CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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PART III: non-local in space models

LECTURE 7: The fractional Laplacian in non-local PDE

Fractional elliptic PDE

Fractional Poisson equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded and regular domain. We consider the following elliptic problem involving the fractional Laplacian

Fractional Poisson equation

$$\begin{cases} (-\Delta)^s u = f, & x \in \Omega \\ u \equiv 0, & x \in \Omega^c. \end{cases} \tag{$\mathcal{P}$}$$

On boundary conditions

In PDE involving the fractional Laplacian on a bounded domain, the boundary condition is actually an **exterior condition** posed on the entire Ω^c.
Indeed, because of the non-locality of the operator, fractional models with standard boundary conditions (given only on $\partial\Omega$) are ill-posed.

M. Warma, *The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets*, Potential Anal., 2015.

Weak solutions

Finite energy solutions

Let $f \in H^{-s}(\Omega)$. A function $u \in H_0^s(\Omega)$ is said to be a **finite energy solution** of $(\mathcal{P})$ if, for every $v \in H_0^s(\Omega)$, it holds the identity

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy = \langle f, v \rangle_{H^{-s}(\Omega), H_0^s(\Omega)}.$$

Proposition

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set and $0 < s < 1$. Then for every $f \in H^{-s}(\Omega)$, the Dirichlet problem $(\mathcal{P})$ has a unique finite energy solution $u \in H_0^s(\Omega)$ and there exists a constant $\mathcal{C} > 0$ such that $\|u\|_{H^s(\Omega)} \leq \mathcal{C} \, \|f\|_{H^{-s}(\Omega)}$. In addition, we can take $\mathcal{C} = \sqrt{2/c_{N,s}}$.

Proof

Consider the bilinear form $\mathcal{E}(\cdot,\cdot) : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy.$$

- $\mathcal{E}(u,v)$ is clearly **symmetric**: $\mathcal{E}(u,v) = \mathcal{E}(v,u)$
- $\mathcal{E}(u,v)$ is **continuous**:

$$|\mathcal{E}(u,v)| = \frac{C_{N,s}}{2} \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{\frac{N}{2}+s}} \cdot \frac{v(x)-v(y)}{|x-y|^{\frac{N}{2}+s}}\, dxdy \right|$$
$$\leq \frac{C_{N,s}}{2} [u]_{H^s(\mathbb{R}^N)} [v]_{H^s(\mathbb{R}^N)} \leq \frac{C_{N,s}}{2} \|u\|_{H^s(\mathbb{R}^N)} \|v\|_{H^s(\mathbb{R}^N)}.$$

- $\mathcal{E}(u,v)$ is **coercive**:

$$|\mathcal{E}(u,u)| = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{\frac{N}{+}2s}}\, dxdy \geq \mathcal{C}(N,s,\Omega) \|u\|^2_{L^2(\Omega)}.$$

Lax-Milgram: for every $f \in H^{-s}(\Omega)$ there exists a unique $u \in H^s_0(\Omega)$ solution of $(\mathcal{P})$.

Taking $v = u$ as a test function: $\mathcal{C}\, \|u\|^2_{H^s(\Omega)} = \langle f, u \rangle_{H^{-s}(\Omega), H^s(\Omega)} \leq \|f\|_{H^{-s}(\Omega)} \|u\|_{H^s(\Omega)}$.

Some remarks

When considering right hand side terms $f \in L^p(\Omega)$ with $p \geq 2$, due to the continuous embedding $L^p(\Omega) \hookrightarrow H^{-s}(\Omega)$, **the notion of weak finite energy solution suffices**.

When $f \in L^p(\Omega)$ with $1 \leq p < 2$, the regularity of the right-hand side term does not suffice to define weak finite energy solutions as above. We shall rather consider those defined by **duality or transposition**.

Transposition solutions

$$\begin{cases} (-\Delta)^s \phi = \psi & \text{in } \Omega \\ \phi \equiv 0 & \text{in } \Omega^c \end{cases} \qquad (1)$$

$$\mathcal{T}(\Omega) = \left\{ \phi \,:\, \phi \textit{ solves } (1) \textit{ with } \psi \in C_0^\infty(\Omega) \right\}.$$

Transposition solutions

Let $f \in L^1(\Omega)$. We say that $u \in L^1(\Omega)$ is a weak duality or transposition solution to the elliptic problem $(\mathcal{P})$, if the identity

$$\int_\Omega u\psi \, dx = \int_\Omega f\phi \, dxdt$$

holds for any $\phi \in \mathcal{T}(\Omega)$ and $\psi \in C_0^\infty(\Omega)$.

T. Leonori, I. Peral, A. Primo and F. Soria, *Basic estimates for solutions of a class of non-local elliptic and parabolic equations*, Discrete Contin. Dyn. Syst., 2015.

Some Hölder regularity results

Proposition

Let Ω be bounded, Lipschitz domain satisfying the exterior ball condition, $f \in L^\infty(\Omega)$, and u be a solution of $(\mathcal{P})$. Then, $u \in C^s(\mathbb{R}^N)$ and $\|u\|_{C^s(\mathbb{R}^N)} \leq \mathcal{C}(\Omega, s)\, \|f\|_{L^\infty(\Omega)}$.

This C^s regularity is optimal:

$$\begin{cases} (-\Delta)^s u = 1, & x \in B_r(x_0) \\ u \equiv 0, & x \in B_r(x_0)^c \end{cases} \Rightarrow u(x) = \frac{2^{-2s}\Gamma\left(\frac{N}{2}\right)\left(r^2 + |x - x_0|^2\right)^s}{\Gamma\left(\frac{N+2s}{2}\right)\Gamma(1+s)} \chi_{B_r(x_0)}(x)$$

u is C^s up to the boundary but it is not C^α for any $\alpha > s$.

Proposition

Let Ω be a bounded and $C^{1,1}$ domain, $f \in L^\infty(\Omega)$, u be a solution of $(\mathcal{P})$, and $\delta(x) = dist(x, \partial\Omega)$. Then, $u/\delta^s\,|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, we have $u/\delta^s \in C^\alpha(\overline{\Omega})$ and

$$\left\| \frac{u}{\delta^s} \right\|_{C^\alpha(\overline{\Omega})} \leq \mathcal{C}(\Omega, s)\, \|f\|_{L^\infty(\Omega)}\,, \quad \alpha > 0, \quad \alpha < min\{s, 1-s\}.$$

X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl., 2014.

