

CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

Umberto Biccari and Enrique Zuazua

Chair of Computational Mathematics, Bilbao, Basque Country, Spain

Chair for Dynamics, Control and Numerics, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany.

Universidad Autónoma de Madrid, Spain.

umberto.biccari@deusto.es
cmc.deusto.es

enrique.zuazua@fau.de
dcn.nat.fau.eu

PART III: non-local in space models

LECTURE 6: The fractional Laplacian



European Research Council
Established by the European Commission



FRACTIONAL SOBOLEV SPACES

Fractional Sobolev spaces

Fractional Sobolev spaces

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy < \infty \right\}$$
$$p \in [1, \infty), \quad s \in (0, 1)$$

Intermediate **Banach** space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Fractional Sobolev norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_{\Omega} |u|^p dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}},$$

$p = 2$: $W^{s,2}(\Omega) = H^s(\Omega)$ is a Hilbert space with scalar product

$$\langle u, v \rangle_{W^{s,2}(\Omega)} := \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy.$$

E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sc. Math., 2012

Fractional Sobolev embeddings

Proposition

Let $p \in [1, +\infty)$ and $0 < s \leq s_1 \leq 1$. Let Ω be an open set in \mathbb{R}^N and $u : \Omega \rightarrow \mathbb{R}$ be a measurable function. Then

$$\|u\|_{W^{s,p}(\Omega)} \leq C(N, s, p, \Omega) \|u\|_{W^{s_1,p}(\Omega)} \Rightarrow W^{s_1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

Theorem

Let $s \in (0, 1)$ and $p \in [1, +\infty)$. Let $\Omega \subset \mathbb{R}^N$ be bounded open and Lipschitz.

$sp < N$: $\|u\|_{L^q(\Omega)} \leq C(N, s, p, \Omega) \|u\|_{W^{s,p}(\Omega)} \Rightarrow W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in \left[1, \frac{Np}{N-sp}\right]$.

$sp = N$: $\|u\|_{L^q(\Omega)} \leq C(N, s, p, \Omega) \|u\|_{W^{s,p}(\Omega)} \Rightarrow W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$.

$sp > N$: $\|u\|_{C^{0,s-\frac{N}{p}}(\Omega)} \leq C(N, s, p, \Omega) \|u\|_{W^{s,p}(\Omega)} \Rightarrow W^{s,p}(\Omega) \hookrightarrow C^{0,s-\frac{N}{p}}(\Omega)$.

High-order fractional Sobolev spaces

High-order fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha \text{ with } |\alpha| = m \right\}$$
$$s = m + \sigma, \quad m \in \mathbb{N}, \quad \sigma \in (0,1).$$

$W^{s,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{W^{m,p}(\Omega)}^p + \sum_{|\alpha|=m} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary. Then $W^{r,p}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ for all $0 < s \leq r$ and $1 < p \leq q < \infty$ such that $r - \frac{N}{p} \geq s - \frac{N}{q}$.

The spaces $W_0^{s,p}(\Omega)$ and $W^{-s,p}(\Omega)$

$W_0^{s,p}(\Omega)$: closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$.

Theorem

For any $s > 0$, the space $C_0^\infty(\mathbb{R}^N)$ of smooth functions with compact support is dense in $W^{s,p}(\Omega)$ if and only if $0 < s \leq 1/p$. As a consequence:

$$0 < s < \frac{1}{p} \Rightarrow W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$$

$$\frac{1}{p} < s < 1 \Rightarrow W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ on } \Omega^c := \mathbb{R}^N \setminus \Omega \right\}$$

Dual space

$W^{-s,p'}(\Omega) := (W_0^{s,p}(\Omega))^*$, $p' := \frac{p}{p-1}$: dual space of $W_0^{s,p}(\Omega)$.

