CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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PART II: non-local in time models

LECTURE 4: fractional calculus and fractional-in-time ODE and PDE

Historical introduction

Origins of fractional calculus

Fractional calculus was born in 1697

In the letters to J. Wallis and J. Bernoulli in 1697, Leibniz mentioned the possible approach to fractional-order differentiation of exponential functions in that sense, that for non-integer values of n the definition could be the following:

$$\frac{d^n}{dx^n}e^{mx} = m^n e^{mx}$$

Origins of fractional calculus

Euler (1730)

For $m, n \in \mathbb{N}$:

$$\frac{d^n}{dx^n}x^m = m(m-1)(m-2)\cdot(m-n+1)x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

Γ: **Euler gamma function**

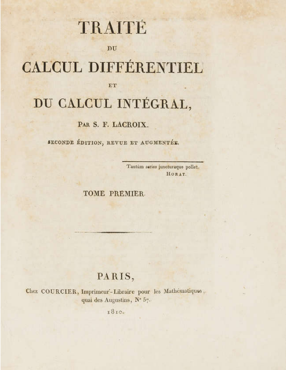
Euler suggested to use this relationship also for **negative** or **non-integer (rational)** values of n. Taking $m = 1$ and $n = 1/2$, he obtained:

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}} = \sqrt{\frac{4x}{\pi}}$$

Origins of fractional calculus

Euler (1730)

Lacroix adopted Euler's derivation for his successful textbook
S. F. Lacroix, *Traité du calcul différentiel et du calcul intégral*, Courcier, 1819.



TRAITÉ
DU
CALCUL DIFFÉRENTIEL
ET
DU CALCUL INTÉGRAL,
PAR S. F. LACROIX.
SECONDE ÉDITION, REVUE ET AUGMENTÉE.

Tantùm series juncturaque pollet.
HORAT.

TOME PREMIER.

PARIS,
Chez COURCIER, Imprimeur-Libraire pour les Mathématiques,
quai des Augustins, N° 57.
1810.

Origins of fractional calculus

Fourier (1820-1822)

The first step to generalization of the notion of differentiation for arbitrary functions was done by Fourier J. B. J. Fourier, *Théorie analytique de la chaleur*, Didot, 1822.

After introducing his famous formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)\left(\int_{-\infty}^{+\infty} \cos(px - pz)\, dp\right) dz,$$

Fourier made a remark that

$$\frac{d^n}{dx^n} f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)\left(\int_{-\infty}^{+\infty} \cos\left(px - pz + \frac{n\pi}{2}\right) dp\right) dz,$$

and this relationship could serve as a definition of the n-th order derivative for non-integer n.

Abel (1823) and Liouville (1826)

They solved the integral equation

$$f(x) = \int_a^x \frac{\phi(t)}{(x-t)^\mu}\, dt, \; x > a, \; 0 < \mu < 1$$

in connection with the **tautochrone problem**. The solution was given for all $0 < \mu < 1$, although the tautochrone problem itself leads to the case $\mu = 1/2$.

Riemann-Liouville fractional integral

Riemann-Liouville fractional integral (1876)

The works of Abel and Liouville led to the definition of the **Rieman-Liouville fractional integral**

$$J^{\alpha}_{a+}f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau, \quad 0<\alpha\in\mathbb{R} \qquad \textbf{RIGHT-SIDED INTEGRAL}$$

$$J^{\alpha}_{b-}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau, \quad 0<\alpha\in\mathbb{R} \qquad \textbf{LEFT-SIDED INTEGRAL}$$

still in use nowadays.

From integer to fractional integrals and derivatives

Towards fractional integrals and derivatives

It all starts from the **fundamental theorem of calculus**

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let the function $F : [a, b] \to \mathbb{R}$ be defined by

$$F(t) = \int_a^t f(\tau)\, d\tau.$$

Then, F is differentiable and $F'(t) = f(t)$.

We have a very close relation between differential operators and integral operators. It is one of the goals of fractional calculus **to retain this relation in a suitably generalized sense**.

