Control and optimization for non-local and fractional differential equations

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PART II: non-local in time models

LECTURE 3: ODE and PDE with memory - the moving control strategy

Historical introduction

Memory-type differential equations

Memory-type differential equations

Introduced in the 1960's, when researchers started understanding that classical mathematical models such as the heat and wave equations may fail to fully capture relevant features of the physical process they describe.

Memory-type differential equations

It is often claimed that the theory of infinitesimal viscoelasticity can be derived from an assumption that on a microscopic level matter can be regarded as composed of linear viscous elements (also called dashpots) and linear elastic elements (called springs) connected together in intricate networks.

... We feel that the physicist's confidence in the usefulness of the theory of infinitesimal viscoelasticity does not stem from a belief that the materials to which the theory is applied are really composed of microscopic networks of springs and dashpots, but comes rather from other considerations. First, there is the observation that the theory works for many real materials. But second, and perhaps more important to theoreticians, is the fact that the theory looks plausible because it seems to be a mathematization of little more than certain intuitive prejudices about smoothness in macroscopic phenomena.

This article tries to make precise these observations... and in so doing seeks to obtain a mathematical derivation of infinitesimal viscoelasticity from plausible macroscopic assumptions. To do this one must first presume a nonlinear theory of the mechanical behavior of materials with memory, and, if the undertaking is to be at all worthwhile, the presumed nonlinear theory must rest on constitutive equations based only on very general physical principles.

B. D. Coleman and W. Noll, Foundations of linear viscoelasticity, Rev. Modern Phys., 1961

Memory-type differential equations

...the classical linear theory of heat conduction for homogeneous and isotropic media is based on the equation

$$\alpha\Delta\theta = \dot{\theta}$$

where $\theta = \theta(x, t)$ *is the absolute temperature,* $\dot{\theta} = \frac{\partial\theta}{\partial t}$*,* Δ *is the Laplacian, and* $\alpha > 0$ *is a constant. This equation, which is parabolic, has a very unpleasant feature: a thermal disturbance at any point in the body is felt instantly at every other point; or in terms more suggestive than precise, the speed of propagation of disturbances is infinite.*

M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, ARMA, 1968

Memory-type differential equations

The works of Coleman & Noll and Gurtin & Pipkin highlight the limitations of standard PDE models in capturing the reality of the phenomena they were supposed to represent.

To overcome these limitation and give a more precise model for viscoelasticity or the heat transfer process, they then modified Fick's and Fourier's law and introduced PDE with memory in the form

PDE with memory

$$y_{tt} - \Delta y - \int_{-\infty}^{t} M(t-s)\Delta y(s)\, ds = 0,$$

$$y_{t} - \Delta y - \int_{-\infty}^{t} M(t-s)\Delta y(s)\, ds = 0,$$

where $M(\cdot)$ is a suitable function, typically called the **relaxation kernel**.

Memory-type differential equations

After these first seminal contributions, the theory of evolution equations with memory has continued expanding both on the physical and on the mathematical level, giving rise to a thick field of research concerned with such systems

Nowadays, evolution equations involving memory terms appear in several different applications to describe natural and social phenomena which, apart from their current state, are influenced also by their history

SOME REFERENCES:

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R. K. Miller and R. L. Wheeler, *Asymptotic behavior for a linear Volterra integral equation in Hilbert space*, JDE, 1977

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P. Markowich and M. Renardy, *Lax--Wendroff methods for hyperbolic history value problems*, SIAM J. Numer. Anal., 1984

M. Renardy, W. J. Hrusa and J. A. Nohel, *Mathematical problems in viscoelasticity*, Longman, 1987

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Control theory for memory-type equations

Control theory for memory-type equations

The raise of interest in the last decades towards differential equations with memory also largely affected the control community.

Many research works considering an ample spectrum of memory-type equations, for which approximate and exact/null controllability properties are analyzed.

SOME REFERENCES:

G. Leugering, *Exact controllability in viscoelasticity of fading memory type*, Appl. Anal., 1984

G. Leugering, *Exact boundary controllability of an integrodifferential equation*, Appl. Math. Optim., 1987

V. Barbu and M. Iannelli, *Controllability of the heat equation with memory*, Differ. Integral Equ., 2000

X. Fu, J. Yong and X. Zhang, *Controllability and observability of a heat equation with hyperbolic memory kernel*, JDE, 2009

P. Loreti, L. Pandolfi and D. Sforza, *Boundary controllability and observability of a viscoelastic string*, SICON, 2012

S. Guerrero and O. Y. Imanuvilov, *Remarks on non controllability of the heat equation with memory*, ESAIM: Control Optim. Calc. Var., 2013

Control theory for memory-type equations

In all the previous references, the controllability issue is addressed focusing only on the steering of the system's state to some given target at time T, without considering that **the presence of the memory introduces additional effects** that make the classical controllability notion not entirely suitable in this context.

ATTENTION!

Driving the solution of a memory-type equation to some specific state, for instance to zero, in general **is not sufficient to guarantee that the dynamics of the system reaches an equilibrium**.

If we were considering an equation without memory, once its solution is driven to rest at time T by a control, then it vanishes for all $t \geq T$ also in the absence of control. This is no longer true when introducing a memory term, which **produces accumulation effects** affecting the definition of an equilibrium point for the system and its overall stability.

Memory-type linear ODE

$$\begin{cases} \dot{\theta}(t) + \int_0^t \theta(s)\, ds = u(t), \quad t \in (0, +\infty) \\ \theta(0) = 1. \end{cases} \tag{1}$$

We look for a control $u \in L^2(0, T)$ such that the solution of (1) is steered to rest at time T.

For this, **it is not enough that $\theta(T) = 0$**.

If we do not pay attention to the accumulated memory, then the solution θ is not guaranteed to stay at the rest after time T as t evolves.

Main reason

Although at a first glance (1) may appear to be a first order equation - since it involves only one time derivative of the state θ - this is actually not the case as **the memory term acts as a hidden component in the system**.

Apply the transformation

$$\zeta(t) := \int_0^t \theta(s)\, ds, \tag{2}$$

to obtain a linear system of two first-order ODE

$$\begin{pmatrix} \dot{\theta}(t) \\ \dot{\zeta}(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta(t) \\ \zeta(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \quad t \in (0, +\infty), \tag{3}$$

with initial datum $(\theta(0), \zeta(0)) = (1, 0)$.

It is then evident that, if we want the system to be completely controlled, both the state $\theta(t)$ and the "velocity" $\zeta(t)$ need to be zero at time T. Therefore, the null controllability of (3) has to be understood as

Memory-type null controllability

$$\theta(T) = 0 \quad \text{and} \quad \zeta(T) = \int_0^T \theta(s)\, ds = 0.$$

To ensure that the original system (1) reaches the equilibrium $\theta(t) = 0$ for $t \geq T$, **it would be also necessary that the memory term reaches the null value**.