The fractional semi-group

For every $f \in H^{-s}(\Omega)$ there exists a unique $u \in H_0^s(\Omega)$ such that

$$\mathcal{E}(u,v) = \langle f, v\rangle_{H^{-s}(\Omega),H_0^s(\Omega)}, \quad \textit{for all } v \in H_0^s(\Omega).$$

This defines an operator $\mathcal{A}_0 : H_0^s(\Omega) \to H^{-s}(\Omega)$ which is continuous and coercive. Let $\mathcal{A}$ be the part of $\mathcal{A}_0$ in $L^2(\Omega)$, in the sense that

$$D(A) = \Big\{u \in H_0^s(\Omega),\ \mathcal{A}u \in L^2(\Omega)\Big\}, \quad Au = \mathcal{A}u.$$

Then $\mathcal{A}$ is the realization in $L^2(\Omega)$ of the operator $(-\Delta)^s$ with the Dirichlet boundary condition $u = 0$ on $\mathbb{R}^N \setminus \Omega$. More precisely,

$$D(A) = \Big\{u \in H_0^s(\Omega),\ (-\Delta)^s u \in L^2(\Omega)\Big\}, \quad Au = (-\Delta)^s u.$$

The operator $\mathcal{A}$ has a **compact** resolvent and its first eigenvalue $\lambda_1 > 0$.

$-\mathcal{A}$ generates a **sub-Markovian strongly continuous** semi-group $(e^{-tA})_{t\geq 0}$ which is also **ultracontractive** in the sense that it maps $L^r(\Omega)$ into $L^m(\Omega)$ for every $t > 0$ and $1 \leq r \leq m \leq \infty$

$$\left\|e^{-tA}f\right\|_{L^m(\Omega)} \leq Ce^{-\lambda_1\left(\frac{1}{r}-\frac{1}{m}\right)}t^{-\frac{N}{2s}\left(\frac{1}{r}-\frac{1}{m}\right)}\|f\|_{L^r(\Omega)}.$$

C. Gal and M. Warma, *Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces*, Commun. PDE, 2017.

A Lebesgue regularity result

Proposition

Assume that $N > 2s$ and let $f \in L^p(\Omega)$ for some $p \geq \frac{2N}{N+2s}$. Then $(\mathcal{P})$ has a unique solution weak solution u. In addition the following assertions hold.

- If $p > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)} .$$

- If $\frac{2N}{N+2s} \leq p \leq \frac{N}{2s}$, then $u \in L^q(\Omega)$ for every q satisfying $p \leq q < \frac{Np}{N-2sp}$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C \|f\|_{L^p(\Omega)} .$$

U. Biccari, M. Warma and E. Zuazua, *Local elliptic regularity for the Dirichlet fractional Laplacian* Adv. Nonlinear Stud., 2017.

A maximal Sobolev regularity result

Theorem

Let $F \in H^{-s}(\mathbb{R}^N)$ and $u \in H^s(\mathbb{R}^N)$ be the weak solution to the fractional Poisson type equation

$$(-\Delta)^s u = F, \quad x \in \mathbb{R}^N.$$

If $F \in L^2(\mathbb{R}^N)$ with $p \geq 2$, then $u \in H^{2s}(\mathbb{R}^N)$

E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

Theorem

Let $f \in H^{-s}(\Omega)$ and $u \in H_0^s(\Omega)$ be the unique finite energy solution of the Dirichlet problem $(\mathcal{P})$. If $f \in L^2(\Omega)$, then $u \in H_{loc}^{2s}(\Omega)$.

U. Biccari, M. Warma and E. Zuazua, *Local elliptic regularity for the Dirichlet fractional Laplacian* Adv. Nonlinear Stud., 2017.

Proof

STEP 1: Cut-off

Given ω and $\widetilde{\omega}$ two open subsets of the domain Ω such that $\widetilde{\omega} \Subset \omega \Subset \Omega$, we introduce a **cut-off function** $\eta \in \mathcal{D}(\omega)$ such that

Cut-off function

$$\begin{cases} \eta(x) \equiv 1 & \text{if } \ x \in \widetilde{\omega} \\ 0 \leq \eta(x) \leq 1 & \text{if } \ x \in \omega \setminus \widetilde{\omega} \\ \eta(x) = 0 & \text{if } \ x \in \mathbb{R}^N \setminus \omega. \end{cases}$$

Given $f \in H^{-s}(\Omega)$ and $u \in H^s_0(\Omega)$ the unique weak solution to the Dirichlet problem $(\mathcal{P})$, the function $u\eta$ belongs to $H^s(\mathbb{R}^N)$. Moreover:

$$(-\Delta)^s u\eta = \eta(-\Delta)^s u + \underbrace{u(-\Delta)^s \eta - I_s(u,\eta)}_{g}.$$

$$I_s(u,\eta) := C_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\, dy.$$

STEP 2: Regularity of the remainder

We have $g \in L^2(\mathbb{R}^N)$. In fact there exists a constant $\mathcal{C} > 0$, independent of u, such that

$$\|g\|_{L^2(\mathbb{R}^N)} \leq \mathcal{C}\, \|u\|_{H_0^s(\Omega)}\,.$$

Indeed, since $u = 0$ on Ω^c and $(-\Delta)^s \eta \in L^\infty(\mathbb{R}^N)$, we have that

$$\|u(-\Delta)^s\eta\|^2_{L^2(\mathbb{R}^N)} = \int_\Omega |u(-\Delta)^s\eta|^2\,dx \leq \|(-\Delta)^s\eta\|^2_{L^\infty(\Omega)}\,\|u\|^2_{L^2(\Omega)} \leq \mathcal{C}\,\|u\|^2_{H_0^s(\Omega)}.$$

As for $I_s(u,\eta)$, for a.e. $x \in \mathbb{R}^N$, we can split

$$I_s(u,\eta)(x) = C_{N,s}\int_\Omega \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\,dy + C_{N,s}\eta(x)\int_{\Omega^c}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = \mathbb{I}_1(x) + \mathbb{I}_2(x).$$

STEP 3.a: Estimate of $\mathbb{I}_1(x)$

Using the Cauchy-Schwartz inequality, we get that

$$|\mathbb{I}_1(x)| \leq C_{N,s} \left(\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dy \right)^{\frac{1}{2}} \left(\int_\Omega \frac{(|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}\, dy \right)^{\frac{1}{2}}.$$

Let $x \in \Omega$ be fixed and $R > 0$ such that $\Omega \subset B(x,R)$. Since η is a smooth function (in particular Lipschitz), there exists a constant $\mathcal{C} > 0$ (depending on η) such that

$$\int_\Omega \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{N+2s}}\, dy \leq \mathcal{C} \int_\Omega \frac{dy}{|x-y|^{N+2s-2}} \leq \mathcal{C} \int_{B(x,R)} \frac{dy}{|x-y|^{N+2s-2}} \leq \mathcal{C}.$$

Hence,

$$\int_{\mathbb{R}^N} |\mathbb{I}_1(x)|^2\, dx \leq \mathcal{C} \int_{\mathbb{R}^N} \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dydx \leq \mathcal{C} \|u\|^2_{H^s_0(\Omega)}.$$

Proof (cont.)

