Control and optimization for non-local and fractional differential equations

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PART III: non-local in space models

LECTURE 9: Control theory for PDE involving the fractional Laplacian

Controllability of the fractional heat equaton

1-d fractional heat equation

$$\begin{cases} y_t + (-d_x^2)^s y = u\chi_\omega & (x,t) \in (-1,1) \times (0,T) \\ y \equiv 0 & (x,t) \in (-1,1)^c \times (0,T), \\ y(\cdot,0) = y_0 & x \in (-1,1). \end{cases} \qquad (\mathcal{FH})$$

Theorem

Given any $y_0 \in L^2(-1,1)$, $\omega \subset (-1,1)$ and $T > 0$, the fractional heat equation $(\mathcal{FH})$ is

- **approximately controllable** at time T with $u \in L^2(\omega \times (0,T))$, for all $s \in (0,1)$.
- **null-controllable** at time T with $u \in L^2(\omega \times (0,T))$, if and only if $s > 1/2$.
- **null-controllable** at time T with $u \in L^\infty(\omega \times (0,T))$, if and only if $s > 1/2$.

U. Biccari and V. Hernández-Santamaría, *Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects*, IMA J. Math. Control Inf., 2018

U. Biccari, M. Warma and E. Zuazua, *Controllability of the one-dimensional fractional heat equation under positivity constraints*, Commun. Pure Appl. Anal., 2019

Previous controllability results

- the 1-d fractional heat equation with **spectral** fractional Laplacian is null controllable at time $T > 0$ if and only if $s > 1/2$.

  S. Micu and E. Zuazua, *On the controllability of a fractional order parabolic equation*, SICON, 2006

- the same result holds in multi-d setting using a **Lebeau-Robbiano** strategy.

  L. Miller, *On the controllability of anomalous diffusions generated by the fractional Laplacian*, Math. Control Signal Systems, 2006

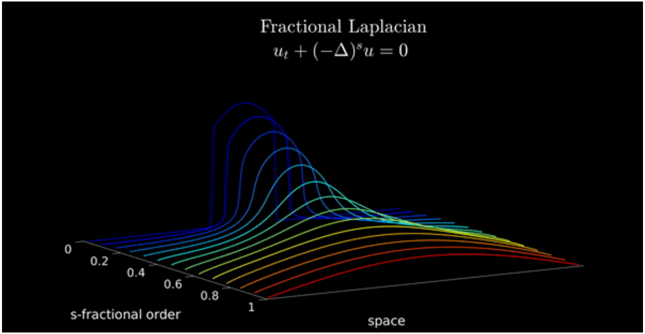
Recall

spectral $\neq$ integral

Why $s > 1/2$?

Fractional heat equation

$$y_t - (-\Delta)^s y = 0.$$



Proof of the approximate controllability

Parabolic unique continuation property

Given $s \in (0,1)$ and $p^T \in L^2(-1,1)$, let p be the unique solution to the adjoint equation

$$\begin{cases} -p_t + (-d_x^2)^s p = 0, & (x,t) \in (-1,1) \times (0,T) \\ p = 0, & (x,t) \in (-1,1)^c \times (0,T) \\ p(x,T) = p_T(x), & x \in (-1,1), \end{cases} \tag{1}$$

Let $\omega \subset (-1,1)$ be an arbitrary open set. If $p = 0$ on $\omega \times (0,T)$, then $p = 0$ on $(-1,1) \times (0,T)$.

Proof of the approximate controllability

To prove this unique continuation, let us observe that for $(x,t) \in (-1,1) \times (0,T)$, p can be expressed in the basis of the eigenfunctions of the fractional Laplacian as

$$p(x,t) = \sum_{k\geq 1} p_k e^{-\lambda_k (T-t)} \phi_k(x), \quad p_k := \langle p_T, \phi_k \rangle_{L^2(-1,1)}, \tag{2}$$

Moreover, p can be analytically extended to the half plane $\mathbf{\Sigma}_T := \{z \in \mathbb{C} \,:\, \Re(z) < T\}$, so that we can write

$$p(x,t) = \sum_{k\geq 1} p_k e^{-\lambda_k (T-t)} \phi_k(x), \qquad \text{for all } (x,t) \in (-1,1) \times (-\infty, T). \tag{3}$$