Towards fractional integrals and derivatives

Definition

We denote by D the operator that maps a differentiable function onto its derivative, i.e.

$$Df(t) = f'(t).$$

We denote by J_a the operator that maps a function f, assumed to be (Riemann) integrable on the compact interval $[a, b]$, onto its primitive centered at a, i.e.

$$J_a f(t) = \int_a^t f(\tau)\, d\tau \quad \text{for } a \leq t \leq b.$$

For $n \in \mathbb{N}$ we use the symbols D^n and J_a^n to denote the n-fold iterates of D and J_a, respectively, i.e. we set

$$\begin{aligned} D^1 &:= D & J_a^1 &:= J_a \\ D^n &:= DD^{n-1} & J_a^n &:= J_a J_a^{n-1} \quad \text{for } n \geq 2. \end{aligned}$$

Key question

How can we extend these concepts to $n \notin \mathbb{N}$?

Some properties

For $n \in \mathbb{N}$, it is well known (and easily proved by induction) that we can replace the recursive definition J_a^n by the following explicit formula

$$J_a^n f(t) = \frac{1}{(n-1)!} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}} \, d\tau$$

for all f Riemann integrable on $[a, b]$ and $a \leq t \leq b$.

Moreover, it is an immediate consequence of the previous identity (and therefore a consequence of the fundamental theorem of calculus) that the following relation holds for the operators D and J_a.

Lemma

Let $m, n \in \mathbb{N}$ such that $m > n$, and let f be a function having a continuous n-th derivative on the interval $[a, b]$. Then,

$$D^n f = D^m J_a^{m-n} f.$$

PROOF: we have $f = D^{m-n} J_a^{m-n} f$. Applying the operator D^n to both sides of this relation and using the fact that $D^n D^{m-n} = D^m$, the statement follows.

Towards fractional integrals and derivatives

Fundamental theorem of calculus in Lebesgue spaces

Let $f \in L^1([a, b])$. Then, $J_a f$ is differentiable almost everywhere in $[a, b]$, and $DJ_a f = f$ holds almost everywhere on $[a, b]$.

Definition

The function $\Gamma : (0, +\infty) \to \mathbb{R}$, defined by

$$\Gamma(t) = \int_0^{+\infty} s^{t-1} e^{-s}\, ds, \quad \Re(t) > 0$$

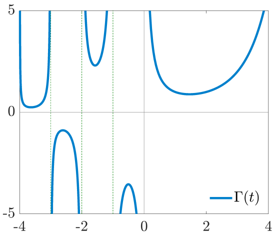
is called **Euler's Gamma function** (or **Euler's integral of the second kind**).
Extended to $\Re(t) < 0$ by analytic continuation to a meromorphic function with simple poles in $t = 0$ and $t \in \mathbb{Z}^-$.

SOME PROPERTIES:

- $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$.
- $\Gamma(t+1) = t\Gamma(t)$ for all $t > 0$.
- $\displaystyle\int_0^t s^{\alpha-1}(t-s)^{\beta-1}\, ds = t^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ for all $\alpha, \beta > 0$.

SPECIFIC VALUES:

- $\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$
- $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$



Definition

Let $0 \leq \alpha \in \mathbb{R}$. The operator J_a^α, defined on $L^1([a,b])$ by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau \quad \text{for } a \leq t \leq b$$

is the **left Riemann-Liouville fractional integral of order α centered at a**.

For $\alpha = 0$, we set $J_a^0 := I$, the identity operator.

For $a = 0$, we simply write $J_0^\alpha = J^\alpha$ for all $0 \leq \alpha \in \mathbb{R}$.

To emphasize the fact that $a < t$, sometimes it is used the notation $J_{a+}^\alpha f(t)$.

Some properties

Theorem

Let $f \in L^1([a,b])$ and $0 < \alpha \in \mathbb{R}$. Then, the integral $J_a^\alpha f(t)$ exists for almost every $t \in [a,b]$. Moreover, $J_a^\alpha f \in L^1([a,b])$.