The discussion is easily extendable to linear ODE of any order.

Memory-type linear ODE of order $k \in \mathbb{N}^*$

$$\begin{cases} \theta^{(k)}(t) + \int_0^t \theta(s)\,ds = u(t), & t \in (0, +\infty) \\ \theta^{(\ell-1)}(0) = 1, & \ell = 1, \ldots, k, \end{cases} \tag{4}$$

where $\theta^{(k)}(t)$ indicates the k-th order derivative of the function θ with respect to the variable t.

Denoting $\theta_k(t) := \theta^{(k-1)}(t)$ and applying again the transformation (2), we obtain from (4) the following linear system of $k+1$ first-order ODE in **Brunovsky canonical form**

$$\begin{pmatrix} \dot\theta_1(t) \\ \dot\theta_2(t) \\ \vdots \\ \dot\theta_k(t) \\ \dot\zeta(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \vdots \\ \theta_k(t) \\ \zeta(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad t \in (0, +\infty),$$

$$\begin{pmatrix} \theta_1(0) \\ \theta_2(0) \\ \vdots \\ \theta_k(0) \\ \zeta(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$$

Memory-type null controllability

$$\theta_\ell(T) = 0 \;\text{ for all }\; \ell \in \{1, \dots, k\} \;\text{ and }\; \zeta(T) = \int_0^T \theta(s)\,ds = 0.$$

Control theory for memory-type equations

Analogous considerations for memory-type PDE models.

Memory-type heat equation

$$\begin{cases} y_t - \Delta y + \displaystyle\int_0^t M(t-s)y(x,s)\,ds = f\chi_\omega, & (x,t) \in \Omega \times (0,T) \\ y = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{5}$$

$z = \int_0^t M(t-s)y(\cdot,s)\,ds$: we can see (5) as a coupled PDE-ODE system

$$\begin{cases} y_t - \Delta y + z = f\chi_\omega, & (x,t) \in \Omega \times (0,T) \\ z_t + M(0)y = 0, & (x,t) \in \Omega \times (0,T) \\ y = z = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), \quad z(x,0) = 0, & x \in \Omega. \end{cases}$$

Memory-type null controllability for (5)

$$y(x,T) = 0 \;\text{ and }\; z(x,T) = \int_0^T M(T-s)y(x,s)\,ds = 0. \tag{6}$$

Control theory for memory-type equations

The same for wave equations with memory.

Memory-type wave equation

$$\begin{cases} y_{tt} - \Delta y + \displaystyle\int_0^t M(t-s)y(x,s)\,ds = f\chi_\omega, & (x,t) \in \Omega \times (0,T) \\ y = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & x \in \Omega. \end{cases} \tag{7}$$

Memory-type null controllability for (7)

$$y(x,T) = 0,\ y_t(x,T) = 0 \ \text{ and } \ \int_0^T M(T-s)y(x,s)\,ds = 0. \tag{8}$$

Control theory for memory-type equations

These controllability notions (6) and (8) have been considered in a limited number of works.

S. Guerrero and O. Y. Imanuvilov, *Remarks on non controllability of the heat equation with memory*, ESAIM: Control Optim. Calc. Var., 2013

Q. Lü, X. Zhang and E. Zuazua, *Null controllability for wave equations with memory*, J. Math. Pures Appl., 2017

F. W. Chaves-Silva, X. Zhang and E. Zuazua, *Controllability of evolution equations with memory*, SICON, 2017

U. Biccari and S. Micu, *Null-controllability properties of the wave equation with a second order memory term*, JDE, 2019

Important difficulties appear when studying the control properties of memory PDE under those notions.

Once one considers (5) and (7) as coupled PDE-ODE systems, it becomes evident that the memory term acts as an hidden ODE component with lack of propagation in the space-variable.

This opens the possibility that **null-controllability properties may fail**, since the non-propagating components of the system cannot not reach the control region ω, except for some trivial case such as when ω coincides with the entire domain of definition Ω.

The moving control strategy

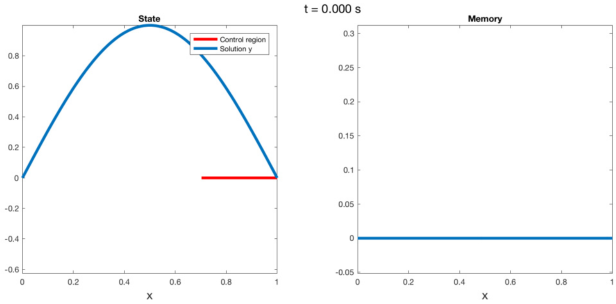
To copy with this issue, we can employ a **moving control strategy** to successfully control PDE with memory.

The equations (5) and (7) are memory-controllable provided the support of the control moves, covering the whole domain where the equations evolves. If the control domain is fixed, (5) and (7) **are not memory-controllable**.

REMARK: clearly, in the case of the wave equation (7), this moving control strategy is effective if the time horizon T is large enough and the control region $\omega(t)$ satisfies the Geometric Control Condition for all $t \in (0, T)$, in analogy with classical hyperbolic PDE without memory.

The moving control strategy

$$y_{tt} - y_{xx} - M \int_0^t y(s)\,ds = u\chi_\omega$$ with $M = 0.1$ and a **fixed** control.

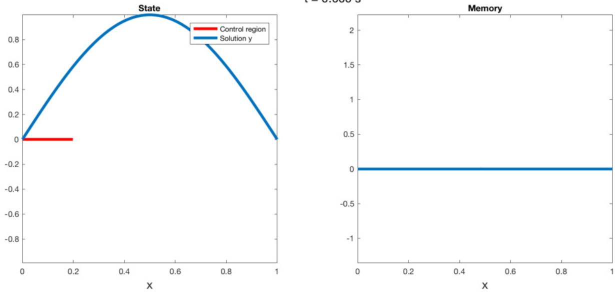


The zero is not reached neither in the state nor in the memory.

The moving control strategy

$$y_{tt} - y_{xx} - M\int_0^t y(s)\,ds = u\chi_{\omega(t)}$$ with $M = 0.1$ and a **moving** control.

t = 0.000 s



Both the state and the memory are driven to zero in time T.