V. Keyantuo and M. Warma, *On the interior approximate controllability for fractional wave equations*, Discr. Cont. Dyn. Syst., 2016

Proof of the approximate controllability

Assume that

$$p = 0 \quad \text{in} \quad \omega \times (0, T) \tag{4}$$

and let $\{\psi_{k_j}\}_{1\leq k\leq m_k}$ be an orthonormal basis of $\ker\,(\lambda_k - (-\Delta)^s)$. Then, we have

$$\phi_k(x) = \sum_{j=1}^{m_k} \phi_{k_j}\psi_{k_j}, \quad (\phi_{k_j})_{j=1}^{m_k} \in \ell^2$$

and (3) can be rewritten as

$$p(x,t) = \sum_{k\geq 1}\left(\sum_{j=1}^{m_k} p_{k_j}\psi_{k_j}(x)\right)e^{-\lambda_k(T-t)}, \quad (x,t) \in (-1,1)\times(-\infty, T),$$

with $p_{k_j} := p_k\phi_{k_j}$.

Proof of the approximate controllability

Let $\sigma \in \mathbb{C}$ with $\eta := \Re(\sigma) > 0$ and let $N \in \mathbb{N}$. Since the functions ψ_{k_j}, $1 \leq j \leq m_k$, $1 \leq k \leq N$ are orthonormal, if we define

$$p_N(x,t) := \sum_{k=1}^{N} \left(\sum_{j=1}^{m_k} p_{k_j} \psi_{k_j}(x) \right) e^{\sigma(t-T)} e^{-\lambda_k (T-t)}.$$

we have that

$$\begin{aligned} \|p_N(x,t)\|^2_{L^2(-1,1)} &\leq \sum_{k\geq 1} \left(\sum_{j=1}^{m_k} |p_{k_j}|^2 \right) e^{2\eta(t-T)} e^{-2\lambda_k (T-t)} \\ &\leq \sum_{k\geq 1} |p_k|^2 \left(\sum_{j=1}^{m_k} |\phi_{k_j}|^2 \right) e^{2\eta(t-T)} \leq C e^{2\eta(t-T)} \|p_T\|^2_{L^2(-1,1)}. \end{aligned}$$

Proof of the approximate controllability

Moreover, we have

$$\int_{-\infty}^{T} e^{\eta(t-T)} \, \|p_T\|_{L^2(-1,1)} \, dt = \frac{1}{\eta} \|p_T\|_{L^2(-1,1)} \int_0^{+\infty} e^{-\tau} \, d\tau = \frac{1}{\eta} \|p_T\|_{L^2(-1,1)}.$$

Therefore, we can apply the Dominated Convergence Theorem and the change of variables $T - t \mapsto \tau$, obtaining for all $x \in (-1,1)$ and $\eta = \Re(\sigma) > 0$

$$\begin{aligned}
\lim_{N\to+\infty} \int_{-\infty}^{T} p_N(x,t) \, dt &= \int_{-\infty}^{T} \lim_{N\to+\infty} p_N(x,t) \, dt \\
&= \int_{-\infty}^{T} e^{\sigma(t-T)} \sum_{k\geq 1} \left(\sum_{j=1}^{m_k} p_{k_j} \psi_{k_j}(x) \right) e^{-\lambda_k (T-t)} \, dt \qquad (5) \\
&= \sum_{k\geq 1} \sum_{j=1}^{m_k} p_{k_j} \psi_{k_j}(x) \int_0^{+\infty} e^{-(\sigma+\lambda_k)\tau} \, d\tau = \sum_{k\geq 1} \sum_{j=1}^{m_k} \frac{p_{k_j}}{\sigma + \lambda_k} \psi_{k_j}(x)
\end{aligned}$$

It follows from (4) and (5) that

$$\sum_{k\geq 1}\sum_{j=1}^{m_k} \frac{p_{k_j}}{\sigma+\lambda_k}\psi_{k_j}(x) = 0, \qquad \text{for all } x\in\omega \text{ and } \Re(\sigma) > 0.$$