STEP 3.b: Estimate of $\mathbb{I}_2(x)$

$\mathbb{I}_2 = 0$ on $\mathbb{R}^N \setminus \omega$. In addition, from the Cauchy-Schwartz inequality we get

$$|\mathbb{I}_2(x)|^2 \leq C_{N,s}^2 \int_{\Omega^c} \frac{\eta^2(x)dy}{|x-y|^{N+2s}} \int_{\Omega^c} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dy.$$

$$y \in \Omega^c \quad \Rightarrow \quad \frac{\eta^2(x)}{|x-y|^{N+2s}} = \frac{\chi_{\overline{\omega}}(x)\eta^2(x)}{|x-y|^{N+2s}} \leq \chi_{\overline{\omega}}(x)\eta^2(x) \sup_{x\in\overline{\omega}} \frac{1}{|x-y|^{N+2s}}.$$

Thus there exists a constant $\mathcal{C} > 0$ such that

$$\int_{\Omega^c} \frac{\eta^2(x)dy}{|x-y|^{N+2s}} \leq \chi_{\overline{\omega}}(x)\eta^2(x) \int_{\Omega^c} \frac{dy}{\text{dist}(y,\partial\overline{\omega})^{N+2s}} \leq \mathcal{C}\chi_{\overline{\omega}}(x)\eta^2(x),$$

where we have used that $dist(\partial\Omega, \partial\overline{\omega}) \geq \delta > 0$, that the distance function it grows linearly as y tends to infinity, and that $N + 2s > N$.

$$\chi_{\overline{\omega}}\eta^2 \in L^\infty(\omega) \quad \Rightarrow \quad \int_{\mathbb{R}^N} |\mathbb{I}_2(x)|^2\, dx = \int_\omega |\mathbb{I}_2(x)|^2\, dx$$

$$\leq \mathcal{C} \int_\omega \int_{\Omega^c} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dydx \leq \mathcal{C}\, \|u\|^2_{H_0^s(\Omega)}.$$

Proof (cont.)

STEP 4: Conclusion We have shown that ηu is a weak solution to the Poisson Equation $(\mathcal{P})$ with

$$F = \eta(-\Delta)^s u + g \in L^2(\mathbb{R}^N).$$

It follows that $(\eta u) \in H^{2s}(\mathbb{R}^N)$. Thus $u \in H^{2s}_{loc}(\Omega)$.

Proof with the heat semi-group definition

The heat semi-group representation of the fractional Laplacian can be used for an alternative proof of the previous result.

$$\varrho(t) := e^{t\Delta}(\eta u), \quad t \geq 0 \quad \Rightarrow \quad \varrho_t - \Delta\varrho = 0, \quad t > 0, \quad \varrho(0) = \eta u.$$

$$\varrho = \phi\eta + z \quad \textit{with} \quad \begin{cases} \phi_t - \Delta\phi = 0, & \phi(0) = u \\ z_t - \Delta z = 2\mathrm{div}(\phi\nabla\eta) - \phi\Delta\eta, & z(0) = 0. \end{cases}$$

$$\begin{aligned} (-\Delta)^s(\eta u) &= \frac{1}{\Gamma(-s)} \int_0^{+\infty} \Big(\rho(t) - \rho(0)\Big) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^{+\infty} \Big(\eta\phi(t) + z(t) - \eta u(t)\Big) \frac{dt}{t^{1+s}} \\ &= \underbrace{\frac{\eta}{\Gamma(-s)} \int_0^{+\infty} \Big(e^{t\Delta}u - u\Big) \frac{dt}{t^{1+s}}}_{\eta(-\Delta)^s u} + \underbrace{\frac{1}{\Gamma(-s)} \int_0^{+\infty} \frac{z(t)}{t^{1+s}}\, dt}_{g}. \end{aligned}$$

By means of sharp estimates on the decay, both at $t = 0$ and at $t \to +\infty$, of the function z we can prove that $g \in L^2(\mathbb{R}^N)$.

The case $p \neq 2$

Theorem

Let $1 < p < +\infty$. Given $F \in L^p(\mathbb{R}^N)$, let u be the weak solution of $(-\Delta)^s u = F$, $x \in \mathbb{R}^N$. Then, $u \in \mathscr{L}^p_{2s}(\mathbb{R}^N)$, where

$$\mathscr{L}^p_{2s}(\mathbb{R}^N) := \Big\{ u \in L^p(\mathbb{R}^N) \ : \ (-\Delta)^s u \in L^p(\mathbb{R}^N) \Big\}.$$

- $1 < p < 2$ and $s \neq 1/2$: $\mathscr{L}^p_{2s}(\mathbb{R}^N) \subset B^{2s}_{p,2}(\mathbb{R}^N)$.
- $1 < p < 2$ and $s = 1/2$: $\mathscr{L}^p_{2s}(\mathbb{R}^N) = \mathscr{L}^p_1(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.
- $2 \le p < +\infty$: $u \in B^{2s}_{p,p}(\mathbb{R}^N) = W^{2s,p}(\mathbb{R}^N)$.