Theorem

Let $0 < \alpha, \beta \in \mathbb{R}$ and $f \in L^1([a,b])$. Then, $J_a^\alpha J_a^\beta f = J_a^{\alpha+\beta} f$ holds almost everywhere on $[a,b]$. If additionally $f \in C([a,b])$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a,b]$.

Corollary

Under the previous assumptions, $J_a^\alpha J_a^\beta f = J_a^\beta J_a^\alpha f$.

Some properties

Riemann-Liouville integral of power functions

For all $\alpha > 0$, $\beta > -1$ and $t > a$

$$J_a^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}$$

Notice that when $\alpha, \beta \in \mathbb{N}$

$$\begin{aligned} J_a^\alpha (t-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta} \\ &= \frac{\beta!}{(\alpha+\beta)!}(t-a)^{\alpha+\beta} \\ &= \frac{(t-a)^{\alpha+\beta}}{(\beta+1)(\beta+2)\cdots(\beta+\alpha)}. \end{aligned}$$

Right Riemann-Liouville integral

Definition

Let $0 \leq \alpha \in \mathbb{R}$. The operator J_{b-}^{α}, defined on $L^1([a,b])$ by

$$J_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau - t)^{1-\alpha}}\, d\tau \quad \text{for } a \leq t \leq b$$

is the **right Riemann-Liouville fractional integral of order α centered at b**.

For $\alpha = 0$, we set $J_{b-}^{0} := I$, the identity operator.

Riemann-Liouville derivatives

Lemma

For all $n \in \mathbb{N}$, we have $D^n J^n = I$ but the inverse is false: $J^n D^n \neq I$. In fact

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k$$

D^n is the **left inverse** (but not the **right inverse**) of J^n.

This motivates defining D^α for $0 < \alpha \in \mathbb{R}$ as the **left inverse of J^α**.

Riemann-Liouville derivatives

Definition

Len $n \in \mathbb{N}$. By $A^n([a, b])$ we denote the set of absolutely continuous functions with absolutely continuous $(n-1)$ derivative on $[a, b]$.

Definition

Let $0 \leq \alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The operator D_a^α, defined on $A^1([a, b])$ by

$$D_a^\alpha f = D^m J_a^{m-\alpha} f$$

is the **Riemann-Liouville fractional derivative of order α centered at a**.

For $\alpha = 0$, we set $D_a^0 := I$, the identity operator.

For $a = 0$, we simply write $D_0^\alpha = D^\alpha$ for all $0 \leq \alpha \in \mathbb{R}$.

Some properties

Theorem

Let $0 < \alpha \in \mathbb{R}$ and $\alpha < m \in \mathbb{N}$. Then, $D_a^\alpha = D_a^m J_a^{m-\alpha}$.

Theorem

Let $f \in A^1([a,b])$, $0 \leq \alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. Then D_a^α exists almost everywhere in $[a,b]$. Moreover $D_a^\alpha f \in L^p([a,b])$ for $1 \leq p < 1/\alpha$ and

$$D_a^\alpha f(t) = \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau\right).$$

Some properties

Riemann-Liouville derivative of power functions

For all $\alpha > 0$, $\beta > -1$ and $t > 0$

$$D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$$

Notice that when $\alpha, \beta \in \mathbb{N}$ with $\beta > \alpha + 1$

$$D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} = \frac{\beta!}{(\beta-\alpha)!} t^{\beta-\alpha} = \beta(\beta-1)\cdots(\beta-\alpha+1) t^{\beta-\alpha}$$

ATTENTION!

The Riemann-Liouville fractional derivative is not zero for the constant functions: for all $0 < \alpha \notin \mathbb{N}$

$$D^{\alpha} 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0.$$

If $0 < \alpha \in \mathbb{N}$, $D^{\alpha} 1 = 0$ due to the poles of the Gamma function.

Caputo fractional derivative

Definition

Let $0 \leq \alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The operator ${}_C D^{\alpha}_{a+}$ defined on $A^1([a,b])$ by

$$ {}_C D^{\alpha}_{a+} f(t) = {}_C D^{\alpha}_{a} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau) - f(a)}{(t-\tau)^{\alpha+1-m}} \, d\tau $$

is the **left Caputo fractional derivative of order α centered at a**.