This holds for every $\sigma \in \mathbb{C} \setminus \{-\lambda_k\}_{k\in\mathbb{N}}$, using the analytic continuation in σ. Hence, taking a suitable small circle around $-\lambda_\ell$ not including $\{-\lambda_k\}_{k\neq\ell}$ and integrating on that circle we get that

$$p_\ell := \sum_{j=1}^{m_\ell} p_{\ell_j}\psi_{\ell_j}(x) = 0, \qquad \text{for all } x\in\omega.$$

According to

M. M. Fall and V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Commun. PDE, 2014

providing a unique continuation property for $(-\Delta)^s$, we have that $p_\ell = 0$ in $(-1,1)$ for every ℓ.

Since $\{\psi_{\ell_j}\}_{1\leq j\leq m_\ell}$ are linearly independent in $L^2(-1,1)$, we get $\varphi_{\ell_j} = 0, 1\leq j\leq m_k$, $\ell\in\mathbb{N}$. It follows that $p_T = 0$ and hence, $p = 0$ in $(-1,1)\times(0,T)$, meaning that *p* **enjoys the parabolic unique continuation property**.

Proof of the approximate controllability

Remark

The elliptic unique continuation property for the fractional Laplacian holds in any space dimension. In view of that, the approximate controllability of the fractional heat equation can be proved also in the case $N > 1$.

On the other hand, we will see that the same does not applies to the null controllability property, since the proof of that result uses arguments specifically designed for one-dimensional problems.

If one would like to analyze the multi-dimensional problem, other tools such as Carleman estimates are needed. As far as we know, these techniques have not been fully developed yet for problems involving the fractional Laplacian on a domain.

Proof of the null controllability in L^2

First of all, let us recall that the result is equivalent to proving the following observability inequality for the solution of the adjoint system (1)

L^2 observability inequality

$$\|p(x,0)\|^2_{L^2(-1,1)} \leq \mathcal{C} \int_0^T \int_\omega |p|^2 \, dxdt. \tag{6}$$

Moreover, using (2), the orthonormality of the eigenfunctions and the change of variables $T - t \mapsto t$, we easily see that (6) is equivalent to

$$\sum_{k\geq 1} |p_k|^2 e^{-2\lambda_k T} \leq \mathcal{C} \int_0^T \int_\omega \left| \sum_{k\geq 1} p_k e^{-\lambda_k t} \phi_k \right|^2 dxdt. \tag{7}$$

We know that, if the eigenvalues $(\lambda_k)_{k\geq 1}$ satisfy

- $\lambda_{k+1} - \lambda : k \geq \gamma > 0$ for all $k \geq 1$
- $\sum_{k\geq 1} \lambda_k^{-1} < +\infty$

then for any sequence $\{c_k\}_{k\geq 1}$ it holds the estimate

$$\sum_{k\geq 1} |c_k| e^{-\lambda_k T} \leq \mathcal{C}(T) \left\| \sum_{k\geq 1} c_k e^{-\lambda_k t} \right\|_{L^2(0,T)}. \tag{8}$$

Therefore, for any $x \in (-1,1)$ fixed, if we take $c_k := p_k \phi_k(x)$ in (8) we get

$$\sum_{k\geq 1} |p_k \phi_k(x)| e^{-\lambda_k T} \leq \mathcal{C}(T) \left\| \sum_{k\geq 1} p_k e^{-\lambda_k t} \phi_k(x) \right\|_{L^2(0,T)}.$$

From this last estimate, we then obtain that

$$\begin{aligned} \sum_{k\geq 1} |p_k \phi_k(x)|^2 e^{-2\lambda_k T} &\leq \left(\sum_{k\geq 1} |p_k \phi_k(x)| e^{-\lambda_k T} \right)^2 \\ &\leq \mathcal{C}(T)^2 \int_0^T \left| \sum_{k\geq 1} p_k e^{-\lambda_k t} \phi_k(x) \right|^2 dt. \end{aligned} \tag{9}$$