Besov space

$$B^s_{p,q}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \left(\int_{\mathbb{R}^N} \frac{\|u(x+y) - u(y)\|^q_{L^p(\mathbb{R}^N)}}{|y|^{N+qs}} \, dy \right)^{\frac{1}{q}} < +\infty \right\}$$

$1 \le p, q \le +\infty, \quad 0 < s < 1.$

Local elliptic regularity

Theorem

Let $1 < p < +\infty$. Given $f \in L^p(\Omega)$, let u be the unique weak solution of

$$\begin{cases} (-\Delta)^s u = f, & x \in \Omega \\ u = 0, & x \in \Omega^c. \end{cases}$$

Then $u \in \mathscr{L}^p_{2s,loc}(\Omega)$. As a consequence:

- If $1 < p < 2$ and $s \neq 1/2$, then $u \in B^{2s}_{p,2,loc}(\Omega)$.
- If $1 < p < 2$ and $s = 1/2$, then $u \in W^{2s,p}_{loc}(\Omega) = W^{1,p}_{loc}(\Omega)$.
- If $2 \leq p < \infty$, then $u \in W^{2s,p}_{loc}(\Omega)$.

$$\mathscr{L}^p_{2s,loc}(\Omega) := \Big\{ u \in L^p(\Omega) \ : \ u\eta \in \mathscr{L}^p_{2s}(\mathbb{R}^N) \textit{ for any } \eta \in \mathcal{D}(\Omega) \Big\}.$$

$$B^{2s}_{p,2,loc}(\Omega) := \Big\{ u \in L^p(\Omega) \ : \ u\eta \in B^{2s}_{p,2}(\mathbb{R}^N) \textit{ for any } \eta \in \mathcal{D}(\Omega) \Big\}.$$

Asymptotic as $s \to 1^-$

Theorem

Let $\mathcal{F}_s = \{f_s\}_{0<s<1} \subset H^{-s}(\Omega)$ be a sequence satisfying the following assumptions:

H1 $\|f_s\|_{H^{-s}(\Omega)} \leq C$, for all $0 < s < 1$ and uniformly with respect to s.

H2 $f_s \rightharpoonup f$ weakly in $H^{-1}(\Omega)$ as $s \to 1^-$.

For all $f_s \in \mathcal{F}_s$, let $u_s \in H_0^s(\overline{\Omega})$ be the unique weak solution of the Dirichlet problem $(\mathcal{P})$. Then, as $s \to 1^-$, $u_s \to u$ strongly in $H_0^{1-\delta}(\Omega)$ for all $0 < \delta \leq 1$. Moreover, $u \in H_0^1(\Omega)$ and verifies

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \forall v \in \mathcal{D}(\Omega),$$

i.e. it is the unique weak solution of

$$\begin{cases} -\Delta u = f, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases}$$

U. Biccari and Hernández-Santamaría, *The Poisson equation from non-local to local*, Electron. J. Differential Equations, 2018

Proof

STEP 1: For any $f \in H^{-1}(\Omega)$ there exists a sequence $\mathcal{F}_s = \{f_s\}_{0<s<1} \subset H^{-s}(\Omega)$ verifying the assumptions **H1** and **H2**. Indeed:

- Any $f \in H^{-1}(\Omega)$ can be written as $f = div(g)$ with $g \in L^2(\Omega)$.
- ρ_ε **standard mollifier**, $g_\varepsilon := g \star \rho_\varepsilon$:

$$\rho_\varepsilon(x) := \begin{cases} C\varepsilon^{-N} \exp\left(\frac{\varepsilon^2}{|x|^2-\varepsilon^2}\right), & |x| < \varepsilon \\ 0, & |x| \geq \varepsilon \end{cases}$$

  (a) g_ε is **well defined**, since $g \in L^2(\Omega)$, hence it is locally integrable.
  (b) $\partial_{x_i} g_\varepsilon$ is **bounded uniformly** with respect to ε for all $i = 1, \ldots, N$.
  (c) $\lim_{\varepsilon \to 0^+} g_\varepsilon = g$, **strongly** in $L^2(\Omega)$.

- $f_\varepsilon := div(g_\varepsilon) \Rightarrow \|f_\varepsilon\|_{H^{-1+\varepsilon}(\Omega)}$ is bounded uniformly w.r.t. ε (property (a)).
- $\partial_{x_i} g_\varepsilon = \rho_\varepsilon \star g_{x_i} \to g_{x_i}$ as $\varepsilon \to 0^+$ (properties (b) and (c)).
- $\lim_{\varepsilon \to 0^+} f_\varepsilon = \lim_{\varepsilon \to 0^+} div(g_\varepsilon) = div(g) = f$ (strong convergence in $H^{-1}(\Omega)$).
- Choose $\varepsilon = 1 - s$.

Proof (cont.)

STEP 2:

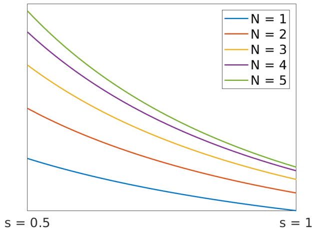
- Assume $s > 1/2$. From **H2** and the definition of weak convergence:

$$\lim_{s\to 1^-} \int_\Omega f_s v\, dx = \int_\Omega f v\, dx, \quad \forall v \in \mathcal{D}(\Omega).$$

- $u_s \in H_0^s(\Omega)$: solution of $(\mathcal{P})$ corresponding to f_s. For s sufficiently close to one:

$$\sqrt{1-s}\, \|u_s\|_{H^s(\Omega)} \le \mathcal{C}(s,N)\, \|f_s\|_{H^{-s}(\Omega)}, \quad \mathcal{C}(s,N) := \sqrt{\frac{2-2s}{c_{N,s}}}$$

- For all N fixed, $\mathcal{C}(s,N)$ is **decreasing** w.r.t. s.



$$\mathcal{C}(s,N) < \mathcal{C}\left(\frac{1}{2}, N\right) = \sqrt{\frac{\pi}{\Gamma\left(\frac{N+1}{2}\right)}}$$

$$\boxed{\sqrt{1-s}\, \|u_s\|_{H^s(\Omega)} \le C(N,\Omega)}$$

Proof (cont.)