For $a = 0$, we simply write ${}_C D^{\alpha}_{0} = {}_C D^{\alpha}$ for all $0 \leq \alpha \in \mathbb{R}$.

Definition

Let $0 \leq \alpha \in \mathbb{R}$ and $m = \lceil \alpha \rceil$. The operator ${}_C D^{\alpha}_{b-}$ defined on $A^1([a,b])$ by

$$ {}_C D^{\alpha}_{b-} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt} \int_t^b \frac{f(\tau) - f(b)}{(\tau-t)^{\alpha+1-m}} \, d\tau $$

is the **right Caputo fractional derivative of order α centered at b**.

Caputo fractional derivative

Definition

For all $0 \leq \alpha \in \mathbb{R}$, $m = \lceil \alpha \rceil$ and $f \in A^m([a,b])$, we have

$$ {}_C D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \, d\tau $$

$$ {}_C D_{b-}^\alpha f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_t^b \frac{f^{(m)}(\tau)}{(\tau-t)^{\alpha+1-m}} \, d\tau $$

In particular

$$ {}_C D_a^\alpha f = J_a^{m-\alpha} D^m f $$

Riemann-Liouville VS Caputo derivative

In general, $D^\alpha f := D^m J^{m-\alpha} f \neq J^{m-\alpha} D^m f =: {}_C D^\alpha f$. In fact, for $0 \leq \alpha \in \mathbb{R}$, $m = \lceil \alpha \rceil$ and $t > 0$, we have

$$D^\alpha f(t) = {}_C D^\alpha f(t) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{\Gamma(k - \alpha + 1)} t^{k-\alpha}.$$

Recalling the formula for the Riemann-Liouville fractional derivative of power functions, we then have

$${}_C D^\alpha f(t) = D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k \right).$$

ATTENTION!

The Caputo fractional derivative of a constant function is zero:

$${}_C D^\alpha 1 = 0, \qquad \text{for all } 0 < \alpha \in \mathbb{R}.$$

Application: solution of integral equations

Abel's equation of the first kind

Abel's equation of the first kind

Given a function $g \in A^1([0,t])$, find $f \in L^1([0,t])$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = g(t).$$

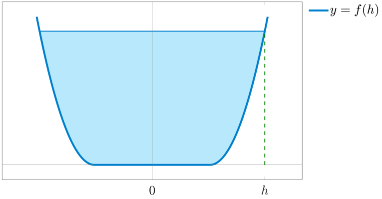
MULTIPLE APPLICATIONS IN DIVERSE FIELDS:

- The mechanical problem of the tautochrone, that is, determining a curve in the vertical plane, such that the time required for a particle to slide down the curve to its lowest point is independent of its initial placement on the curve.
- Evaluation of spectroscopic measurements of cylindrical gas discharges
- Study of the solar or a planetary atmosphere
- Star densities in a globular cluster
- Inversion of travel times of seismic waves for determination of terrestrial sub-surface structure
- Inverse boundary value problems in partial di¤erential equations

Example

Let the colored area in the figure represent the cross section of a weir notch, whose form is determined by the function $y = f(h)$ for $h \geq 0$ the cross section being symmetric w.r.t. the y axis. The quantity of flow through the notch per unit time is given by

$$Q = \int_0^h f(\xi)\sqrt{h-\xi}\, d\xi.$$



Problem

Determine f so that the flow per unit of time is proportional to a given power of the depth of the strem, i.e. $Q = kh^m$ for $m > 0$.