Proof of the null controllability in L^2

Finally, we can show that there exists a positive constant $\beta > 0$, independent of k, such that the eigenfunctions of the fractional Laplacian satisfy the estimate

$$\|\phi_k\|_{L^2(\omega)} \geq \beta|\omega|^{-1}, \quad \text{for all } k \geq 1 \text{ and } \omega \subset (-1,1).$$

Then, integrating over ω, we finally obtain from (9) that

$$\begin{aligned}\beta|\omega|^{-1}\sum_{k\geq 1}|p_k|^2 e^{-2\lambda_k T} &\leq \int_\omega \sum_{k\geq 1}|p_k\phi_k(x)|^2 e^{-2\lambda_k T}\,dx \\ &\leq \mathcal{C}(T)^2\int_0^T\int_\omega\left|\sum_{k\geq 1}p_k e^{-\lambda_k t}\phi_k(x)\right|^2 dxdt,\end{aligned}$$

from which (7) follows immediately.

Proof of the null controllability in L^2

To conclude our proof, it only remains to check that the eigenvalues $(\lambda_k)_{k\geq 1}$ satisfy

$$\lambda_{k+1} - \lambda_k \geq \gamma > 0 \text{ for all } k \geq 1 \qquad \sum_{k\geq 1} \lambda_k^{-1} < +\infty.$$

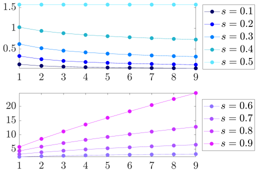
This is true if and only if $s > 1/2$.

In fact, in this case, the eigenvalues are simple and, therefore, fulfill the gap condition.

Moreover, we have the following asymptotic behavior:

$$\lambda_k = \left(\frac{k\pi}{2} + \frac{(1-s)\pi}{4}\right)^{2s} + \mathcal{O}\left(\frac{1}{k}\right)$$

as $k \to +\infty$.



M. Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the interval*, J. Funct. Anal., 2012

We then have that the above sum is convergent if and only if $s > 1/2$.

By duality, the controllability of $(\mathcal{FH})$ wit controls $u \in L^\infty(\omega \times (0,T))$ is equivalent to the following observability inequality for the solution of the adjoint system (1)

L^1 observability inequality

$$\|p(x,0)\|^2_{L^2(-1,1)} \leq \mathcal{C} \left(\int_0^T \int_\omega |p| \, dxdt \right)^2 .$$

The proof of this last inequality is the same as the L^2 case, and uses

- $\sum_{k\geq 1} |c_k| e^{-\lambda_k T} \leq \mathcal{C}(T) \left\| \sum_{k\geq 1} c_k e^{-\lambda_k t} \right\|_{L^1(0,T)}$
  that holds under the same assumptions on the eigenvalues $\{\lambda_k\}_{k\geq 1}$
- $\|\phi_k\|_{L^1(\omega)} \geq \beta > 0$

Proof of the null controllability in L^∞

Remark

The L^1 observability inequality implies that the fractional heat equation $(\mathcal{FH})$ is null controllable at time $T > 0$ with controls $u \in L^\infty(\omega \times (0,T))$ such that

$$\|u\|_{L^\infty(\omega\times(0,T))} = \|p\|_{L^1(\omega\times(0,T))}$$

This does not necessary implies that such controls are of bang-bang nature, i.e. in the form

$$u = \text{sign}(p)\, \|p\|_{L^\infty(\omega\times(0,T))}$$

To have bang-bang controls, **we need the zero set of the solutions of the adjoint equation to be of null measure**, so that the sign of the adjoint state is well defined.

This is true in the case of the classical heat equation, as a consequence of the space-time analyticity properties of the solutions, but is unknown (and a very challenging PDE analysis problem) for the fractional heat equation for which **only time-analyticity of solutions is known**.

Controllability under positivity constraints

Constrained controllability

We have seen that

1. The fractional heat equation $(\mathcal{FH})$ is null controllable in any time $T > 0$ by means of a control $u \in L^2(\omega \times (0, T))$ (or $u \in L^\infty(\omega \times (0, T))$), if and only if $s > 1/2$. Besides, the equation being linear, by translation the same result holds if the final target is a trajectory $\widehat{y}$.
2. The fractional heat equation preserves positivity: if y_0 is a given non-negative initial datum in $L^2(-1, 1)$ and u is a non-negative function, then so it is for the solution y of $(\mathcal{FH})$.

