CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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PART II: non-local in time models

LECTURE 5: control theory for fractional-in-time ODE and PDE

Controllability of fractional in time differential equations

Null controllability

Recall the notion of null controllability for classical abstract differential equations

$$\begin{cases} y_t - Ay = Bu & t \in (0,T) \\ y(0) = y_0 \in H \end{cases}$$

The system is null controllable at time T if for any $y_0 \in H$ there is a control $u \in L^2(0,T;U)$ such that the corresponding solution y satisfies $y(T) = 0$.

Fractional-in-time differential equation

Consider now a fractional-in-time differential equation

$$\begin{cases} {}_cD^{\alpha}y - Ay = Bu \quad t \in (0,T),\ \alpha \in (0,1) \\ y(0) = y_0 \in H \end{cases} \tag{1}$$

The previous standard definition of controllability, when translated as such for this kind of fractional systems, would lead to a notion of **partial but not of full null controllability**.

For models involving fractional-in-time derivatives, due to **memory effects induced by the integral term**, the fact that the solution y reaches the null value at time $t = T$ does not guarantee that the solution stays at rest for $t \geq T$ when the control action stops.

Fractional-in-time differential equation

M. Bettayeb and S. Djennoune, *New results on the controllability and observability of fractional dynamical systems*, J. Vib. Control., 2008.

K. Li, J. Peng and J. Gao, *Controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in Banach spaces*, Rep. Math. Phys., 2013.

D. Matignon and B. d'Andréa-Novel, *Some results on controllability and observability of finite-dimensional fractional differential systems*, Proc. IEEE Conference on systems, man and cybernetics, 1996.

Some references where the **partial null controllability problem** for fractional-in-time ODE was studied.

Similar to the classical controllability result for ODE, the authors show that **the Kalman rank condition of (A, B) is a sufficient and necessary condition for the partial null controllability**.

Fractional-in-time differential equation

Null controllability

System (1) is null controllable at time $T > 0$ if for any $y_0 \in H$, there is a control $u \in L^2(0, T; U)$ such that the corresponding solution y satisfies that $y(t) = 0$ for all $t \geq T$.

In this definition, we implicitly assume that the control u that has its support in $t \in [0, T]$ and vanishes afterwards, i.e. $u(t) = 0$ for all $t \geq T$.

Negative controllability result

Q. Lü and E. Zuazua, *On the lack of controllability of fractional in time ODE and PDE*, Math. Control Signals Syst, 2016.

Lack of controllability for (1)

The null controllability property in the strict sense of the previous definition fails systematically due to the memory effects induced by the integral entering in the fractional in time derivative.

Proof

Let us assume that the system (1) is null controllable for some $T > 0$ in the sense of the previous definition. Then, since $y(t) = 0$ for all $t \geq T$, we have that from the definition of Caputo derivative that

$$\int_0^t \frac{y'(\tau)}{(t-\tau)^\alpha}\, d\tau = \int_0^T \frac{y'(\tau)}{(t-\tau)^\alpha}\, d\tau = 0, \qquad \text{for all } t \geq T.$$

Thus, for any $\phi \in D(A^*)$, we have that

$$\int_0^T \frac{\langle y'(\tau), \phi\rangle_{D(A^*)', D(A^*)}}{(t-\tau)^\alpha}\, d\tau = 0, \qquad \text{for all } t \geq T.$$

By taking derivatives of the above equality with respect to t, we get that

$$\int_0^T \frac{\langle y'(\tau), \phi\rangle_{D(A^*)', D(A^*)}}{(t-\tau)^{\alpha+j}}\, d\tau = 0, \qquad \text{for all } t \geq T, j \in \mathbb{N}.$$

Proof

Let $\sigma := (t-\tau)^{-1}$. From the last identity, we have that

$$\int_{\frac{1}{t}}^{\frac{1}{t-T}} \left\langle y\left(t-\frac{1}{\sigma}\right), \phi\right\rangle_{D(A^*)',D(A^*)} \sigma^{\alpha+j-2}\, d\sigma = 0, \quad \text{for all } t \geq T, j \in \mathbb{N}.$$

Let

$$f(\sigma) := \left\langle y\left(t-\frac{1}{\sigma}\right), \phi\right\rangle_{D(A^*)',D(A^*)} \sigma^{\alpha-2}.$$

We then get that

$$\int_{\frac{1}{t}}^{\frac{1}{t-T}} f(\sigma)\sigma^{j}\, d\sigma = 0, \quad \text{for all } t \geq T, j \in \mathbb{N}.$$

Weierstrass approximation theorem

The polynomials are dense in the space of continuous functions.