Hence, we must find f from an integral equation of the form

$$\int_0^h f(\xi)\sqrt{h-\xi}\, d\xi = kh^m \quad \xrightarrow{\frac{d}{dh}} \quad \underbrace{\int_0^h \frac{f(\xi)}{\sqrt{h-\xi}}\, d\xi = 2kmh^{m-1} = g(h)}_{\text{Abel's equation of first kind}}.$$

W. C. Brenke, *An application of Abel's integral equation*, Amer. Math. Monthly, 1922.

Abel's equation of the first kind - solution

Abel's equation of the first kind

Given a function $g \in A^1([0,t])$, find $f \in L^1([0,t])$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau = g(t).$$

The equation is immediately solved by observing that it can be written in the form

$$J^{\alpha} f(t) = g(t) \quad \rightarrow \quad f(t) = D^{\alpha} g(t)$$

Abel's equation of the first kind - solution

Coming back to the previous example, we have

$$\int_0^h \frac{f(\xi)}{\sqrt{h-\xi}}\,d\xi = 2kmh^{m-1} \quad \to \quad J^{\frac{1}{2}} f(h) = \frac{2km}{\Gamma\left(\frac{1}{2}\right)} h^{m-1}.$$

Hence

$$f(h) = \frac{2km}{\Gamma\left(\frac{1}{2}\right)} D^{\frac{1}{2}} h^{m-1} = \frac{2km}{\sqrt{\pi}} \frac{\Gamma(m)}{\Gamma\left(m-\frac{1}{2}\right)} h^{m-\frac{3}{2}}$$

Abel's equation of the second kind

Abel's equation of the second kind

Given a function $g \in A^1([0,t])$, find $f \in L^1([0,t])$ such that

$$f(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = g(t), \quad \alpha > 0,\ \lambda \in \mathbb{C}.$$

Most often applied in problems of heat and mass transfer.

Abel's equation of the second kind

Abel's equation of the second kind

In terms of the Riemann-Liouville integral operator:

$$\Big(1+\lambda J^{\alpha}\Big) f(t) = g(t), \quad \alpha > 0,\ \lambda \in \mathbb{C}.$$

FORMAL SOLUTION:

$$f(t) = \Big(1+\lambda J^{\alpha}\Big)^{-1} g(t) = \left(1 + \sum_{k\geq 1} (-\lambda)^k J^{\alpha k}\right) g(t)$$

Noting that $J^{\alpha k} f(t) = \Big(\Phi_{\alpha k} * f\Big)(t)$ with $\Phi_{\alpha k}(t) = \frac{t^{\alpha k-1}}{\Gamma(\alpha k)}$, we then get

$$f(t) = g(t) + \left(\sum_{k\geq 1} (-\lambda)^k \frac{t^{\alpha k-1}}{\Gamma(\alpha k)}\right) * g(t)$$

Relation with the Mittag-Leffler function

Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad 0 < \alpha, \beta \in \mathbb{R}$$

- $\beta = 1$: $E_{\alpha,1}(z) = E_\alpha(z)$
- $\alpha = 0$: the series above equals the Taylor expansion of the geometric series and consequently

$$E_{0,\beta}(z) = \frac{1}{\Gamma(\beta)} \frac{1}{1-z}$$

- $\alpha = \beta = 1$: the series above equals the Taylor expansion of the exponential and consequently

$$E_1(z) = e^z$$

- $\alpha = 2, \ \beta = 1$: the series above equals the Taylor expansion of the hyperbolic cosine evaluated in $\sqrt{z}$ and consequently

$$E_2(z) = \cosh(\sqrt{z})$$

In particular $E_2(-z^2) = \cosh(iz) = \cos(z)$

Relation with the Mittag-Leffler function

We have that

$$\sum_{k\geq 1}(-\lambda)^k\frac{t^{\alpha k-1}}{\Gamma(\alpha k)} = \frac{d}{dt}\sum_{k\geq 0}(-\lambda)^k\frac{t^{\alpha k}}{\Gamma(\alpha k+1)} = \frac{d}{dt}\sum_{k\geq 0}\frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k+1)}$$
$$= \frac{d}{dt}E_{\alpha}(-\lambda t^{\alpha}) = E'_{\alpha}(-\lambda t^{\alpha})$$