The Poincaré inequality in H^s

Proposition

Let Ω be a star-shaped domain with respect to a ball B . For any $u \in H^s(\Omega)$ with $0 < s < 1$ we call $\bar{u} := \frac{1}{|\Omega|} \int_{\Omega} u \, dx$. Then we have

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C d(\Omega)^s [u]_{H^s(\Omega)}.$$

PROOF:

$$\begin{aligned} \int_{\Omega} (u(x) - \bar{u}(x))^2 \, dx &= \frac{1}{|\Omega|^2} \int_{\Omega} \left(u(x) \int_{\Omega} 1 \, dy - \int_{\Omega} u(y) \, dy \right)^2 \, dx \\ &= \frac{1}{|\Omega|^2} \int_{\Omega} \left(\int_{\Omega} (u(x) - u(y)) \, dy \right)^2 \, dx \\ &\leq \frac{1}{|\Omega|} \int_{\Omega} \int_{\Omega} (u(x) - u(y))^2 \, dy \, dx \\ &\leq \frac{d(\Omega)^{N+2s}}{|\Omega|} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx \end{aligned}$$

- $d(B)^N \sim |B| < |\Omega| \Rightarrow \|u - \bar{u}\|_{L^2(\Omega)}^2 \leq C d(\Omega)^s [u]_{H^s(\Omega)}$, $C \sim [d(\Omega)/d(B)]^{\frac{N}{2}}$.

The Poincaré inequality in H^s

Proposition

Let Ω be a bounded and regular domain and denote

$$\mathbb{V} := \left\{ u \in \Omega : u = 0 \text{ in } \Omega^c \right\}.$$

Then, for all $u \in \mathbb{V}$, there exists a constant $C = C(\Omega, N, s) > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C[u]_{H^s(\mathbb{R}^N)}.$$

PROOF: E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Lemma 6.1

There exists some constant $c(N, s) > 0$ such that for all $x \in \Omega$,

$$c(N, s)|\Omega|^{-\frac{2s}{N}} \leq \int_{\Omega^c} \frac{dy}{|x - y|^{N+2s}}.$$

On the other hand, since $u = 0$ in Ω^c , we know that

$$u(x)^2 = (u(x) - u(y))^2 \quad \text{for all } x \in \Omega, y \in \Omega^c.$$

So, we obtain

$$\begin{aligned} c(N, s)|\Omega|^{-\frac{2s}{N}} \int_{\Omega} |u(x)|^2 dx &\leq \int_{\Omega} \int_{\Omega^c} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy = [u]_{H^s(\mathbb{R}^N)}. \end{aligned}$$

THE FRACTIONAL LAPLACIAN

The fractional Laplacian

\mathcal{S} : Schwartz space.

For $u \in \mathcal{S}(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

Fractional Laplacian

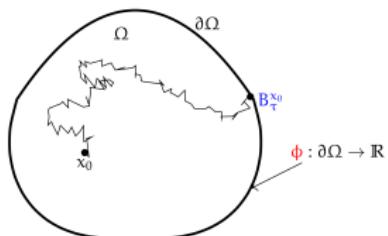
$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N.$$

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta \right)^{-1} = \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

Motivation: elliptic PDE via stochastic processes

Brownian motion \rightarrow 2nd order PDE

Expected payoff at $\partial\Omega$.



$$u(x) = \mathbb{E}(\phi(B_\tau^{x_0})) \text{ solves}$$

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

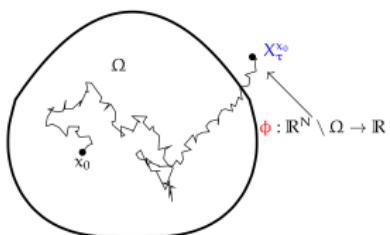
$-\Delta$ is a **local operator**, that is:
 $\text{supp}(-\Delta u) \subset \text{supp}(u)$.

- $B_\tau^{x_0}$: Brownian motion in \mathbb{R}^N starting at x_0 .
- τ : stopping time - first time at which $B_\tau^{x_0}$ hits $\partial\Omega$.

Motivation: elliptic PDE via stochastic processes

Lévy process → integro-differential equations

Expected payoff at Ω^c .