The moving control strategy

2D heat equation witout memory

$$y_t - \Delta y = u\chi_\omega$$

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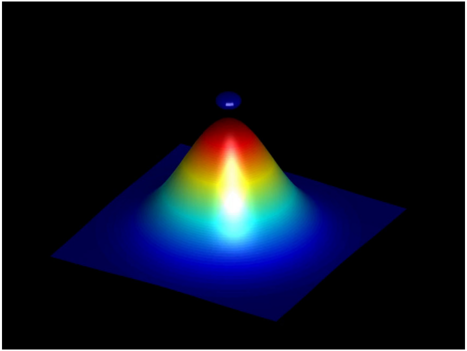
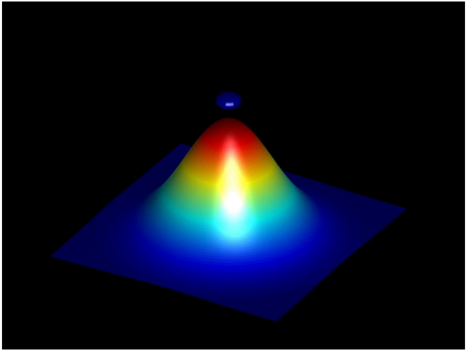


The moving control strategy

2D heat equation with memory and fixed control

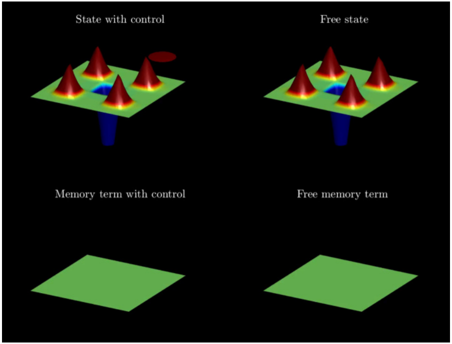
$$y_t - \Delta y - \int_0^t y(s)\,ds = u\chi_\omega.$$



The moving control strategy

2D heat equation with memory and moving control

$$y_t - \Delta y - \int_0^t y(s)\, ds = u\chi_{\omega(t)}.$$



High-order memory terms

Physically relevant memory-type PDE involve high-order memory terms in the form

$$\int_0^t M(t-s)\Delta y(x,s)\,ds. \tag{9}$$

Such a memory term is quite delicate to be handled and may raise important difficulties when addressing controllability properties.

- F. W. Chaves-Silva, X. Zhang and E. Zuazua, *Controllability of evolution equations with memory*, SICON, 2017

By means of a Carleman approach it is shown that the system

$$\begin{cases} y_t - \Delta y + \int_0^t M(t-s)\Delta y(x,s)\,ds = f\chi_{\omega(t)}, & (x,t)\in\Omega\times(0,T) \\ y=0, & (x,t)\in\partial\Omega\times(0,T) \\ y(x,0)=y_0(x), & x\in\Omega. \end{cases}$$

with initial datum $y_0 \in L^2(\Omega)$ is null controllable at any time $T>0$.

High-order memory terms

The situation changes drastically when considering hyperbolic equations with high-order memory, in which case (9) naturally yields to observability inequalities involving the H^2-norm of the adjoint state, which is not compact with respect to the usual L^2-norm.

- The approach of

  Q. Lü, X. Zhang and E. Zuazua, *Null controllability for wave equations with memory*, J. Math. Pures Appl., 2017

  fails in providing a controllability property.

- Controllability for high-order memory-type wave equations have been obtained in space dimension one by employing biorthogonal sequences.

  U. Biccari and S. Micu, *Null-controllability properties of the wave equation with a second order memory term*, JDE, 2019

Null controllability of a one-dimensional wave equation with memory

Problem formulation

High-order memory-type wave equation

$$\begin{cases} y_{tt} - y_{xx} + M \int_0^t y_{xx}(s)\, ds = u\mathbf{1}_{\omega(t)}, & (x,t) \in \mathbb{T} \times (0,T) := Q \\ y(x,0) = y^0(x), \;\; y_t(x,0) = y^1(x), & x \in \mathbb{T}, \;\; M \in \mathbb{R} \setminus \{0\}. \end{cases}$$

- $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$: one-dimensional torus.
- $\omega(t) = \omega_0 - ct, \quad c \in \mathbb{R}$.
- $u \in L^2(\mathcal{O})$
- $\mathcal{O} = \left\{ (x,t) \,\middle|\, t \in (0,T),\, x \in \omega(t) \right\}$.

Functional setting

- $L^2_p(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} \,\middle|\, \int_{-\pi}^{\pi} |f(x)|^2 \, dx < +\infty, \; \int_{-\pi}^{\pi} f(x) \, dx = 0 \right\}.$
- $H^1_p(\mathbb{T}) := \left\{ f \in L^2_p(\mathbb{T}) \,\middle|\, f_x \in L^2_p(\mathbb{T}), \; \int_{-\pi}^{\pi} f(x) \, dx = 0 \right\}.$
- $\|f\|_{L^2_p(\mathbb{T})} := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$
- $\|f\|_{H^1_p(\mathbb{T})} := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f_x(x)|^2 \, dx \right)^{\frac{1}{2}} = \|f_x\|_{L^2(-\pi,\pi)}.$
- $H^{-1}_p(\mathbb{T}) = (H^1_p(\mathbb{T}))'$: dual of $H^1_p(\mathbb{T})$ with pivot space $L^2_p(\mathbb{T})$.

Proposition

For any $(y^0, y^1) \in H^1_p(\mathbb{T}) \times L^2_p(\mathbb{T})$ and $u \in L^2(\mathcal{O})$ the system admits a unique solution $y \in C([0,T]; H^1_p(\mathbb{T})) \cap C^1([0,T]; L^2_p(\mathbb{T}))$. Moreover,

$$\|y\|_{C([0,T];H^1_p(\mathbb{T})) \cap C^1([0,T];L^2_p(\mathbb{T}))} \leq \mathcal{C}(T) \left(\left\| (y^0, y^1) \right\|_{H^1_p(\mathbb{T}) \times L^2_p(\mathbb{T})} + \|u\|_{L^2(\mathcal{O})} \right).$$

Proof

$$\mathcal{Z} := C([0,T]; H^1_p(\mathbb{T})) \cap C^1([0,T]; L^2_p(\mathbb{T}))$$

$$\|y\|_{\mathcal{Z}} := \left(\left\| e^{-\alpha t} y \right\|^2_{C([0,T];H^1_p(\mathbb{T}))} + \left\| e^{-\alpha t} y_t \right\|^2_{C^1([0,T];L^2_p(\mathbb{T}))} \right)^{\frac{1}{2}},$$

α positive real number to be specified.

- $e^{-\alpha T} \|y\|_{C([0,T];H^1_p(\mathbb{T}))\cap C^1([0,T];L^2_p(\mathbb{T}))} \leq \|y\|_{\mathcal{Z}} \leq \|y\|_{C([0,T];H^1_p(\mathbb{T}))\cap C^1([0,T];L^2_p(\mathbb{T}))}$.
- $\mathcal{Z}$ is a **Banach space** with the norm $\|\cdot\|_{\mathcal{Z}}$
- $\mathcal{Z}$ equals $C([0,T]; H^1_p(\mathbb{T})) \cap C^1([0,T]; L^2_p(\mathbb{T}))$ algebraically and topologically.

Proof (cont.)