Hence, the solution of the Abel's integral equation of second kind is given by

$$f(t) = g(t) + E'_{\alpha}(-\lambda t^{\alpha}) * g(t)$$

Fractional-in-time ODE and PDE

The Laplace transform

Laplace tranfsorm

If $f \in L^1(\mathbb{R}_+)$, we define its **Laplace transform** (usually denoted by $\mathcal{L}f$ or $\tilde{f}$) as

$$\mathcal{L}f(s) = \tilde{f}(s) = \int_0^{+\infty} f(t) e^{-st}\, dt$$

Properties of the Laplace transform

Generalized linearity $$\mathcal{L}\left(\sum_{k\geq 0} a_k f_k(t)\right)(s) = \sum_{k\geq 0} a_k \mathcal{L}f_k(s)$$

LT of powers $$\mathcal{L}(t^k)(s) = \frac{\Gamma(k+1)}{s^{k+1}}$$

LT of derivatives $$\mathcal{L}\left(\frac{d^n}{dt^n}f(t)\right)(s) = s^n \mathcal{L}f(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}f(0)$$

LT of integrals $$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)(s) = \frac{\mathcal{L}f(s)}{s}$$

LT of convolution $$\mathcal{L}\Big(f(t) * g(t)\Big)(s) = \Big(\mathcal{L}f(s)\Big)\Big(\mathcal{L}g(s)\Big)$$

LT of R-L integral $$\mathcal{L}J^{\alpha}f(t) = \frac{\mathcal{L}f(s)}{s^{\alpha}}$$

LT of R-L derivative $$\mathcal{L}D^{\alpha}f(t) = s^{\alpha}\mathcal{L}f(s) - \sum_{k=0}^{m-1} D^k J^{m-\alpha} f(0) s^{m-1-k}, \quad m = \lceil \alpha \rceil$$

LT of Caputo derivative $$\mathcal{L}_c D^{\alpha}f(t) = s^{\alpha}\mathcal{L}f(s) - \sum_{k=0}^{m-1} f^{(k)}(0) s^{\alpha-1-k}, \quad m = \lceil \alpha \rceil$$

Fourier transform

If $f \in L^1(\mathbb{R})$, we define its **Fourier transform** (usually denoted by $\mathcal{F}f$ or $\hat{f}$) as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx$$

Generalized linearity $$\mathcal{F}\left(\sum_{k\geq 0} a_k f_k(x)\right)(\xi) = \sum_{k\geq 0} a_k \mathcal{F}f_k(\xi)$$

LT of derivatives $$\mathcal{F}\left(\frac{d^n}{dt^n}f(x)\right)(\xi) = (i\xi)^n \mathcal{F}f(\xi)$$

LT of convolution $$\mathcal{F}\Big(f(x) * g(x)\Big)(\xi) = \Big(\mathcal{F}f(\xi)\Big)\Big(\mathcal{F}g(\xi)\Big)$$

Fractional-in-time ODE

Linear first-order ODE

$$u'(t) = -u(t), \quad t \geq 0$$
$$u(0) = u_0$$

SOLUTION: $u(t) = u_0 e^{-t}$

Linear first-order ODE

$$u''(t) = -u(t) + q(t), \quad t \geq 0$$
$$u(0) = u_0, \; u'(0) = u_1$$

SOLUTION: $u(t) = u_0 \cos(t) + u_1 \sin(t)$

Linear second-order ODE

What about fractional order ODE?

Fractional-in-time ODE

Linear α-order ODE

$$
{}_{C}D^{\alpha}u(t) = D^{\alpha}\left(u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(0)}{k!}t^{k}\right) = -u(t), \quad t \geq 0
$$

$$
u^{(k)}(0) = u_k, \quad k \in \{0, \ldots, m-1\}
$$

$$
0 < \alpha \in \mathbb{R}, \quad m = \lceil \alpha \rceil
$$

SOLUTION BY LAPLACE TRANSFORM

Applying the operator J^α on both sides of the equation, we get

$$u(t) = \sum_{k=0}^{m-1} \frac{u^{(k)}(0)}{k!} t^k - J^\alpha u(t)$$

Applying the Laplace transform on both sides of the equation, we get

$$\tilde{u}(s) = \sum_{k=0}^{m-1} \frac{u^{(k)}(0)}{k!} \frac{1}{s^{k+1}} - \frac{1}{s^\alpha} \tilde{u}(s) \quad \rightarrow$$

$$\tilde{u}(s) = \sum_{k=0}^{m-1} \frac{u^{(k)}(0)}{k!} \frac{s^{\alpha-k-1}}{s^\alpha + 1}$$