STEP 3:

- $u_s \to u$ strongly in $H_0^{1-\delta}(\Omega)$ for all $0 < \delta \leq 1$ and $u \in H_0^1(\Omega)$.
  J. Bourgain, H. Brezis and P. Mironescu, *Another look at Sobolev spaces*, in Optimal Control and Partial Differential Equations, 2001.
- For all $\phi \in H_0^s(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$:

$$\left\langle (-\Delta)^s \phi, \psi \right\rangle = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}} \, dxdy$$

$$= \left\langle \phi, (-\Delta)^s \psi \right\rangle \Rightarrow \left\langle u_s, (-\Delta)^s v \right\rangle = \int_\Omega f_s v \, dx$$

- As $s \to 1^-$

$$\left| \langle u_s, \ (-\Delta)^s v \rangle - \langle u, -\Delta v \rangle \right| = \left| \langle u_s, (-\Delta)^s v - (-\Delta v) \rangle + \langle u_s - u, -\Delta v \rangle \right|$$

$$\leq \|u_s\|_{L^2(\Omega)} \left\| (-\Delta)^s v - (-\Delta v) \right\|_{L^2(\Omega)} + \|-\Delta v\|_{L^2(\Omega)} \|u_s - u\|_{L^2(\Omega)} \to 0$$

$$\Rightarrow \lim_{s \to 1^-} \left\langle u_s, (-\Delta)^s v \right\rangle = \left\langle u, -\Delta v \right\rangle = -\int_\Omega u \Delta v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

- Hence: $\displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$, for all $v \in \mathcal{D}(\Omega)$.

Weakening the assumptions

A convergence result as $s \to 1^-$ can be obtained under weaker assumption on the sequence $\mathcal{F}_s$.

Theorem

Let $\mathcal{F}_s = \{f_s\}_{0<s<1} \subset H^{-1}(\Omega)$ be a sequence such that $f_s \rightharpoonup f$ weakly in $H^{-1}(\Omega)$. For all $f_s \in \mathcal{F}_s$, let u_s be the corresponding solution to $(\mathcal{P})$. Then, as $s \to 1^-$, $u_s \rightharpoonup u$ weakly in $L^2(\Omega)$, with u solution to $(\mathcal{P})$ in the transposition sense.

Proof

The right-hand side f_s belongs to $H^{-1}(\Omega)$, which is strictly greater than $H^{-s}(\Omega)$. Hence, we cannot apply Lax-Milgram and we shall define the solution by **transposition**.

For all $\phi \in L^2(\Omega)$, let y be solution of the elliptic problem

$$\begin{cases} (-\Delta)^s y = \phi, & x \in \Omega \\ y \equiv 0, & x \in \Omega^c. \end{cases}$$

- For all $\varepsilon > 0$, $y \in H_0^{2s-\varepsilon}(\Omega) \hookrightarrow H_0^1(\Omega)$, with continuous and compact embedding.
- The map $\Lambda : \phi \mapsto y$ is **linear and continuous** from $L^2(\Omega)$ into $H_0^{2s-\varepsilon}(\Omega)$. Thus, Λ is **compact** from $L^2(\Omega)$ into $H_0^1(\Omega)$ and Λ^* is **compact** from $H^{-1}(\Omega)$ into $L^2(\Omega)$.

$$\langle f_s, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle f_s, \Lambda\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = (\Lambda^* f_s, \phi)_{L^2(\Omega)}.$$

- Therefore, $u_s := \Lambda^* f_s \in L^2(\Omega)$ is a solution of $(\mathcal{P})$ defined by **transposition**:

$$\int_\Omega u_s \phi \, dx = \langle f_s, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

  Moreover, we have

$$\|u_s\|_{L^2(\Omega)} \leq C \|f_s\|_{H^{-1}(\Omega)} \leq C.$$

Proof (cont.)

In particular, $\{u_s\}_{0<s<1}$ is a **bounded sequence** in $L^2(\Omega)$, which implies that $u_s \rightharpoonup u$ **weakly** in $L^2(\Omega)$.

Using the definition of weak limit we have

$$\int_\Omega u\phi\,dx = \lim_{s\to 1^-}\int_\Omega u_s\phi\,dx = \lim_{s\to 1^-}\langle f_s, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle f, y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

i.e. u is a solution by transposition of

$$\begin{cases} -\Delta u = f, & x\in\Omega \\ u = 0, & x\in\partial\Omega. \end{cases}$$

Since the $L^2(\Omega)$-regularity of u_s cannot be improved, its convergence to a solution of the above Poisson equation can be expected only in the weak sense.

Fractional heat equation

Fractional heat equation

$$\begin{cases} u_t + (-\Delta)^s u = f, & (x,t) \in \Omega \times (0,T) \\ u = 0, & (x,t) \in \Omega^c \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \qquad (\mathcal{H})$$

It represents processes involving anomalous diffusion. Applications in:

- **PHYSICS** (plasma models).
- **ECOLOGY** (population dynamics).
- **FINANCE**.

Weak solutions

Weak solutions

$u \in L^2(0,T;H_0^s(\Omega)) \cap C([0,T],L^2(\Omega))$ with $u_t \in L^2(0,T;H^{-s}(\Omega))$ is a weak solution for $(\mathcal{H})$ with $f \in L^2(0,T;H^{-s}(\Omega))$ and $u_0 \in L^2(\Omega)$ if it satisfies

$$\int_0^T \int_\Omega u_t v \, dxdt + \int_0^T a(u,v)\, dt = \int_0^T \langle f, v\rangle_{-s,s}\, dt,$$

for any $v \in L^2(0,T;H_0^s(\Omega))$.

The bilinear form $a(\cdot,\cdot) : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$ is defined as

$$a(u,v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\, dxdy.$$

Some remarks

When considering right hand side terms $f \in L^p(\Omega \times (0,T))$ with $p \geq 2$, due to the continuous embedding $L^p(\Omega \times (0,T)) \hookrightarrow L^2((0,T); H^{-s}(\Omega))$, **the notion of weak finite energy solution suffices**.

When $f \in L^p(\Omega \times (0,T))$ with $1 \leq p < 2$, the regularity of the right-hand side term does not suffice to define weak finite energy solutions as above. We shall rather consider those defined by **duality or transposition**.

Transposition solutions

$$\begin{cases} -p_t + (-\Delta)^s p = \psi & (x,t) \in \Omega \times (0,T) =: \Omega_T, \\ p \equiv 0 & (x,t) \in \Omega^c \times (0,T), \\ p(\cdot, T) \equiv 0 & x \in \Omega. \end{cases} \tag{$\mathcal{P}$}$$

$$\mathcal{P}(\Omega_T) = \Big\{ p(\cdot,t) \in C^1((0,T), C_0^\beta(\Omega)) \;:\; p \textit{ solves } (\mathcal{P}) \textit{ with } \psi \in C_0^\infty(\Omega_T) \Big\}.$$

Transposition solutions

Let $f \in L^1(\Omega \times (0,T))$. We say that $u \in C([0,T]; L^1(\Omega))$ is a weak duality or transposition solution to the parabolic problem $(\mathcal{H})$, if the identity

$$\int_0^T \int_\Omega u\psi \, dxdt = \int_0^T \int_\Omega fp \, dxdt$$

holds, for any $p \in \mathcal{P}(\Omega_T)$ and $\psi \in C_0^\infty(\Omega_T)$.