Hence,

$$f = 0 \text{ in } \left[\frac{1}{t}, \frac{1}{t-T}\right) \quad \Rightarrow \quad y = 0 \text{ in } [0, T).$$

The dual observability problem

Controllability and observability properties are in duality. Thus, it is natural to analyze the signification of the negative result on null controllability in what concerns the dual observability property.

Adjoint problem

$$\begin{cases} {}_cD^{\alpha}_{T-}p - A^*p = \dfrac{\alpha p_0}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{d\tau}{(t-\tau)^{1+\alpha}}, & \text{in } (0,T),\ p_0 \in D(A^*) \\ p(T) = p_T \in D(A^*) \end{cases} \tag{2}$$

The observability problem

Proposition

System (1) is null controllable in time T if and only if system (2) is observable in the sense that there is a constant $\mathcal{C} > 0$ such that for any $\tau \in (T, +\infty)$ and $p_0 \in D(A^*)$, it holds

$$\frac{1}{\Gamma(1-\alpha)} \left\| \int_0^T \left(\frac{p(t)}{t^\alpha} + \frac{p_0}{\tau^\alpha} \right) dt \right\|_H \leq \mathcal{C} \left\| B^* p \right\|_{L^2(0,T;U)} .$$

Theorem

System (2) is not observable.

Stabilization

Stabilization

The stabilization problem for the system (1) consists in finding a **feedback control** $u = F(y)$, with a suitable linear map F, to accelerate the speed of the decay of solutions of the free system as $t \to +\infty$.

One typically seeks for **exponential decay** properties although, in some cases, the decay achieved can be slow, either **polynomial** or **logarithmic**.

N. Burq and M. Hitrik, *Energy decay for damped wave equations on partially rectangular domains*, Math. Res. Lett., 2007.

G. Lebeau and L. Robbiano, *Stabilisation de l'équation des ondes par le bord*, Duke Math. J., 2007.

K.-D. Phung, *Polynomial decay rate for the dissipative wave equation*, JDE, 2007.

Stabilization

In the context of fractional in time models, feedback operators $F \in \mathcal{L}(H, U)$, may not suffice to achieve the exponential decay of the system (1).

$$\begin{cases} {}_cD^\alpha y = cy, & \text{in } (0,T), c \in \mathbb{R},\ c < 0 \\ y(0) = y_0 \end{cases} \quad \Rightarrow \quad y(t) = y_0 E_\alpha(ct^\alpha) = \mathcal{O}\left(\frac{1}{ct^\alpha}\right).$$

No matter what the value of c, the solution decays **polynomially**. This example also shows that if $F \in \mathcal{L}(H, U)$, the rate of the polynomial decay depends only on α, which cannot be improved by the choice of F.

Approximate controllability

Laplace tranfsorm

What about approximate controllability?

We have seen that system (1) is not null controllable, due to the memory effects introduced by the Caputo derivative. But is it approximately controllable?

Recall that approximate controllability is a weaker property than null/exact controllability. Can we expect some approximate controllability result to hold?

Approximate controllability

$$\begin{cases} {}_cD^\alpha y + Ay = f\chi_\omega & \text{in } \Omega \times (0,T),\ \alpha \in (1,2) \\ y = 0 & \text{in } \partial\Omega \times (0,T) \\ y(0) = y_0,\ y_t(0) = y_1 & \text{in } \Omega \end{cases} \tag{3}$$

V. Keyantuo and M. Warma, *On the interior approximate controllability for fractional wave equations*, Discr. Cont. Dyn. Syst., 2016.

Theorem

The system (3) is approximately controllable at time $T > 0$ with control function $f \in C_0^\infty(\omega \times (0,T))$.

Proof

Four main ingredients are required.

The adjoint system:

Adjoint system

$$\begin{cases} {}_cD^{\alpha}_{T-}p + Ap = 0 & \text{in } \Omega \times (0,T),\ \alpha \in (1,2) \\ p = 0 & \text{in } \partial\Omega \times (0,T) \\ J^{2-\alpha}_{T-}p(T) = p_0,\ {}_cD^{\alpha-1}_{T-}p(T) = p_1 & \text{in } \Omega \end{cases} \tag{4}$$

Proof

Four main ingredients are required.

Spectral decomposition of the solution:

$V_\gamma := D(A^\gamma)$ equipped with the norm $\|y\|_{V_\gamma} := \|A^\gamma y\|_{L^2(\Omega)}$.