Fractional-in-time ODE

LAPLACE TRANSFORM OF MITTAG-LEFFLER: $\mathcal{L}\left(J^k E_\alpha(-\lambda t^\alpha)\right)(s) = \dfrac{s^{\alpha-k-1}}{s^\alpha+\lambda}$

Hence

$$\begin{aligned}\tilde{u}(s) &= \sum_{k=0}^{m-1} \frac{u^{(k)}(0)}{k!} \mathcal{L}\left(J^k E_\alpha(-t^\alpha)\right)(s) \\ &= \sum_{k=0}^{m-1} \frac{u_k}{k!} \mathcal{L}\left(J^k E_\alpha(-t^\alpha)\right)(s) = \mathcal{L}\left(\sum_{k=0}^{m-1} \frac{u_k}{k!} J^k E_\alpha(-t^\alpha)\right)(s)\end{aligned}$$

Inverting the Laplace transform we finally obtain

Solution of the equation

$$u(t) = \sum_{k=0}^{m-1} \frac{u_k}{k!} J^k E_\alpha(-t^\alpha)$$

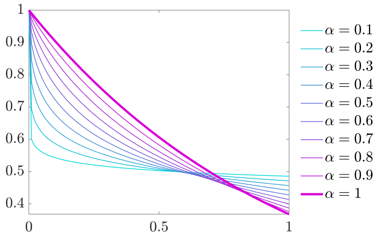
Solution of the equation

$$u(t) = \sum_{k=0}^{m-1} \frac{u_k}{k!} J^k E_\alpha(-t^\alpha)$$

$\alpha \in (0,1) \rightarrow m = 1$: we have $u(t) = u_0 J^0 E_\alpha(-t^\alpha) = u_0 E_\alpha(-t^\alpha)$

When $\alpha = 1$ $\left(\text{recall that } {}_C D^\alpha u(t) = {}_C D^1 u(t) = u'(t)\right)$:

$$u(t) = u_0 E_1(-t) = u_0 e^{-t}$$



Fractional-in-time ODE

Solution of the equation

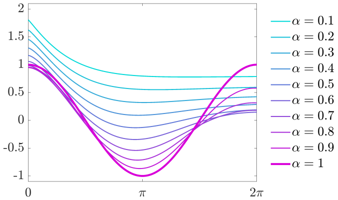
$$u(t) = \sum_{k=0}^{m-1} \frac{u_k}{k!} J^k E_\alpha(-t^\alpha)$$

$\alpha \in (1,2) \;\rightarrow\; m = 2$: we have $u(t) = u_0 E_\alpha(-t^\alpha) + u_1 J E_\alpha(-t^\alpha)$

When $\alpha = 2$ $\left(\text{recall that } {}_cD^\alpha u(t) = {}_cD^2 u(t) = u''(t)\right)$:

$$u(t) = u_0 E_2(-t^2) + u_1 J E_2(-t^2) = u_0 \cosh(it) + u_1 J \cosh(it)$$

$$= u_0 \cosh(it) + u_1 \int_0^t \cosh(i\tau)\, d\tau = u_0 \cos(t) + u_1 \int_0^t \cos(\tau)\, d\tau = u_0 \cos(t) + u_1 \sin(t)$$



Fractional-in-time PDE

Fractional diffusion equation

$$
\begin{aligned}
&{}_cD^{\alpha}u(x,t) = \Delta u(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \quad \alpha \in (0,1) \\
&u(x,0) = u_0(x)
\end{aligned}
$$

Obtained from the standard diffusion equation by replacing the first-order time derivative with a Caputo derivative of order $\alpha \in (0,1)$