Define the map $\mathcal{F} : \mathcal{Z} \to \mathcal{Z}, \widetilde{y} \mapsto \widehat{y}$, where $\widehat{y}$ is the solution to

$$\begin{cases} \widehat{y}_{tt}(x,t) - \widehat{y}_{xx}(x,t) = -M \int_0^t \widetilde{y}_{xx}(x,s)\, ds + u(x,t)\mathbf{1}_{\omega(t)}, & (x,t) \in Q \\ \widehat{y}(x,0) = y^0(x), \;\; \widehat{y}_t(x,0) = y^1(x), & x \in \mathbb{T}. \end{cases}$$

Wave equation with a non-homogeneous term in $L^2(0,T; H_p^{-1}(\mathbb{T}))$. It has a unique solution $\widehat{y} \in C([0,T]; H_p^1(\mathbb{T})) \cap C^1([0,T]; L_p^2(\mathbb{T}))$ such that

$$\begin{aligned} \|\widehat{y}\|_{\mathcal{Z}} &\le \|\widehat{y}\|_{C([0,T];H_p^1(\mathbb{T}))\cap C^1([0,T];L_p^2(\mathbb{T}))} \\ &\le \mathcal{C}\left[\left\|(y^0, y^1)\right\|_{H_p^1(\mathbb{T})\times L_p^2(\mathbb{T})} + \|u\|_{L^2(\mathcal{O})} + \left\| M \int_0^t \widetilde{y}_{xx}(s)\, ds \right\|_{H_0^1([0,T];H_p^{-1}(\mathbb{T}))} \right] \\ &= \mathcal{C}\left[\left\|(y^0, y^1)\right\|_{H_p^1(\mathbb{T})\times L_p^2(\mathbb{T})} + \|u\|_{L^2(\mathcal{O})} + |M| \, \|\widetilde{y}\|_{L^2([0,T];H_p^1(\mathbb{T}))} \right], \end{aligned}$$

where $\mathcal{C}$ is a positive constant depending only on T. Hence, $\mathcal{F}(\mathcal{Z}) \subset \mathcal{Z}$.

Given $\widetilde{\widetilde{y}} \in \mathcal{Z}$, let $\widehat{\widehat{y}} = \mathcal{F}(\widetilde{\widetilde{y}})$ be the corresponding solution to the original equation.

$$
\begin{aligned}
& e^{-\alpha t}\left(\left\|\mathcal{F}(\widetilde{y})(t) - \mathcal{F}(\widetilde{\widetilde{y}})(t)\right\|_{H^1_p(\mathbb{T})} + \left\|\mathcal{F}(\widetilde{y})_t(t) - \mathcal{F}(\widetilde{\widetilde{y}})_t(t)\right\|_{L^2_p(\mathbb{T})}\right) \\
&\quad \leq e^{-\alpha t}\left\|\widehat{y} - \widehat{\widehat{y}}\right\|_{C([0,t];H^1_p(\mathbb{T}))\cap C^1([0,t];L^2_p(\mathbb{T}))} \leq \mathcal{C}|M|e^{-\alpha t}\left(\int_0^t \left\|\widetilde{y}(s) - \widetilde{\widetilde{y}}(s)\right\|^2_{H^1_p(\mathbb{T})} ds\right)^{\frac{1}{2}} \\
&\quad = \mathcal{C}|M|\left(\int_0^t e^{-2\alpha(t-s)}\left\|e^{-\alpha s}\left(\widetilde{y}(s) - \widetilde{\widetilde{y}}(s)\right)\right\|^2_{H^1_p(\mathbb{T})} ds\right)^{\frac{1}{2}} \\
&\quad \leq \mathcal{C}|M|\left(\int_0^t e^{-2\alpha(t-s)} ds\right)^{\frac{1}{2}}\left\|\widetilde{y} - \widetilde{\widetilde{y}}\right\|_{\mathcal{Z}} = \mathcal{C}|M|\left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right)^{\frac{1}{2}}\left\|\widetilde{y} - \widetilde{\widetilde{y}}\right\|_{\mathcal{Z}}.
\end{aligned}
$$

This implies

$$
\left\|\mathcal{F}(\widetilde{y}) - \mathcal{F}(\widetilde{\widetilde{y}})\right\|_{\mathcal{Z}} \leq \frac{\mathcal{C}|M|}{\sqrt{2\alpha}}\left\|\widetilde{y} - \widetilde{\widetilde{y}}\right\|_{\mathcal{Z}}.
$$

Proof (conclusion)

Taking $\alpha = 2\mathcal{C}^2M^2$ we obtain

$$\left\|\mathcal{F}(\widetilde{y}) - \mathcal{F}(\widetilde{\widetilde{y}})\right\|_{\mathcal{Z}} \leq \frac{1}{2}\left\|\widetilde{y} - \widetilde{\widetilde{y}}\right\|_{\mathcal{Z}},$$

i.e. $\mathcal{F}$ is a **contractive map**. Therefore, it admits a unique fixed point, which is the solution to the equation.

Let now y be this unique solution. We have that

$$\begin{aligned}
&\|y(t)\|^2_{H^1_p(\mathbb{T})} + \|y_t(t)\|^2_{L^2_p(\mathbb{T})} \\
&\quad \leq \mathcal{C}\left(\left\|(y^0, y^1)\right\|^2_{H^1_p(\mathbb{T})\times L^2_p(\mathbb{T})} + \|u\|^2_{L^2(\mathcal{O})} + |M|\int_0^t \|y(s)\|^2_{L^2_p(\mathbb{T})}\,ds\right) \\
&\quad \leq \mathcal{C}\left(\left\|(y^0, y^1)\right\|^2_{H^1_p(\mathbb{T})\times L^2_p(\mathbb{T})} + \|u\|^2_{L^2(\mathcal{O})}\right) + \mathcal{C}|M|\int_0^t \left(\|y(s)\|^2_{H^1_p(\mathbb{T})} + \|y_s(s)\|^2_{L^2_p(\mathbb{T})}\,ds\right).
\end{aligned}$$

Using **Gronwall's inequality** we get

$$\|y(t)\|^2_{H^1_p(\mathbb{T})} + \|y_t(t)\|^2_{L^2_p(\mathbb{T})} \leq \mathcal{C}\left(1 + \mathcal{C}|M|e^{\mathcal{C}|M|t}\right)\left(\left\|(y^0, y^1)\right\|^2_{H^1_p(\mathbb{T})\times L^2_p(\mathbb{T})} + \|u\|^2_{L^2()}\right).$$

On the necessity of a moving control

We want to prove a null controllability result for our equation. Before doing that, we discuss further the necessity of a moving control.

- We show that the spectrum has an accumulation point.
- We show that there are solutions which are localized along vertical characteristics and, therefore, do not propagate in time.