Theorem

Let $1 < \alpha < 2$, $T > 0$ and $\gamma = 1/\alpha$. Then, for every $p_0 \in V_\gamma$ and $p_1 \in L^2(\Omega)$, (4) has a unique solution p given by

$$p(\cdot, t) = \sum_{k\geq 1} p_{0,k}(T-t)^{\alpha-2} E_{\alpha,\alpha-1}\Big(-\lambda_k (T-t)^\alpha\Big)\phi_k + \sum_{k\geq 1} p_{1,k}(T-t)^{\alpha-1} E_{\alpha,\alpha}\Big(-\lambda_k (T-t)^\alpha\Big)\phi_k$$

with $p_{0,k} := (p_0, \phi_k)$ and $p_{1,k} := (p_1, \phi_k)$

Proof

Four main ingredients are required.

Unique continuation property for the operator A:

Unique continuation for A

If λ_k is an eigenvalue of A, $(A-\lambda)u = 0$ in Ω and $u = 0$ in ω, then $u = 0$ in Ω.

Examples of operators fulfilling this unique continuation property

- Variable coefficients Laplacian in $\mathbb{R}^N$

$$Ay(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{i,j}(x)\frac{\partial y}{\partial x_i}(x)\right) + b(x)u(x)$$

  with **Dirichlet** or **Robin** boundary conditions.

  D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, 2001.

- The fractional Laplacian $Ay(x) = (-\Delta)^s y(x)$.

  M. M. Fall and V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Commun. PDE, 2014

Proof

Four main ingredients are required.

Unique continuation property for the solution of (4):

Unique continuation for (4)

Let $1 < \alpha < 2$, $\gamma = 1/\alpha$, $p_0 \in V_\gamma$ and $p_1 \in L^2(\Omega)$. Assume that A has the unique continuation property in the sense of the previous definition. Let p be the unique solution to the system (4). If $p = 0$ on $\omega \times (0, T)$, then $p = 0$ on $\Omega \times (0, T)$.

Proof of the unique continuation for (4)

Assume that $p = 0$ in $\omega \times (0, T)$. Since $p : [0, T) \to L^2(\Omega)$ can be analytically extended to the half plane $\boldsymbol{\Sigma}_T := \{z \in \mathbb{C} \,:\, \Re(z) < T\}$, it follows that

$$p(x,t) = \sum_{k\geq 1} p_{0,k}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}\Big(-\lambda_k(T-t)^{\alpha}\Big)\phi_k(x) \qquad (5)$$
$$+\sum_{k\geq 1} p_{1,k}(T-t)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_k(T-t)^{\alpha}\Big)\phi_k(x), \quad (x,t) \in \omega \times (-\infty, T).$$

Proof of the unique continuation for (4)

Let $\{\lambda_k\}_{k\geq 1}$ be the set of eigenvalues of the operator A and let $\{\psi_{k_j}\}_{1\leq k\leq m_k}$ be an orthonormal basis of $\ker\,(\lambda_k - A)$. Then, we have

$$\phi_k(x) = \sum_{j=1}^{m_k} \phi_{k_j}\psi_{k_j}, \quad (\phi_{k_j})_{j=1}^{m_k} \in \ell^2$$

and (5) can be rewritten as

$$p(x,t) = \sum_{k\geq 1}\left(\sum_{j=1}^{m_k} p_{0,k_j}\psi_{k_j}(x)\right)(T-t)^{\alpha-2}E_{\alpha,\alpha-1}\Big(-\lambda_k(T-t)^{\alpha}\Big) \qquad (6)$$
$$+\sum_{k\geq 1}\left(\sum_{j=1}^{m_k} p_{1,k_j}\psi_{k_j}(x)\right)(T-t)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_k(T-t)^{\alpha}\Big),\ (x,t)\in\omega\times(-\infty,T),$$

with $p_{0,k_j} := p_{0,k}\phi_{k_j}$ and $p_{1,k_j} := p_{1,k}\phi_{k_j}$.