Fractional-in-time PDE

SOLUTION BY FOURIER/LAPLACE TRANSFORM IN SPACE/TIME

1. Taking the Laplace transform in time, we obtain

$$s^{\alpha}\tilde{u}(x,s) - u_0(x)s^{\alpha-1} = \Delta\tilde{u}(x,s)$$

2. Taking the Fourier transform in space, we obtain

$$s^{\alpha}\hat{\tilde{u}}(\xi,s) - \hat{u}_0(\xi)s^{\alpha-1} = -|\xi|^2\hat{\tilde{u}}(\xi,s) \quad \to \quad \hat{\tilde{u}}(\xi,s) = \hat{u}_0(\xi)\frac{s^{\alpha-1}}{s^{\alpha}+|\xi|^2}$$

3. Inverting the Laplace transform, we obtain

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)E_{\alpha}(-|\xi|^2t^{\alpha})$$

4. Inverting the Fourier transform, we obtain

$$u(x,t) = \frac{1}{(2\pi)^{\frac{N}{2}}}\left[u_0 * \mathcal{F}^{-1}\Big(E_{\alpha}(-|\cdot|^2t^{\alpha})\Big)\right](x)$$

The Wright function

Wright function

$$M_\nu(z) = \sum_{k\geq 0} \frac{(-z)^k}{k!\Gamma(-\nu k + 1 - \nu)} = \frac{1}{\pi} \sum_{k\geq 0} \frac{(-z)^{k-1}}{(k-1)!} \Gamma(\nu k) \sin(\pi \nu k)$$

F. Mainardi and A. Consiglio, *The Wright functions of the second kind in Mathematical Physics*, Mathematics, 2020.

FOURIER TRANSFORM OF THE WRIGHT FUNCTION:

$$\mathcal{F}\Big(t^{-\nu} M_\nu(|x| t^{-\nu})\Big)(\xi) = 2E_{2\nu}(-|\xi|^2 t^\nu)$$

Solution of the PDE

We then have

$$E_\alpha(-|\xi|^2 t^\alpha) = E_\alpha\Big(-|\xi|^2 (t^2)^{\frac{\alpha}{2}}\Big) = \frac{1}{2} F\Big(t^{-\frac{\alpha}{2}} M_{\frac{\alpha}{2}}\left(|x| t^{-\frac{\alpha}{2}}\right)\Big)(\xi)$$

Therefore, the solution of the fractional diffusion equation is given by

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{\frac{N}{2}}}\Big[u_0 * \mathcal{F}^{-1}\Big(E_\alpha(-|\cdot|^2 t^\alpha)\Big)\Big](x) \\ &= \frac{1}{2}\frac{t^{-\frac{\alpha}{2}}}{(2\pi)^{\frac{N}{2}}}\Big[u_0 * M_{\frac{\alpha}{2}}\left(|\cdot| t^{-\frac{\alpha}{2}}\right)\Big](x) \end{aligned}$$

Fundamental solution

Fundamental solution of the fractional diffusion equation

As for the case of the standard heat equation, the fundamental solution is obtained when considering as initial datum the Dirac delta distribution:

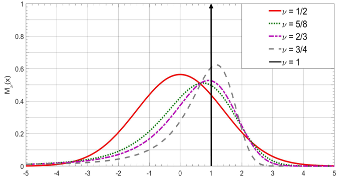
$$u_0(x) = \delta(x)$$

Since δ is the identity element of convolution, we obtain

$$u(x,t) = \frac{1}{(2\pi)^{\frac{N}{2}}} \mathcal{F}^{-1}\Big(E_\alpha(-|\xi|^2 t^\alpha)\Big)(x) = \frac{1}{2}\frac{t^{-\frac{\alpha}{2}}}{(2\pi)^{\frac{N}{2}}} M_{\frac{\alpha}{2}}\left(|x| t^{-\frac{\alpha}{2}}\right)$$

$\alpha = 1$: fundamental solution of the diffusion equation

$$u(x,t) = \frac{\mathcal{F}^{-1}\Big(e^{-|\xi|^2 t}\Big)(x)}{(2\pi)^{\frac{N}{2}}} = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{N}{2}}}$$



The Wright function for different values of ν.

Source: F. Mainardi and A. Consiglio, 2020.

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