Question

Can we control the fractional heat dynamics $(\mathcal{FH})$ from any initial datum $y_0 \in L^2(-1, 1)$ to any positive trajectory $\widehat{y}$, under positivity constraints on the control and/or the state?

Constrained controllability

Theorem

Let $s > 1/2$, $y_0 \in L^2(-1,1)$ and let $\widehat{y}$ be a positive trajectory, i.e., a solution of $(\mathcal{FH})$ with initial datum $0 < \widehat{y}_0 \in L^2(-1,1)$ and right hand side $\widehat{u} \in L^\infty(\omega \times (0,T))$. Assume that there exists $\nu > 0$ such that $\widehat{u} \geq \nu$ a.e in $\omega \times (0,T)$. Then, the following assertions hold.

1. There exist $T > 0$ and a non-negative control $u \in L^\infty(\omega \times (0,T))$ such that the corresponding solution y of $(\mathcal{FH})$ satisfies $y(x,T) = \widehat{y}(x,T)$ a.e. in $(-1,1)$. Moreover, if $y_0 \geq 0$, $y(x,t) \geq 0$ a.e. in $(-1,1) \times (0,T)$.

2. Define the minimal controllability time by

$$T_{min}(y_0, \widehat{y}) := \inf \Big\{ T > 0 : \exists \;\; 0 \leq u \in L^\infty(\omega \times (0,T)) \; s.\,t. \qquad y(\cdot,0) = y_0 \; and \; y(\cdot,T) = \widehat{y}(\cdot,T) \Big\}.$$

   For $T = T_{min}$, there exists a non-negative control $u \in \mathcal{M}(\omega \times (0,T_{min}))$, the space of Radon measures on $\omega \times (0,T_{min})$, such that the corresponding solution of $(\mathcal{FH})$ satisfies $y(x,T) = \widehat{y}(x,T)$ a.e. in $(-1,1)$.

U. Biccari, M. Warma and E. Zuazua, *Controllability of the one-dimensional fractional heat equation under positivity constraints*, Commun. Pure Appl. Anal., 2019

Proof of the constrained controllability

Two main ingredients

1. Controllability through $L^\infty(\omega \times (0,T))$ controls.
2. Dissipativity of the fractional heat semi-group.

Proof of the constrained controllability

STEP 1: reduction to null-controllability

Subtracting $\widehat{y}$ in the equation, if we denote $w := u - \widehat{u}$, we have that $\xi := y - \widehat{y}$ fulfills

$$\begin{cases} \xi_t + (-d_x^2)^s \xi = w\chi_\omega, & (x,t) \in (-1,1) \times (0,T) \\ \xi = 0, & (x,t) \in (-1,1)^c \times (0,T) \\ \xi(\cdot,0) = y_0(\cdot) - \widehat{y}_0(\cdot), & x \in (-1,1) \\ \xi(\cdot,T) = 0, & x \in (-1,1) \end{cases}$$

It is enough to show that $v \in L^\infty(\omega \times (0,T))$ fulfills $v > -\nu$ a.e. in $\omega \times (0,T)$.

Proof of the constrained controllability

STEP 2: controllability $\equiv$ observability

The controllability of the previous system for the function ξ is is equivalent to the observability inequality

$$\|p(\cdot,\tau)\|^2_{L^2(-1,1)} \leq \mathcal{C}(T-\tau)\left(\int_\tau^T\int_\omega |p(x,t)|\,dxdt\right)^2$$

STEP 3: dissipativity

Using that the eigenvalues $\{\lambda_k\}_{k\geq 1}$ form a non-decreasing sequence, and the dissipativity of the fractional heat semi-group, we have

$$\|p(\cdot,0)\|^2_{L^2(-1,1)} \leq e^{-2\lambda_1\tau}\,\|p(\cdot,\tau)\|^2_{L^2(-1,1)} \leq e^{-2\lambda_1\tau}\mathcal{C}(T-\tau)\left(\int_0^T\int_\omega |p(x,t)|\,dxdt\right)^2$$