$u(x) = \mathbb{E}(\phi(X_\tau^{x_0}))$ solves

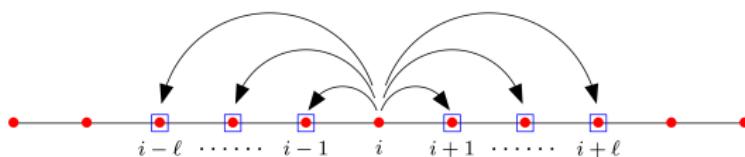
$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega \\ u = \phi & \text{in } \Omega^c \end{cases}$$

$(-\Delta)^s$ is a **non-local operator**,
that is: $\text{supp}((-\Delta)^s) \not\subset \text{supp}(u)$.

- $X_\tau^{x_0}$: Lévy process with discontinuous sample paths.
- τ : stopping time - first time at which $X_\tau^{x_0}$ is in Ω^c .

Long-jump random walks

One can derive the fractional Laplacian through **Long-jump random walks**



$$u(x, t + \tau) = \sum_{i \in \mathbb{Z}^N} \mu(i) u(x + ik, t)$$

- By setting $\mu(\cdot) = |\cdot|^{-(N+2s)}$ and $s \in (0, 1)$, and taking $\tau = h^{2s}$, in the limit $h \rightarrow 0^+$ we get

$$u_t(x, t) + (-\Delta)^s u(x, t) = 0.$$

The fractional Laplacian

\mathcal{S} : Schwartz space.

For $u \in \mathcal{S}(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

Fractional Laplacian

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N.$$

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta \right)^{-1} = \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

The fractional Laplacian

\mathcal{S} : Schwartz space.

For $u \in \mathcal{S}(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

Fractional Laplacian

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N.$$

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta \right)^{-1} = \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}.$$

Some properties

- For any $u \in \mathcal{S}(\mathbb{R}^N)$ we have $(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s} \hat{u}(\xi)), \xi \in \mathbb{R}^N$.

$$\triangleright [u]_{H^s(\mathbb{R}^N)}^2 = 2C_{N,s}^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi.$$

$$\triangleright [u]_{H^s(\mathbb{R}^N)}^2 = 2C_{N,s}^{-1} \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^N)}^2.$$

- For all $u, v \in H^s(\mathbb{R}^N)$ such that $u \equiv v \equiv 0$ in Ω^c

$$\int_{\Omega} u(-\Delta)^s v dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx = \int_{\Omega} v(-\Delta)^s u dx$$

$$\int_{\Omega} u(-\Delta)^s v dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy$$

- $\lim_{s \rightarrow 0^+} (-\Delta)^s u = u$ for all $u \in C_0^\infty(\mathbb{R}^N)$.

- $\lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$ for all $u \in C_0^\infty(\mathbb{R}^N)$.

The spectral fractional Laplacian

There exists an alternative notion of fractional Laplacian, which is based on spectral theory, the so-called **spectral** fractional Laplacian.

$\{\psi_k\}_{k \in \mathbb{N}} \subset H_0^1(\Omega)$: normalized eigenfunctions of the Dirichlet Laplacian on Ω . $\{\lambda_k\}_{k \in \mathbb{N}}$: corresponding eigenvalues.

Spectral fractional Laplacian

$$(-\Delta)_S^s = \sum_{k \geq 1} \langle u, \psi_k \rangle \lambda_k^s \psi_k(x).$$

ATTENTION!

The spectral fractional Laplacian and the fractional Laplacian defined in integral form **are two different operators**.

R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators*, Proc. Roy Soc. Edinburgh Sec. A, 2014.