Well-posedness theorems

Theorem

Assume $f \in L^2(0,T;H^{-s}(\Omega))$. Then for any $u_0 \in L^2(\Omega)$, problem $(\mathcal{H})$ has a unique weak solution. Moreover, if f is also a non-negative function and $u_0 > 0$, such a solution is non-negative too.

Theorem

Let $f \in L^1(\Omega\times(0,T))$ and $u_0 \in L^1(\Omega)$. Then there exists a unique transposition solution of $(\mathcal{H})$. Moreover:

- $u \in L^q(\Omega \times (0,T))$ for all $q \in \left(1, \frac{N+2s}{N}\right)$.
- $|(-\Delta)^s u| \in L^r(\Omega \times (0,T))$ for all $r \in \left(1, \frac{N+2s}{N+s}\right)$.

T. Leonori, I. Peral, A. Primo and F. Soria, *Basic estimates for solutions of a class of non-local elliptic and parabolic equations*, Discrete Contin. Dyn. Syst., 2015.

Maximum principle

Proposition

Let $f \in L^2(\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$ be non-negative. Then the corresponding solution u of the system $(\mathcal{H})$ is also non-negative.

U. Biccari, M. Warma and E. Zuazua, *Local regularity for fractional heat equations*, in Recent advances in PDEs: analysis, numerics and control, 2019

PROOF: $(-\Delta)^s$ is a **self-adjoint** operator in $L^2(\Omega)$ associated with the bilinear form

$$a(\varphi,\psi) = \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dxdy, \quad \varphi,\psi \in H_0^s(\Omega).$$

$(-\Delta)^s$ is a **resolvent positive** operator. Indeed, let $\lambda > 0$ be a real number, $g \in L^2(\Omega)$ and set

$$\phi := \Big(\lambda + (-\Delta)^s\Big)^{-1} g.$$

Then, ϕ belongs to $H_0^s(\Omega)$ and is a weak solution of the Dirichlet problem

$$\begin{cases} (-\Delta)^s\phi + \lambda\phi = g, & x \in \Omega \\ \phi = 0, & x \in \Omega^c \end{cases}$$

in the sense that

$$\mathcal{E}(\phi, v) + \lambda\int_\Omega \phi v\,dx = \int_\Omega gv\,dx, \quad \forall\, v \in H_0^s(\Omega).$$

Maximum principle (cont.)

There is a constant $C > 0$ such that

$$\lambda \int_{\Omega}^{1} |v|^2 \, dx + \mathcal{E}(v, v) \geq C \|v\|^2_{H^s_0(\Omega)}, \quad \forall v \in H^s_0(\Omega).$$

Assume that $g \leq 0$ a.e. in Ω and define $\phi^+ := \max\{\phi, 0\}$ and $\phi^- := \max\{-\phi, 0\}$. We have

$$\begin{aligned}
&\left(\phi^-(x) - \phi^-(y)\right)\left(\phi^+(x) - \phi^+(y)\right) \\
&\qquad = \phi^-(x)\phi^+(x) - \phi^-(x)\phi^+(y) - \phi^-(y)\phi^+(x) + \phi^-(y)\phi^+(y) \\
&\qquad = -\left(\phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x)\right) \leq 0 \Rightarrow \mathcal{E}(\phi^-, \phi^+) \leq 0.
\end{aligned}$$

Hence,

$$\mathcal{E}(\phi, \phi^+) = \mathcal{E}(\phi^+ - \phi^-, \phi^+) = \mathcal{E}(\phi^+, \phi^+) - \mathcal{E}(\phi^-, \phi^+) \geq 0$$

$$0 \leq \lambda \int_{\Omega} \phi\phi^+ \, dx + \mathcal{E}(\phi, \phi^+) = \int_{\Omega} g\phi^+ \, dx \leq 0.$$

Therefore, $\phi^+ = 0$, that is, $\phi \leq 0$ almost everywhere.

Sobolev regularity

Theorem

Let $1 < p < \infty$ and $f \in L^p(\Omega \times (0,T))$. Then, problem $(\mathcal{H})$ has a unique weak solution $u \in C([0,T]; L^p(\Omega))$ such that $u \in L^p\Big((0,T); \mathscr{L}^p_{2s,loc}(\Omega)\Big)$ and $u_t \in L^p(\Omega \times (0,T))$. As a consequence:

- If $1 < p < 2$ and $s \neq 1/2$, $u \in L^p\Big((0,T); B^{2s}_{p,2,loc}(\Omega)\Big)$.
- If $1 < p < 2$ and $s = 1/2$,
  $u \in L^p\Big((0,T); W^{2s,p}_{loc}(\Omega)\Big) = L^p\Big((0,T); W^{1,p}_{loc}(\Omega)\Big)$.
- If $2 \leq p < +\infty$, $u \in L^p\Big((0,T); W^{2s,p}_{loc}(\Omega)\Big)$.

U. Biccari, M. Warma and E. Zuazua, *Local regularity for fractional heat equations*, in Recent advances in PDEs: analysis, numerics and control, 2019

Proof

Direct consequence of the elliptic regularity and the following result.

Theorem

Let (Ω, Σ, m) be a measure space and let A be the generator of a strongly continuous semi-group of linear operators $(\mathbb{T}_t)_{t\geq 0}$ on $L^2(\Omega, \Sigma, m)$ satisfying the following hypothesis:

- The semi-group $(\mathbb{T}_t)_{t\geq 0}$ is analytic and bounded on $L^2(\Omega, \Sigma, m)$.
- For every $p \in [1, \infty]$ and $\phi \in L^p(\Omega) \cap L^2(\Omega)$ we have the estimate

$$\|\mathbb{T}_t \phi\|_{L^p(\Omega)} \leq \|\phi\|_{L^p(\Omega)}, \text{ for all } t \geq 0.$$

Let $p \in (1, \infty)$. If $f \in L^p(\Omega \times (0, T))$, then the system

$$\begin{cases} u_t - Au = f, & t \in (0, T) \\ u(0) = 0 \end{cases}$$

admits a solution $u \in C([0, T]; L^p(\Omega))$, such that $u_t, Au \in L^p(\Omega \times (0, T))$.