Proof of the unique continuation for (4)

Let $\sigma \in \mathbb{C}$ with $\eta := \Re(\sigma) > 0$ and let $N \in \mathbb{N}$. Since the functions ψ_{k_j}, $1 \leq j \leq m_k$, $1 \leq k \leq N$ are orthonormal, we have

$$\left\| \sum_{k=1}^{N} \left(\sum_{j=1}^{m_k} p_{0,k_j} \psi_{k_j}(x) \right) e^{\sigma(t-T)} (T-t)^{\alpha-2} E_{\alpha,\alpha-1}\Big(-\lambda_k (T-t)^\alpha \Big) \right\|_{L^2(\Omega)}^2$$

$$\leq \sum_{k\geq 1} \left(\sum_{j=1}^{m_k} |p_{0,k_j}|^2 \right) e^{2\eta(t-T)} \left| (T-t)^{\alpha-2} E_{\alpha,\alpha-1}\Big(-\lambda_k (T-t)^\alpha \Big) \right|^2$$

$$\leq \mathcal{C} e^{2\eta(t-T)} (T-t)^{2(\alpha-2)} \|p_0\|_{V_\gamma}^2$$

and

$$\left\| \sum_{k=1}^{N} \left(\sum_{j=1}^{m_k} p_{1,k_j} \psi_{k_j}(x) \right) e^{\sigma(t-T)} (T-t)^{\alpha-2} E_{\alpha,\alpha-1}\Big(-\lambda_k (T-t)^\alpha \Big) \right\|_{L^2(\Omega)}^2$$

$$\leq \mathcal{C} e^{2\eta(t-T)} (T-t)^{2(\alpha-1)} \|p_1\|_{L^2(\Omega)}^2$$

Proof of the unique continuation for (4)

Hence, if we define

$$p_N(x,t) := \sum_{k=1}^{N}\left(\sum_{j=1}^{m_k} p_{0,k_j}\psi_{k_j}(x)\right) e^{\sigma(t-T)}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}\Big(-\lambda_k(T-t)^{\alpha}\Big)$$
$$+\sum_{k=1}^{N}\left(\sum_{j=1}^{m_k} p_{1,k_j}\psi_{k_j}(x)\right) e^{\sigma(t-T)}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}\Big(-\lambda_k(T-t)^{\alpha}\Big)$$

we have that

$$\|p_N(x,t)\|_{L^2(\Omega)} \leq \mathcal{C}e^{2\eta(t-T)}\Big[(T-t)^{\alpha-2}\|p_0\|_{V_\gamma} + (T-t)^{\alpha-1}\|p_1\|_{L^2(\Omega)}\Big].$$

Proof of the unique continuation for (4)

Moreover, since $1 < \alpha < 2$, we have that

$$\begin{aligned}
&\int_{-\infty}^{T} e^{2\eta(t-T)} \left[(T-t)^{\alpha-2} \|p_0\|_{V_\gamma} + (T-t)^{\alpha-1} \|p_1\|_{L^2(\Omega)} \right] dt \\
&\qquad = \|p_0\|_{V_\gamma} \int_0^{+\infty} e^{-\tau} \frac{\tau^{\alpha-2}}{\eta^{\alpha-1}} \, d\tau + \|p_1\|_{L^2(\Omega)} \int_0^{+\infty} e^{-\tau} \frac{\tau^{\alpha-1}}{\eta^{\alpha}} \, d\tau \\
&\qquad = \frac{\Gamma(\alpha-1)}{\eta^{\alpha-1}} \|p_0\|_{V_\gamma} + \frac{\Gamma(\alpha)}{\eta^{\alpha}} \|p_1\|_{L^2(\Omega)} .
\end{aligned}$$

Therefore, we can apply the Dominated Convergence Theorem and the fact that

$$\int_{-\infty}^{T} e^{\sigma(t-T)}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}\Big(-\lambda_k(T-t)^{\alpha}\Big)\,dt = \frac{\sigma}{\sigma^{\alpha}+\lambda_k}$$

$$\int_{-\infty}^{T} e^{\sigma(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_k(T-t)^{\alpha}\Big)\,dt = \frac{1}{\sigma^{\alpha}+\lambda_k}$$

to conclude that for all $x \in (-1,1)$ and $\eta = \Re(\sigma) > 0$

$$\lim_{N\to+\infty}\int_{-\infty}^{T} p_N(x,t)\,dt = \int_{-\infty}^{T}\lim_{N\to+\infty} p_N(x,t)\,dt = \sum_{k\geq 1}\sum_{j=1}^{m_k}\left(\frac{\sigma p_{0,k_j}+p_{1,k_j}}{\sigma^{\alpha}+\lambda_k}\right)\psi_{k_j}(x) \tag{7}$$

Proof of the unique continuation for (4)