Reduction to a coupled PDE/ODE system

$$z = \int_0^t y(s)\,ds \quad \Rightarrow \begin{cases} y_{tt} - y_{xx} + M z_{xx} = u\mathbf{1}_{\omega(t)}, & (x,t) \in Q \\ z_t = y, & (x,t) \in Q \\ y(0) = y^0, \;\; y_t(0) = y^1, \;\; z(0) = 0, & x \in \mathbb{T}. \end{cases}$$

Our original equation can be equivalently written as a first order system.

Reduction to a first order system

$$\begin{cases} Y'(t) + \mathcal{A}Y = 0, & t \in (0,T) \\ Y(0) = Y^0 \end{cases}, \quad Y(t) = \begin{pmatrix} y \\ w \\ z \end{pmatrix}, \quad Y^0 = \begin{pmatrix} y^0 \\ y^1 \\ 0 \end{pmatrix}$$

$$\mathcal{A} : D(\mathcal{A}) \to X_{-\sigma}, \quad \mathcal{A}\begin{pmatrix} y \\ w \\ z \end{pmatrix} = \begin{pmatrix} -w \\ -y_{xx} + Mz_{xx} \\ -y \end{pmatrix}.$$

The spaces $X_{-\sigma}$ and $D(\mathcal{A})$ are given respectively by

$$X_{-\sigma} = H_p^{-\sigma}(\mathbb{T}) \times H_p^{-\sigma-1}(\mathbb{T}) H_p^{-\sigma}(\mathbb{T})$$

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} y \\ w \\ z \end{pmatrix} \in X_{-\sigma} : y - Mz \in H_p^{-\sigma-1}(\mathbb{T}) \right\}.$$

Theorem

The spectrum of the operator $(D(\mathcal{A}), \mathcal{A})$ is given by

$$\sigma(\mathcal{A}) = \left(\mu_n^j\right)_{n\in\mathbb{N}^*,\, 1\leq j\leq 3},$$

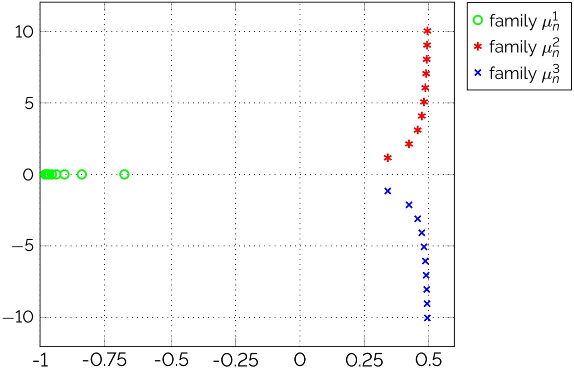
where the eigenvalues μ_n^j verify the following estimates

$$\begin{cases} \mu_n^1 = -M + \dfrac{M^3}{n^2} + \mathcal{O}\left(\dfrac{1}{n^4}\right), & n \in \mathbb{N}^* \\ \mu_n^2 = \dfrac{M}{2} - \dfrac{M^3}{2n^2} + in + i\,\dfrac{3M^2}{8n} + \mathcal{O}\left(\dfrac{1}{n^3}\right), & n \in \mathbb{N}^* \\ \mu_n^3 = \overline{\mu_n^2}, & n \in \mathbb{N}^*. \end{cases}$$

Each eigenvalue $\mu_n^j \in \sigma(\mathcal{A})$ is double and has two associated eigenvectors

$$\Phi_{\pm n}^j = \begin{pmatrix} 1 \\ -\mu_n^j \\ \dfrac{1}{\mu_n^j} \end{pmatrix} e^{\pm inx}, \qquad j \in \{1,2,3\},\ n \in \mathbb{N}^*.$$

Spectral analysis



Distribution of the eigenvalues μ_n^j for $n \in \{1, \dots, 10\}$, corresponding to $M = 1$. The accumulation of the family μ_n^1 zeros at $-M$ appears.

Proof

$$\mathcal{A}\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix} = \begin{pmatrix}-\phi^2\\ -\phi^1_{xx} + M\phi^3_{xx}\\ \phi^1\end{pmatrix} = \mu\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\end{pmatrix} \Rightarrow \begin{cases}\phi^2 = -\mu\phi^1\\ \phi^3 = \mu^{-1}\phi^1\\ -(\mu - M)\phi^1_{xx} = -\mu^3\phi^1\\ \phi^1 \in H^{2+\sigma}_p(\mathbb{T}).\end{cases}$$

Plugging the first two equations in the third one, we immediately get

$$-\phi^1_{xx} = \left(-\frac{\mu^3}{\mu - M}\right)\phi^1.$$

Consequently, ϕ^1 takes the form

$$\phi^1(x) = e^{inx}, \quad n \in \mathbb{Z}^*,$$

and μ is an eigenvalue of the operator $\mathcal{A}$ corresponding to ϕ^1 if and only if verifies the **characteristic equation**

$$\mu^3 + n^2\mu - Mn^2 = 0.$$

Proof (cont.)

For each $n \geq 1$, the characteristic equation has a unique real root $\mu_n^1 \in (0, M)$.

$$\mu_n^1 = M - \frac{M(\mu_n^1)^2}{(\mu_n^1)^2 + n^2} \Rightarrow \mu_n^1 = M - \frac{M^3}{n^2} + \mathcal{O}\left(\frac{1}{n^4}\right), \qquad n \geq 1.$$

In particular, M is an **accumulation point**.

Consider now that $\mu = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Plugging this in the characteristic equation and setting both the real an the imaginary part to zero, we have

$$\begin{cases} \beta^2 = 3\alpha^2 + n^2 \\ -8\alpha^3 - 2\alpha n^2 - Mn^2 = 0. \end{cases}$$

Hence, -2α is the real solution of the characteristic equation and we deduce that

$$\begin{cases} \mu_n^{2,3} = \alpha_n^{2,3} + i\beta_n^{2,3}, & n \in \mathbb{N}^* \\ \alpha_n^2 = \alpha_n^3 = -\dfrac{\mu_n^1}{2}, & n \in \mathbb{N}^* \\ \beta_n^2 = \sqrt{3\left(\dfrac{\mu_n^1}{2}\right)^2 + n^2}, \quad \beta_n^3 = -\sqrt{3\left(\dfrac{\mu_n^1}{2}\right)^2 + n^2}, & n \in \mathbb{N}^*. \end{cases}$$

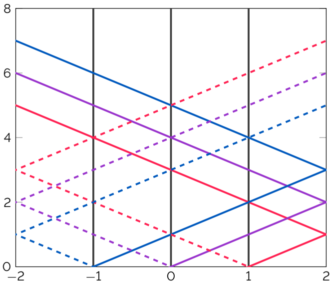
Construction of a localized solution

Adjoint equation

$$p_{tt} - p_{xx} + M \int_0^t p_{xx}(s)\,ds + M q^0_{xx}, \quad (x,t) \in \mathbb{R} \times (0,T)$$

