

Based on three main steps.

Step 1

Use a local Carleman estimate for the operator $\partial_t^2 + \Delta$ in order to deduce the interpolation inequality: for any $T > 0$ and all $\alpha \in (0, T/2)$, there exists $\gamma \in (0,1)$ such that

$$\|\phi_k\|_{L^2(\Omega\times(\alpha,T-\alpha))} \leq C\,\|\phi_k\|_{H^1(Q)}^{\gamma} \left(\left\|(\partial_t^2 + \Delta)\phi_k\right\|_{L^2(Q)} + \|\phi_{k,t}(x,0)\|_{L^2(\omega)}\right)^{1-\gamma}$$

Step 2

Use the interpolation inequality to obtain the spectral inequality

$$\sum_{\lambda_k \leq 2^{2j}} |a_k|^2 \leq \mathcal{C}_1 e^{2^{2j}\mathcal{C}_2} \left\| \sum_{\lambda_k \leq 2^{2j}} a_k\phi_k(x) \right\|_{L^2(\omega)}$$

to obtain the the observability of low-frequency solutions.

The Lebeau-Robbiano strategy

Step 3

Use an iterative strategy alternating the observability of low-frequency and decay of the heat semi-group to obtain the final observability result.

G. Lebeau and L. Robbiano, *Contrôle exact de léquation de la chaleur*, Commun. PDE, 1995

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