Heat semi-group representation

Heat semi-group representation of $(-\Delta)^s$

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}}, \quad s \in (0, 1)$$

$v(x, t) = e^{t\Delta} u(x)$: solution of the heat equation on \mathbb{R}^N with initial datum u

$$\begin{cases} v_t - \Delta v = 0, & (x, t) \in \mathbb{R}^N \times (0, +\infty) \\ v(x, 0) = u(x), & x \in \mathbb{R}^N \end{cases}$$

The definition is inspired by the identity

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}.$$

P. R. Stinga and J. L. Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Comm. PDE, 2010.

Heat semi-group representation

Proposition

For $u \in \mathcal{S}(\mathbb{R}^N)$ and $s \in (0, 1)$ we have

$$\frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}} = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.$$

PROOF (sketch): by Fubini's Theorem and the change of variables $\tau \mapsto t|\xi|^2$ we have

$$\begin{aligned} & \frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^{+\infty} \left(\int_{\mathbb{R}^N} (e^{-t|\xi|^2} - 1) \hat{u}(\xi) e^{i\xi \cdot x} d\xi \right) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^N} \left(\int_0^{+\infty} \frac{e^{-t|\xi|^2} - 1}{t^{1+s}} dt \right) \hat{u}(\xi) e^{i\xi \cdot x} d\xi \\ &= \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^N} \left(\int_0^{+\infty} \frac{e^{-\tau} - 1}{\tau^{1+s}} d\tau \right) |\xi|^{2s} \hat{u}(\xi) e^{i\xi \cdot x} d\xi \\ &= \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi) e^{i\xi \cdot x} d\xi = (-\Delta)^s u(x). \end{aligned}$$

The Caffarelli-Silvestre extension

Given $x \in \mathbb{R}^N$ and a function $f : \mathbb{R}^N \rightarrow \mathbb{R}$, we consider $u : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$ that satisfies the equation

$$\begin{cases} \operatorname{div}(y^\alpha \nabla u(x, y)) = 0, & (x, y) \in \mathbb{R}^{N+1} \\ u(x, 0) = f(x), & x \in \mathbb{R}^N. \end{cases}$$

Then we have

$$d_s(-\Delta)^s f(x) = - \lim_{y \rightarrow 0^+} y^\alpha \partial_y u.$$

Paying the price of increasing by one the dimension of the problem analyzed, this extension procedure has the advantage of allowing to work in a local framework. Since its first introduction, it has been employed for several different applications, including unique continuation properties or for the built of algorithms for the finite element discretization of PDEs the fractional Laplacian.

L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. PDE, 2008

Sketch of the proof

The extension problem can be rewritten as

$$\begin{array}{l} \Delta_x u + \frac{\alpha}{y} u_y + u_{yy} = 0 \\ u(x, 0) = f(x) \end{array} \xrightarrow{z = [(1-\alpha)^{-1}y]^{1-\alpha}} \begin{array}{l} \Delta_x u + z^\beta u_{zz} = 0 \\ \beta := -\frac{2\alpha}{1-\alpha} \\ y^\alpha u_y = u_z \end{array}$$

Poisson kernel

$$P(x, z) = \frac{1}{4} \pi^{N+\alpha} \Gamma\left(\frac{N-2+\alpha}{2}\right) z \left(|x|^2 + (1-\alpha)^2 |z|^{\frac{2}{1-\alpha}}\right)^{\frac{1-N-\alpha}{2}}$$

$$\begin{aligned} u_z(x, 0) &= \lim_{z \rightarrow 0^+} \frac{u(x, z) - u(x, 0)}{z} = \lim_{z \rightarrow 0^+} \frac{1}{z} \int_{\mathbb{R}^N} P(x - \xi, z) (f(\xi) - f(x)) d\xi \\ &= \lim_{z \rightarrow 0^+} \int_{\mathbb{R}^N} \frac{\frac{1}{4} \pi^{N+\alpha} \Gamma\left(\frac{N-2+\alpha}{2}\right) (f(\xi) - f(x))}{\left(|x - \xi|^2 + (1-\alpha)^2 |z|^{\frac{2}{1-\alpha}}\right)^{\frac{N-1+\alpha}{2}}} d\xi \\ &= \frac{1}{4} \pi^{N+\alpha} \Gamma\left(\frac{N-2+\alpha}{2}\right) P.V. \int_{\mathbb{R}^N} \frac{f(\xi) - f(x)}{|x - \xi|^{N-1+\alpha}} d\xi = C(-\Delta)^{\frac{1-\alpha}{2}} f(x). \end{aligned}$$