Characteristics

$$p_{ttt} - p_{xxt} + M p_{xx} = 0$$
$$\Downarrow$$
$$\mathcal{P}(x,t,\xi,\tau) = \tau^3 - \tau|\xi|^2 = \tau(\tau^2 - |\xi|^2)$$



Construction of a localized solution

Theorem

For any $x_0 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$, let the functions p^{ε} be defined as

$$p^{\varepsilon}(x,t) := \frac{1}{x_0}\left(\frac{2}{\pi}\right)^{\frac{1}{4}} \varepsilon^{\frac{7}{8}} e^{\frac{i}{\varepsilon}x - \frac{1}{\sqrt{\varepsilon}}(x-x_0)^2 + Mt - M^3\varepsilon^2 t}.$$

- The p^{ε} are approximate solutions, with the choice

$$q^0(x) := \frac{1}{M - M^3\varepsilon^2} p^{\varepsilon}(x,0).$$

- The initial energy of p^{ε} satisfies

$$E^{\varepsilon}(0) := E(p^{\varepsilon})(0) = 1 + \mathcal{O}(\sqrt{\varepsilon}),$$

  i.e. it is bounded as $\varepsilon \to 0$.
- The energy of p^{ε} is exponentially small off the vertical ray (t, x_0):

$$\int_{|x-x_0| > \varepsilon^{\frac{1}{8}}} \left(|p_x^{\varepsilon}|^2 + |p_t^{\varepsilon}|^2\right) dx = \mathcal{O}(e^{-2\varepsilon^{-\frac{1}{4}}})$$

Construction of a localized solution

The adjoint system

Adjoint system

$$\begin{cases} p_{tt} - p_{xx} + Mq_{xx} = 0, & (x,t) \in Q \\ -q_t = p, & (x,t) \in Q \\ p(T) = p^0, \quad p_t(T) = p^1, & x \in \mathbb{T} \\ q(T) = q^0, & x \in \mathbb{T} \end{cases}$$

Change of variables $p(x,t) = \varphi(x+ct,t), \quad q(x,t) = \psi(x+ct,t)$

$$\begin{cases} \varphi_{tt} - (1-c^2)\varphi_{xx} + 2c\varphi_{xt} + M\psi_{xx} = 0, & (x,t) \in Q \\ -\psi_t - c\psi_x = \varphi, & (x,t) \in Q \\ \varphi(T) = p^0, \quad \varphi_t(T) = p^1 - cp^0_x, & x \in \mathbb{T} \\ \psi(T) = q^0, & x \in \mathbb{T}. \end{cases}$$

P. Martin, L. Rosier and P. Rouchon, *Null controllability of the structurally damped wave equation with moving control* SICON, 2013.

Reduction to a first order system

$$\begin{cases} \Phi'(t) + \mathcal{A}_c \Phi = 0, \quad t \in (0, T) \\ \Phi(0) = \Phi^0 \end{cases},$$

$$\Phi(t) = \begin{pmatrix} \varphi \\ \eta \\ \psi \end{pmatrix}, Y^0 = \begin{pmatrix} p^0 \\ p^1 - cp_x^0 \\ q^0 \end{pmatrix}$$

$$\mathcal{A}_c \begin{pmatrix} \varphi \\ \varphi_t \\ \psi \end{pmatrix} = \begin{pmatrix} -\eta \\ -(1-c^2)\varphi_{xx} + 2c\eta_x + M\psi_{xx} \\ c\psi_x + \varphi \end{pmatrix}.$$

The domain $D(\mathcal{A}_c)$ is the same as $D(\mathcal{A})$.

Spectral analysis

Theorem

The spectrum of the operator $(D(\mathcal{A}_c), \mathcal{A}_c)$ is given by

$$\sigma(\mathcal{A}_c) = \left(\lambda_n^j\right)_{n\in\mathbb{Z}^*,\, 1\leq j\leq 3},$$

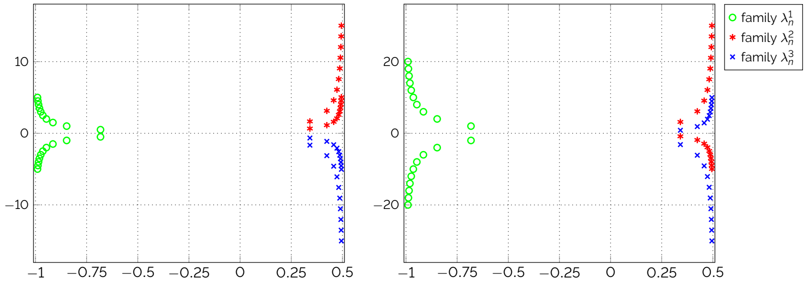
where the eigenvalues λ_n^j are defined as follows

$$\lambda_n^j = cni + \mu_{|n|}^j, \qquad n\in\mathbb{Z}^*,\ 1\leq j\leq 3.$$

Each eigenvalue $\lambda_n^j \in \sigma(\mathcal{A}_c)$ is simple and has an associated eigenvector of the form

$$\Psi_n^j = \begin{pmatrix} 1 \\ -\lambda_n^j \\ \dfrac{1}{(cni-\lambda_n^j)} \end{pmatrix} e^{inx}, \qquad j\in\{1,2,3\},\ n\in\mathbb{Z}^*.$$

Spectral analysis



Distribution of the eigenvalues λ_n^j for $n \in \{1, \ldots, 10\}$, corresponding to $M = 1$ and $c = 0.5$ (left) and $c = 2$ (right).

Lemma

Let $0 \leq \sigma < \infty$. The equation is memory-type null controllable in time T if and only if, for each initial data $(y^0, y^1) \in H_p^{\sigma+1}(\mathbb{T}) \times H_p^{\sigma}(\mathbb{T})$, there exits a control $\widetilde{u} \in L^2(Q)$ verifying

$$\int_{-\pi}^{\pi} \mathbf{1}_{\omega_0} \widetilde{u}(t,x)\, dx = 0 \qquad t \in (0,T),$$

such that, for any $(p^0, p^1, q^0) \in L_p^2(\mathbb{T}) \times H_p^{-1}(\mathbb{T}) \times L_p^2(\mathbb{T})$, it holds the identity

$$\int_0^T \int_{\omega_0} \widetilde{u}(t,x)\bar{\varphi}(t,x)\, dxdt = \langle y^0(\cdot), \varphi_t(0,\cdot)\rangle_{H_p^1(\mathbb{T}), H_p^{-1}(\mathbb{T})} - \int_{\mathbb{T}} \left(y^1(x) + c y_x^0(x)\right) \bar{\varphi}(0,x)\, dx,$$

where (φ, ψ) is the unique solution to the corresponding adjoint system.