It follows from (6) and (7) that

$$\sum_{k\geq 1}\sum_{j=1}^{m_k}\left(\frac{\sigma p_{0,k_j}+p_{1,k_j}}{\sigma^\alpha+\lambda_k}\right)\psi_{k_j}(x)=0, \quad \text{for all } x\in\omega \text{ and } \Re(\sigma)>0.$$

Moreover, letting $\zeta=\sigma^\alpha$, we have

$$\sum_{k\geq 1}\sum_{j=1}^{m_k}\left(\frac{\zeta^{\frac{1}{\alpha}}p_{0,k_j}+p_{1,k_j}}{\zeta+\lambda_k}\right)\psi_{k_j}(x)=0, \quad \text{for all } x\in\omega \text{ and } \Re(\zeta)>0.$$

This holds for every $\zeta\in\mathbb{C}\setminus\{-\lambda_k\}_{k\in\mathbb{N}}$, using the analytic continuation in ζ. Hence, taking a suitable small circle around $-\lambda_\ell$ not including $\{-\lambda_k\}_{k\neq\ell}$ and integrating on that circle we get that

$$p_\ell:=\sum_{j=1}^{m_\ell}\left[(-\lambda_\ell)^{\frac{1}{\alpha}}p_{0,\ell_j}+p_{1,\ell_j}\right]\psi_{\ell_j}(x)=0, \quad \text{for all } x\in\omega,$$

where

$$(-\lambda_\ell)^{\frac{1}{\alpha}}=e^{\frac{1}{\alpha}\ln(-\lambda_\ell)}=e^{\frac{1}{\alpha}(\ln(\lambda_\ell)+i\pi)}=\lambda_\ell^{\frac{1}{\alpha}}\left[\cos\left(\frac{\pi}{\alpha}\right)+i\sin\left(\frac{\pi}{\alpha}\right)\right].$$

Proof of the unique continuation for (4)

Then, from the unique continuation property for A, we can conclude that $p_\ell = 0$ in Ω for every ℓ.

Since $\{\psi_{\ell_j}\}_{1\leq j\leq m_\ell}$ are linearly independent in $L^2(\Omega)$, we get that

$$\left((-\lambda_\ell)^{\frac{1}{\alpha}} p_0 + p_1, \varphi_{\ell_j}\right) = 0, \quad \text{for all } 1 \leq j \leq m_k,\ \ell \in \mathbb{N}.$$

This implies that

$$0 = (-\lambda_\ell)^{\frac{1}{\alpha}} p_0 + p_1 = \lambda_\ell^{\frac{1}{\alpha}} \left[\cos\left(\frac{\pi}{\alpha}\right) + i \sin\left(\frac{\pi}{\alpha}\right)\right] p_0 + p_1.$$

It follows that $p_0 = 0 = p_1$ and hence, $p = 0$ in $\Omega \times (0, T)$.

Proof of the approximate controllability

First of all, notice that the approximate controllability for (3) is equivalent to the fact that the set

$$\mathbb{U} := \Big\{ (y(\cdot,T), y_t(\cdot,T)) \,:\, f \in C_0^\infty(\omega \times (0,T)) \Big\}$$

is dense in $V_\gamma \times L^2(\Omega)$, that is,

$$\overline{\mathbb{U}}^{V_\gamma \times L^2(\Omega)} = V_\gamma \times L^2(\Omega). \tag{8}$$

By Hahn-Banach Theorem, (8) is equivalent to show that, if $(p_0, p_1) \in V_\gamma \times L^2(\Omega)$ are such that

$$\int_\Omega \Big[y_t(x,T)p_0(x) + y(x,T)p_1(x) \Big] \, dx = 0, \tag{9}$$

for any $f \in C_0^\infty(\omega \times (0,T))$, then $p_0 = 0 = p_1$.

Proof of the approximate controllability

Now, multiplying (3) by (4) and integrating over $\Omega \times (0, T)$, we can show that

$$\int_{\Omega} \Big[y_t(x,T) p_0(x) + y(x,T) p_1(x) \Big] \, dx = \int_0^T \int_{\omega} fp \, dxdt.$$

Hence, if (9) holds, we have that

$$\int_0^T \int_{\omega} fp \, dxdt = 0, \quad \text{for any } f \in C_0^{\infty}(\omega \times (0,T)) \quad \Rightarrow \quad p = 0 \text{ in } \omega \times (0,T).$$

It then follows from the unique continuation of (4) that $p = 0$ in $\Omega \times (0, T)$. Since the solution of (4) is unique, we can then conclude that $p_0 = 0 = p_1$.

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