D. Lamberton, *Equations d'évolution linéaires associées à des semi-groupes de contractions dans les espaces L^p*, J. Funct. Anal., 1987

Asymptotic as $s \to 1^-$

Theorem

Let $\mathcal{G}_s := \{g_s\}_{0<s<1} \subset L^2(0,T;H^{-s}(\Omega))$ satisfy for all $0 < t < T$:

K1 $\|g_s(t)\|_{H^{-s}(\Omega)} \leq C$, for all $0 < s < 1$ and uniformly with respect to s.

K2 $g_s(t) \rightharpoonup g(t)$ weakly in $H^{-1}(\Omega)$ as $s \to 1^-$.

For any $f_s \in \mathcal{G}_s$, let $\phi_s \in L^2(0,T;H_0^s(\Omega))$ be the unique weak solution of

$$\begin{cases} \partial_t \phi_s + (-\Delta)^s \phi_s = g_s, & (x,t) \in \Omega \times (0,T) \\ \phi_s \equiv 0, & (x,t) \in \Omega^c \times (0,T) \\ \phi_s(x,0) = 0, & x \in \Omega, \end{cases}$$

Then, $s \to 1^-$, $(\phi_s, \partial_t \phi_s) \to (\phi, \partial_t \phi)$ strongly in $L^2(0,T;H_0^{1-\delta}(\Omega)) \times L^2(0,T;H^{-1}(\Omega))$ for any $0 < \delta \leq 1$. Moreover, $\phi \in L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H^{-1}(\Omega))$ and verifies

$$\int_0^T \int_\Omega \partial_t \phi \psi \, dxdt + \int_0^T \int_\Omega \nabla \phi \cdot \nabla \psi \, dxdt = \int_0^T \int_\Omega g \psi \, dxdt, \quad \forall \psi \in \mathcal{D}(\Omega \times (0,T)),$$

i.e. it is the unique weak solution of

$$\begin{cases} \partial_t \phi - \Delta \phi = g, & (x,t) \in \Omega \times (0,T) \\ \phi = 0, & (x,t) \in \partial\Omega \times (0,T) \\ \phi(x,0) = 0, & x \in \Omega. \end{cases}$$

U. Biccari and Hernández-Santamaría, *The Poisson equation from non-local to local*, Electron. J. Differential Equations, 2018

Proof

- A sequence $\mathcal{G}_s$ verifying **K1** and **K2** exists. It can be constructed following the methodology of the elliptic case, since both properties are independent of the time variable.
- We shall only analyze the first term on the left-hand side of the variational formulation. Indeed:
  - ▷ The functional space in which the integration in time is carried out **does not depend** on s. Therefore, the limit process does not affect the regularity in the time variable.
  - ▷ For the remaining two terms, the limit as $s \to 1^-$ can be addressed in an analogous way as in the elliptic case.
- Multiplying the equation by ϕ_s and integrating by parts we obtain the **energy estimate**:

$$\|\phi_s\|_{L^2(0,T;H_0^s(\Omega))} + \|\partial_t \phi_s\|_{L^2(0,T;H^{-s}(\Omega))} \leq C\,\|g_s\|_{L^2(0,T;H^{-s}(\Omega))}$$

Proof (cont.)

- Analogously as in the elliptic case, we can show that $\phi_s \rightarrow \phi$ **strongly** in $L^2(0, T; H_0^{1-\delta}(\Omega))$ for all $0 < \delta \leq 1$ as $s \rightarrow 1^-$.
- **Energy estimate**: $\{\partial_t \phi_s\}$ bounded in $L^2(0, T; H^{-s}(\Omega)) \hookrightarrow L^2(0, T; H^{-1}(\Omega))$ **compactly**. Thus, as $s \rightarrow 1^-$, $\partial_t \phi_s \rightarrow \partial_t \phi$ strongly in $L^2(0, T; H^{-1}(\Omega))$ and $(\phi_s, \partial_t \phi_s) \rightarrow (\phi, \partial_t \phi)$ **strongly** in $L^2(0, T; H_0^{1-\delta}(\Omega)) \times L^2(0, T; H^{-1}(\Omega))$ for all $0 < \delta \leq 1$. In particular:

$$\lim_{s \rightarrow 1^-} \int_0^T \int_\Omega \partial_t \phi_s \psi \, dxdt = \int_0^T \int_\Omega \partial_t \phi \psi \, dxdt.$$

- This, together with the above remarks, implies that the function ϕ satisfies

$$\int_0^T \int_\Omega \partial_t \phi \psi \, dxdt + \int_0^T \int_\Omega \nabla \phi \cdot \nabla \psi \, dxdt = \int_0^T \int_\Omega g \psi \, dxdt,$$

for all $\psi \in \mathcal{D}(\Omega \times (0, T))$.

The fractional Laplacian with exterior conditions

The fractional Laplacian with exterior conditions

Fractional Poisson equation with non-homogeneous exterior condition

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = g & \text{in } \Omega^c. \end{cases} \tag{2}$$

Fractional heat equation with non-homogeneous exterior condition

$$\begin{cases} y_t + (-\Delta)^s y = 0 & \text{in } \Omega \times (0, T) \\ y = g & \text{in } \Omega^c \times (0, T) \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases} \tag{3}$$

The fractional normal derivative

Fractional normal derivative

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy, \quad x \in \Omega^c. \tag{4}$$

Clearly, $\mathcal{N}_s$ is a non-local operator. Moreover, it is well defined on $H^s(\mathbb{R}^N)$ as the following result shows.

Lemma

The non-local normal derivative $\mathcal{N}_s$ maps $H^s(\mathbb{R}^N)$ continuously into $H^s_{loc}(\Omega^c) \subset L^2_{loc}(\Omega^c)$.

T. Ghosh, M. Salo and G. Uhlmann, *The Calderón problem for the fractional Schrödinger equation*, Anal. PDE, 2020.

Some properties

Even if $\mathcal{N}_s$ is defined on the unbounded domain Ω^c, it is still denoted **normal derivative**. This is due to similarity with the classical normal derivative.