Lemma

For each initial data

$$\begin{pmatrix} \varphi(T,x) \\ \varphi_t(T,x) \\ \psi(T,x) \end{pmatrix} = \begin{pmatrix} \varphi^0(x) \\ \varphi^1(x) \\ \psi^0(x) \end{pmatrix} = \sum_{(n,j)\in S} b_n^j \Psi_n^j(x) \in L_p^2(\mathbb{T}) \times H_p^{-1}(\mathbb{T}) \times L_p^2(\mathbb{T}),$$

there exists a unique solution of the adjoint equation given by

$$\begin{pmatrix} \varphi(t,x) \\ \varphi_t(t,x) \\ \psi(t,x) \end{pmatrix} = \sum_{(n,j)\in S} b_n^j e^{\lambda_n^j (T-t)} \Psi_n^j(x),$$

where we denoted $S := \{(n,j) : n \in \mathbb{Z}^*, \;\; 1 \leq j \leq 3\}$

The controllability problem

Lemma

Let $0 \leq \sigma < \infty$. The equation is memory-type null controllable at time T if, for each initial data $(y^0, y^1) \in H_p^{\sigma+1}(\mathbb{T}) \times H_p^{\sigma}(\mathbb{T})$,

$$y^0(x) = \sum_{n\in\mathbb{Z}^*} y_0^n e^{inx}, \quad y^1(x) = \sum_{n\in\mathbb{Z}^*} y_1^n e^{inx},$$

there exists $\widehat{u} \in L^2(Q)$ such that the following relations hold

$$\int_0^T \int_{\omega_0} \widehat{u}(t,x) e^{-inx} e^{-\bar{\lambda}_n^j t}\, dxdt = -2\pi\left(\bar{\mu}_{|n|}^j y_n^0 + y_n^1\right), \qquad (n,j) \in S$$

$$\int_0^T \int_{\omega_0} \widehat{u}(t,x) e^{-\bar{\lambda}_n^j t}\, dxdt = 0, \qquad (n,j) \in S.$$

Some spectral properties

Theorem

Each eigenvalue $\lambda_n^j \in \sigma(\mathcal{A}_c)$ has an associated eigenvector of the form

$$\Psi_n^j = \begin{pmatrix} 1 \\ -\lambda_n^j \\ \dfrac{1}{\lambda_n^j - icn} \end{pmatrix} e^{inx}, \qquad j \in \{1,2,3\},\ n \in \mathbb{Z}^*.$$

Moreover, for any $\sigma \geq 0$, the set $\left(|n|^\sigma \Psi_n^j\right)_{n\in\mathbb{Z}^*,\,1\leq j\leq 3}$ forms a Riesz basis in the spaces $X_{-\sigma}$.

RIESZ BASIS: there exist two positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that

$$\mathcal{C}_1 \sum_{(n,j)\in S} |a_n^j|^2 \leq \left\| \sum_{(n,j)\in S} a_n^j |n|^\sigma \Psi_n^j \right\|_{X_{-\sigma}}^2 \leq \mathcal{C}_2 \sum_{(n,j)\in S} |a_n^j|^2.$$

Some spectral properties

Lemma

All the elements of the spectrum $\sigma(\mathcal{A}_c) = \left(\lambda_n^j\right)_{(n,j)\in S}$ are well separated one from another except for the following special cases:

- If $c \in (0,1)$ the eigenvalues λ_m^2 and $\lambda_{-n_m}^2$ have a distance at least of order $\frac{1}{m^2}$ between them and a similar relation holds for λ_m^3 and $\lambda_{-n_m}^3$.
- If $c \in (1,\infty)$ the eigenvalues λ_m^2 and $\lambda_{n_m}^3$ have a distance at least of order $\frac{1}{m^2}$ between them and a similar relation holds for λ_{-m}^3 and $\lambda_{-n_m}^2$.
- If $c \in \mathcal{V}$, there exists a unique double eigenvalues $\lambda_{-n_c}^2 = \lambda_{n_c}^3$, where we denoted

$$\mathcal{V} = \left\{ \sqrt{1 + 3\left(\frac{\mu_n^1}{2n}\right)^2} \; : \; n \in \mathbb{N}^* \right\}.$$

A biorthogonal family

Theorem

Let $c \in \mathbb{R} \setminus \{-1, 0, 1\}$ and let define

$$P(z) = z^3 \lim_{R\to\infty} \prod_{\substack{(n,j)\in S \\ |\lambda_n^j|\leq R}} \left(1 + \frac{z}{i\overline{\lambda}_n^j}\right).$$

- P is an entire function of exponential type $\left(\frac{1}{|c|} + \frac{1}{|1+c|} + \frac{1}{|1-c|}\right)\pi$.
- For each $\delta > 0$, there exists a positive constant $\mathcal{C}(\delta) > 0$ such that

  $$|P(z)| \leq C(\delta), \qquad z = x + iy,\ x, y \in \mathbb{R},\ |y| \leq \delta.$$

  Moreover, there exists a constant $\mathcal{C}_1 > 0$ such that, for any $(m, k) \in S$,

  $$\left|\frac{P(x)}{x + i\overline{\lambda}_m^k}\right| \leq \frac{C_1}{1 + \left|x + \Im(\lambda_m^k)\right|}, \qquad z = x + iy,\ x, y \in \mathbb{R},\ |y| \leq \delta.$$

- Each $-i\overline{\lambda}_m^j$ is a simple zero of P and there exists a positive constant $\mathcal{C}_2$ such that

  $$\left|P'(-i\overline{\lambda}_m^j)\right| \geq \frac{C_2}{m^2}, \qquad (m, j) \in S.$$

A biorthogonal family

Theorem

Let $c \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $T > 2\pi\left(\frac{1}{|c|} + \frac{1}{|c-1|} + \frac{1}{|c+1|}\right)$. There exist a biorthogonal sequence $(\theta_m^k)_{(m,k)\in S}$ to the family of complex exponentials $\left(e^{-\lambda_n^j t}\right)_{(n,j)\in S}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ and a positive constant C with the property that

$$\left\|\sum_{(m,k)\in S} \beta_m^k \theta_m^k\right\|^2_{L^2(-\frac{T}{2},\frac{T}{2})} \leq C \sum_{(m,k)\in S} m^4 \left|\beta_m^k\right|^2,$$

for any finite sequence of complex numbers $(\beta_m^k)_{(m,k)\in S}$.