Pohozaev identity

Proposition

Let Ω be a bounded $C^{1,1}$ domain of \mathbb{R}^N , $s \in (0, 1)$ and $\delta(x) = \text{dist}(x, \partial\Omega)$, with $x \in \Omega$, be the distance of a point x from $\partial\Omega$. Let $u \in H_0^s(\Omega)$ satisfy the following:

- $u \in C^s(\mathbb{R}^N)$ and, for every $\beta \in [s, 1 + 2s]$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C\rho^{s-\beta}, \quad \text{for all } \rho \in (0, 1).$$

- $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\gamma \in (0, 1)$ such that $u/\delta^s \in C^\gamma(\overline{\Omega})$. In addition, for all $\beta \in [\gamma, s + \gamma]$ it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x \in \Omega | \delta(x) \geq \rho\})} \leq C\rho^{\gamma-\beta}, \quad \text{for all } \rho \in (0, 1).$$

- $(-\Delta)^s u$ is point-wise bounded in Ω .

Then, the following identity holds

$$\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{2s - N}{2} \int_{\Omega} u (-\Delta)^s u \, dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (x \cdot \nu) \, d\sigma,$$

where ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

Sketch of the proof (star-shaped domains)

If Ω is star-shaped with respect to the origin, we can rewrite

$$\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{d}{d\lambda} \Big|_{\lambda=1^+} \int_{\Omega} u_{\lambda} (-\Delta)^s u \, dx,$$

where, for $\lambda > 1$, $u_{\lambda} = u(\lambda x)$. Notice that, since Ω is star-shaped, $u_{\lambda} \equiv 0$ in Ω^c . We have

$$\int_{\Omega} u_{\lambda} (-\Delta)^s u \, dx = \int_{\Omega} (-\Delta)^{\frac{s}{2}} u_{\lambda} (-\Delta)^{\frac{s}{2}} u \, dx = \lambda^{\frac{2s-N}{2}} \int_{\Omega} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy,$$

where $w(x) = (-\Delta)^{\frac{s}{2}} u(x)$ and we employed the change of variables $y = \sqrt{\lambda}x$. Thus,

$$\begin{aligned} \int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u \, dx &= \frac{d}{d\lambda} \Big|_{\lambda=1^+} \left(\lambda^{\frac{2s-N}{2}} \int_{\Omega} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy \right) \\ &= \frac{2s-N}{2} \int_{\mathbb{R}^N} w^2 \, dx + \frac{d}{d\lambda} \Big|_{\lambda=1^+} \int_{\Omega} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy \\ &= \frac{2s-N}{2} \int_{\mathbb{R}^N} u(-\Delta)^s u \, dx + \frac{1}{2} \frac{d}{d\lambda} \Big|_{\lambda=1^+} \int_{\Omega} w_{\lambda} w_{1/\lambda} \, dy. \end{aligned}$$

For obtaining the final result it is enough to show that

$$\frac{1}{2} \frac{d}{d\lambda} \Big|_{\lambda=1^+} \int_{\Omega} w_{\lambda} w_{1/\lambda} \, dy = -\frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (x \cdot \nu) \, d\sigma.$$

THANK YOU FOR YOUR ATTENTION!

Funding

- European Research Council (ERC): grant agreements NO: 694126-DyCon and No.765579-ConFlex.
- MINECO (Spain): Grant PID2020-112617GB-C22 KILEARN
- Alexander von Humboldt-Professorship program
- DFG (Germany): Transregio 154 Project “Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks”
- COST Action grant CA18232, “Mathematical models for interacting dynamics on networks”.



European Research Council
Established by the European Commission