Proposition

Divergence theorem: let $u \in C^2(\mathbb{R}^N)$ vanishing at $\pm\infty$. Then

$$\int_\Omega (-\Delta)^s u \, dx = -\int_{\Omega^c} \mathcal{N}_s u \, dx.$$

Integration by parts formula: let $u \in H^s(\mathbb{R}^N)$ be such that $(-\Delta)^s u \in L^2(\Omega)$ and $\mathcal{N}_s u \in L^2(\Omega^c)$. Then, for every $v \in H^s(\mathbb{R}^N)$ we have

$$\int_\Omega v(-\Delta)^s u \, dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, dxdy - \int_{\Omega^c} v \mathcal{N}_s u \, dx,$$

where $\mathbb{R}^{2N} \setminus (\Omega^c)^2 = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega)$.

Limit as $s \uparrow 1^-$: let $u, v \in C^2(\mathbb{R}^N)$ vanishing at $\pm\infty$. Then

$$\lim_{s\uparrow 1^-} \int_{\Omega^c} v \mathcal{N}_s u \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\sigma.$$

S. Dipierro, X. Ros-Oton and E. Valdinoci, *Nonlocal problems with Neumann boundary conditions*, Rev. Mat. Iberoam., 2017.

Existence and uniqueness of solutions

Weak solutions

Let $g \in L^2(\Omega^c)$, $f \in H^{-s}(\Omega)$ and $G \in H^s(\mathbb{R}^N)$ be such that $G|_{\Omega^c} = g$. A function $u \in H^s(\mathbb{R}^N)$ is said to be a weak solution to (2) if $u - G \in H^s_0(\Omega)$ and the identity

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = \langle f, v\rangle_{-s,s}$$

holds for every $v \in H^s_0(\Omega)$.

Theorem

Let $f \in H^{-s}(\Omega)$ and $g \in L^2(\Omega^c)$. Then, (2) has a unique weak solution $u \in H^s(\mathbb{R}^N)$, and there is a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C\left(\|f\|_{H^{-s}(\Omega)} + \|g\|_{L^2(\Omega^c)}\right).$$

H. Antil, R. Khatri and M. Warma, *External optimal control of nonlocal PDEs*, Inv. Problems, 2019.

Existence and uniqueness of solutions

Very weak (transposition) solutions

Let $g \in L^2(\Omega^c)$ and $f \in H^{-s}(\Omega)$. A function $u \in L^2(\mathbb{R}^N)$ is said to be a solution by transposition to (2) if the identity

$$\int_\Omega u(-\Delta)^s v\,dx = \langle f, v\rangle_{-s,s} - \int_{\Omega^c} g\mathcal{N}_s v\,dx$$

holds for every $v \in V := \left\{v \in H_0^s(\Omega)\ :\ (-\Delta)^s v \in L^2(\Omega)\right\}$.

Theorem

Let $f \in H^{-s}(\Omega)$ and $g \in L^2(\Omega^c)$. Then, (2) has a unique solution by transposition $u \in L^2(\mathbb{R}^N)$, and there is a constant $C > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^N)} \leq C\left(\|f\|_{H^{-s}(\Omega)} + \|g\|_{L^2(\Omega^c)}\right).$$

H. Antil, R. Khatri and M. Warma, *External optimal control of nonlocal PDEs*, Inv. Problems, 2019.

The parabolic case

Remark

Let $y_0 \in L^2(\Omega)$, $g \in L^2((0,T);H^s(\Omega^c))$ and consider the following two systems:

$$\begin{cases} \xi_t + (-\Delta)^s \xi = 0 & \text{in } \Omega \times (0,T) \\ \xi = 0 & \text{in } \Omega^c \times (0,T) \\ \xi(\cdot,0) = y_0 & \text{in } \Omega. \end{cases} \tag{5}$$

and

$$\begin{cases} z_t + (-\Delta)^s z = 0 & \text{in } \Omega \times (0,T) \\ z = g & \text{in } \Omega^c \times (0,T) \\ z(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

Then, the solution of (3) is given by $y = \xi + z$.

The parabolic case

Theorem

Let $(\phi_k)_{k\in\mathbb{N}}$ be the normalized eigenfunctions of the operator $(-\Delta)^s$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$. For every $y_0 \in L^2(\Omega)$, define $y_{0,k} := \langle y_0, \phi_k\rangle_{L^2(\Omega)}$. Then, there is a unique function

$$\xi \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^s(\Omega)) \cap H^1((0,T); H^{-s}(\Omega))$$

satisfying (5) which is given for a.e. $x \in \Omega$ and every $t \in [0,T]$ by

$$\xi(x,t) = \sum_{j\geq 1} y_{0,k} e^{-\lambda_k t} \phi_k(x).$$

The parabolic case

Weak solutions of (6)

Let $g \in L^2((0,T);H^s(\Omega^c))$. By a weak solution of (6) we mean a function $z \in L^2((0,T);H^s(\mathbb{R}^N))$ such that $z = g$ a.e. in $\Omega^c \times (0,T)$ and the identity

$$\int_0^T \langle -w_t + (-\Delta)^s w, z\rangle_{-s,s}\, dt = \int_\Omega z(x,T)w(x,T)\, dx + \int_0^T \int_{\Omega^c} g\mathcal{N}_s w\, dxdt$$

holds for every $w \in C([0,T];L^2(\Omega)) \cap L^2((0,T);H_0^s(\Omega)) \cap H^1((0,T);H^{-s}(\Omega))$ with $\mathcal{N}_s w \in L^2((0,T)\times\Omega^c)$.

Theorem

For every $g \in L^2((0,T);H^s(\Omega^c))$, (6) has a unique weak solution $z \in L^2((0,T);H^s(\mathbb{R}))$ given by

$$z(x,t) = \sum_{k\geq 1}\left(\int_0^t \left(g(\cdot,t-\tau),\mathcal{N}_s\phi_k\right)_{L^2(\Omega^c)} e^{-\lambda_k\tau}\, d\tau\right)\phi_k(x).$$

M. Warma, *Approximate controllability from the exterior of space-time fractional diffusive equations*, SICON, 2019.

The parabolic case

Theorem

For every $y_0 \in L^2(\Omega)$ and $g \in L^2((0,T); H^s(\Omega^c))$, the system (3) has a unique weak solution $y \in L^2((0,T) \times \mathbb{R}^d)$ given by

$$y(x,t) = \sum_{k \geq 1} y_{0,k} e^{-\lambda_k t} \phi_k + \sum_{k \geq 1} \left(\int_0^t \left(g(\cdot, t-\tau), \mathcal{N}_s \phi_k\right)_{L^2(\Omega^c)} e^{-\lambda_k \tau} \, d\tau \right) \phi_k(x).$$

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