BIORTHOGONAL SEQUENCE: $(\theta_m^k)_{(m,k)\in S}$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m^k(t) e^{-\overline{\lambda}_n^j t}\, dt = \delta_{m,k}^{n,j}.$$

Corollary

Let $c \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $T > 2\pi \left(\frac{1}{|c|} + \frac{1}{|c-1|} + \frac{1}{|c+1|} \right)$. For any finite sequence of scalars $(a_n^j)_{(n,j)\in S} \subset \mathbb{C}$, it holds the inequality

$$\sum_{(n,j)\in S} \frac{\left|a_n^j\right|^2}{n^4} \leq C \left\| \sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t} \right\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)}^2 .$$

PROOF: By taking into account the orthogonality properties of $\left(\theta_m^k\right)_{(m,k)\in S}$ we deduce

$$\sum_{(n,j)\in S} \frac{\left|a_n^j\right|^2}{n^4} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{\left(\sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t} \right)} \left(\sum_{(m,k)\in S} \frac{a_m^k}{m^4} \theta_m^k(t) \right) dt$$

$$\leq \left\| \sum_{(n,j)\in S} a_n^j e^{-\lambda_n^j t} \right\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} \left\| \sum_{(m,k)\in S} \frac{a_m^k}{m^4} \theta_m^k \right\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} .$$

A technical result

Theorem

Let $(f_n)_n$ be a sequence of vectors belonging to a Hilbert space H and $(c_n)_n$ a sequence of scalars. In order that the equations

$$(f, f_n) = c_n$$

shall admit at least one solution $f \in H$ for which $\|f\|_H \leq M$, it is necessary and sufficient that

$$\left|\sum_n a_n \bar{c}_n\right| \leq M \left\|\sum_n a_n f_n\right\|_H \qquad (\mathcal{Y})$$

for every finite sequence of scalars $(a_n)_n$.

R. M. Young, *An Introduction to Non-Harmonic Fourier Series*, Elsevier, 2001.

Controllability result

Theorem

Let $M \neq 0$, $c \in \mathbb{R} \setminus \{-1, 0, 1\}$, $T > 2\pi \left(\frac{1}{|c|} + \frac{1}{|1-c|} + \frac{1}{|1+c|} \right)$, ω a non void open set in $\mathbb{T}$ and

$$\omega(t) = \omega_0 - ct, \qquad t \in [0, T].$$

For each initial data $(y^0, y^1) \in H^3_p(\mathbb{T}) \times H^2_p(\mathbb{T})$ there exists a control $u \in L^2(\mathcal{O})$ such that the solution (y, y_t) of our original problem verifies

$$y(T, x) = y_t(T, x) = \int_0^T y_{xx}(s, x)\, ds = 0.$$

Proof

It is enough to show that, for each initial data $(y^0, y^1) \in H^3_p(\mathbb{T}) \times H^2_p(\mathbb{T})$ in the form

$$y^0(x) = \sum_{n\in\mathbb{Z}^*} y_0^n e^{inx}, \quad y^1(x) = \sum_{n\in\mathbb{Z}^*} y_1^n e^{inx},$$

there exists $\widehat{u} \in L^2(Q)$ such that

$$\int_0^T \int_{\omega_0} \widehat{u}(t,x) e^{-inx} e^{-\bar{\lambda}_n^j t}\, dxdt = -2\pi\left(\bar{\mu}_{|n|}^j y_n^0 + y_n^1\right), \qquad (n,j) \in S$$

$$\int_0^T \int_{\omega_0} \widehat{u}(t,x) e^{-\bar{\lambda}_n^j t}\, dxdt = 0, \qquad (n,j) \in S.$$

This is equivalent to show that the following inequality holds

$$\left|\sum_{(n,j)\in S} (\mu_{|n|}^j \bar{y}_n^0 + \bar{y}_n^1) a_n^j\right|^2 \leq C \int_0^T \int_{\omega_0} \left|\sum_{(n,j)\in S} b_n^j e^{-\lambda_n^j t} + \sum_{(n,j)\in S} a_n^j e^{inx} e^{-\lambda_n^j t}\right|^2 dxdt,$$

for all finite sequences $(a_n^j)_{(n,j)\in S} \cup (b_n^j)_{(n,j)\in S} \subset \mathbb{C}$.

Proof (cont.)

Notice that

$$\left|\sum_{(n,j)\in S}(\mu^j_{|n|}\bar{y}^0_n+\bar{y}^1_n)a^j_n\right|^2 \le \left(\sum_{(n,j)\in S} n^4\left|\mu^j_{|n|}\bar{y}^0_n+\bar{y}^1_n\right|^2\right)\left(\sum_{(n,j)\in S}\frac{|a^j_n|^2}{n^4}\right)$$

$$\le \mathcal{C}\left\|(y^0,y^1)\right\|^2_{H^3_p(\mathbb{T})\times H^2_p(\mathbb{T})}\left(\sum_{(n,j)\in S}\frac{|a^j_n|^2}{n^4}\right).$$

On the other hand, taking into account that we can have at most one double eigen-value $\lambda^2_{-n_c}$ (if $c\in\mathcal{V}$), we deduce that

$$\int_0^T\int_{\omega_0}\left|\sum_{(n,j)\in S} b^j_n e^{-\lambda^j_n t}+\sum_{(n,j)\in S} a^j_n e^{inx}e^{-\lambda^j_n t}\right|^2 dxdt$$

$$=\int_{\omega_0}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{(n,j)\in S}\left(a^j_n e^{inx}+b^j_n\right)e^{\lambda^j_n\frac{T}{2}}e^{-\lambda^j_n t}\right|^2 dtdx$$

$$\ge \mathcal{C}\Bigg(\sum_{(n,j)\in S\setminus\{(-n_c,2),\,(n_c,3)\}}\frac{1}{n^4}\int_{\omega_0}\left|\left(a^j_n e^{inx}+b^j_n\right)e^{\lambda^j_n\frac{T}{2}}\right|^2 dx$$

$$+\int_{\omega_0}\left|a^2_{-n_c}e^{\lambda^2_{-n_c}\frac{T}{2}}e^{-in_cx}+a^3_{n_c}e^{\lambda^3_{n_c}\frac{T}{2}}e^{in_cx}+b^2_{-n_c}e^{\lambda^2_{-n_c}\frac{T}{2}}+b^3_{n_c}e^{\lambda^3_{n_c}\frac{T}{2}}\right|^2 dx\Bigg).$$

Proof (conclusion)

Notice that, if $c \notin \mathcal{V}$, all the eigenvalues are simple and the separation of the second term in the last inequality is not needed.

Since the maps

$$\mathbb{C}^2 \ni (a,b) \mapsto \left(\int_{\omega_0} \left| a e^{inx} + b \right|^2 dx \right)^{\frac{1}{2}},$$

$$\mathbb{C}^3 \ni (a',a'',b) \mapsto \left(\int_{\omega_0} \left| a' e^{-in_c x} + a'' e^{in_c x} + b \right|^2 dx \right)^{\frac{1}{2}},$$

are norms in $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively, it follows that

$$\int_0^T \int_{\omega_0} \left| \sum_{(n,j)\in S} b_n^j e^{-\lambda_n^j t} + \sum_{(n,j)\in S} a_n^j e^{inx} e^{-\lambda_n^j t} \right|^2 dxdt \geq C \sum_{(n,j)\in S} \frac{|a_n^j|^2}{n^4}.$$

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