Proof of the constrained controllability

STEP 4: duality

By duality, the control w can be chosen such that

$$\|w\|^2_{L^\infty(\omega\times(0,T))} \leq e^{-2\lambda_1\tau}\mathcal{C}(T-\tau)\,\|y_0-\widehat{y}_0\|^2_{L^2(-1,1)}\,.$$

Taking $\tau = T/2$, we obtain

$$\|w\|^2_{L^\infty(\omega\times(0,T))} \leq e^{-\lambda_1 T}\mathcal{C}(T)\,\|y_0-\widehat{y}_0\|^2_{L^2(-1,1)}\,.$$

The observability constant $\mathcal{C}(T)$ is **uniformly bounded** for any $T>0$ (although it blows-up exponentially as $T\to 0^+$). Hence, for T large enough we have

$$\|w\|^2_{L^\infty(\omega\times(0,T))} < \nu \quad \to \quad w > -\nu$$

Therefore, the control $w>-\nu$ steers ξ from $y_0-\widehat{y}_0$ to zero in time T, provided T is large enough. Consequently, $u > w+\widehat{u}\geq 0$ steers y from y_0 to $\widehat{y}(\cdot,T)$ in time T

If $y_0 \geq 0$, thanks to the maximum principle, we also have $y(x,t)\geq 0$ for every $(x,t)\in(-1,1)\times(0,T)$.

Minimal-time control

Constrained controllability of $(\mathcal{FH})$ holds in the minimal time T_{min} with controls in the (Banach) space of the Radon measures $\mathcal{M}(\omega \times (0, T_{min}))$ endowed with the norm

$$\|\mu\|_{\mathcal{M}(\omega\times(0,T_{min}))} = \sup \left\{ \int_0^{T_{min}} \int_\omega \varphi(x,t)\, d\mu(x,t) : \right.$$
$$\left. \varphi \in C(\overline{\omega} \times [0, T_{min}], \mathbb{R}),\ \max_{\overline{\omega}\times[0,T_{min}]} |\varphi| = 1 \right\}.$$

Minimal-time control

Solutions of $(\mathcal{FH})$ with controls in $\mathcal{M}(\omega \times (0, T_{min}))$ are defined by transposition

Transposition solution

Given $y_0 \in L^2(-1,1)$, $T > 0$, and $u \in \mathcal{M}(\omega \times (0,T))$, $y \in L^1((-1,1) \times (0,T))$ is a solution of $(\mathcal{FH})$ defined by transposition if

$$\int_0^T \int_\omega p(x,t)\, du(x,t) = \langle y(\cdot,T), p_T \rangle - \int_{-1}^1 y_0(x) p(x,0)\, dx,$$

where, for every $p_T \in L^\infty(-1,1)$, the function $p \in L^\infty(Q)$ is the unique solution of the adjoint equation.

The existence of a unique transposition solution of $(\mathcal{FH})$ is a consequence of the **maximum principle** together with **duality** and **approximation** arguments.

J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Dunod, 1968

Minimal-time control

Denote $T_k := T_{min} + \frac{1}{k}$, $k \geq 1$. There exists a sequence of non-negative controls $\{u^{T_k}\}_{k\geq 1} \subset L^\infty(\omega \times (0, T_k))$ such that the corresponding solution y^k of $(\mathcal{FH})$ with $y^k(x,0) = y_0(x)$ a.e. in $(-1,1)$ satisfies $y^k(x, T_k) = \widehat{y}(x, T_k)$ a.e. in $(-1,1)$.

Extend these controls by $\widehat{u}$ on $(T_k, T_{min} + 1)$ to get a new sequence in the space $L^\infty(\omega \times (0, T_{min+1}))$.

Take $p_T > 0 \;\Rightarrow\; p(x,t) \geq \theta > 0$ for all $(x,t) \in (-1,1) \times (0, T_{min} + 1)$. Then,

$$\begin{aligned}
\theta \left\| u^{T_k} \right\|_{L^1(\omega\times(0,T_{min}+1))} &= \theta \int_0^{T_{min}+1} \int_\omega u^{T_k}\, dxdt \\
&\leq \int_0^{T_{min}+1} \int_{-1}^{1} p u^{T_k}\, dxdt \\
&= \langle y(\cdot, T), p_T \rangle - \int_{-1}^{1} y_0 p(\cdot, 0)\, dx \leq M.
\end{aligned}$$

Minimal-time control

$\{u^{T_k}\}_{k\geq 1}$ is **bounded** in $L^1(\omega\times(0,T_{min+1}))$, hence, it is bounded in $\mathcal{M}(\omega\times(0,T_{min+1}))$. Thus, extracting a sub-sequence, we have:

$$u^{T_k} \overset{*}{\rightharpoonup} \widetilde{u} \quad \textit{weakly} - * \textit{ in } \mathcal{M}(\omega\times(0,T_{min+1})) \textit{ as } k\to+\infty.$$

The limit control $\widetilde{u}$ satisfies the non-negativity constraint.

For any k large enough and $T_{min} < T_0 < T_{min+1}$, we have

$$\int_0^{T_0}\int_\omega p\, du^{T_k} = \langle \widehat{y}(\cdot,T_0), p_T\rangle - \int_{-1}^1 y_0 p(\cdot,0)\,dx.$$

p_T: first eigenfunction of $(-d_x^2)^s \Rightarrow p\in C([0,T];D((-d_x^2)^s)) \hookrightarrow C([0,T]\times[-1,1])$.

By definition of *weak** limit, letting $k\to+\infty$, we obtain

$$\int_0^{T_0}\int_\omega p d\widetilde{u} = \langle \widehat{y}(\cdot,T_0), p_T\rangle - \int_{-1}^1 y_0 p(\cdot,0)\,dx \quad\to\quad y(\cdot,T_0) = \widehat{y}(\cdot,T_0) \text{ a.e. } in(-1,1).$$

Taking the limit as $T_0\to T_{min}$ and using the fact that

$$|\widetilde{u}|(\omega\times(T_{min},T_0)) = |\hat{u}|(\omega\times(T_{min},T_0)) = 0, \quad \textit{as } T_0\to T_{min}$$

we deduce that $z(\cdot,T_{min}) = \widehat{z}(\cdot,T_{min})$ a.e. in $(-1,1)$.

Lower bounds for the minimal controllability time

What about lower bounds for the minimal controllability time T_{min}?

Lower bounds for the minimal controllability time

General approach for the case of the classical linear and semi-linear heat equations.

D. Pighin and E. Zuazua, *Controllability under positivity constraints of semilinear heat equations*, Math. Control. Relat. Fields, 2018

1. By a translation argument, we can consider the case zero initial datum. Then, from the definition of transposition solutions we have

$$\langle y(\cdot, T), p_T \rangle - \int_0^T \int_\omega p(x,t)\, du(x,t) = 0.$$

2. The idea is now to find $T_0 > 0$ and $p_T \in L^2(-1,1)$ such that the corresponding solution of the adjoint system satisfies

$$\begin{cases} p \geq 0, & \text{in } \omega \times (0, T_0), \\ \langle \widehat{y}(\cdot, T), p_T \rangle < 0, & \text{for all } T \in [0, T_0). \end{cases} \tag{10}$$

   Then, an explicit lower bound of T_{min} is obtained by analyzing sharply the conditions required for (10) to hold.

Lower bounds for the minimal controllability time

The choice of a suitable initial datum such that (10) holds is not obvious.

For linear and semi-linear heat equations with a boundary control, we can take

$$p_T = -\phi_1 + 2(1-\zeta)\phi_1 \quad \text{or} \quad p_T = -\alpha\phi_1 + \beta\phi_3,$$

where ϕ_1 and ϕ_3 are respectively the first and third eigenfunction of the Dirichlet Laplacian, α and β are suitable positive constants, and ζ is a cut-off function supported outside the control region.

With these choices, a lower estimate for T_{min} is obtained by employing the positivity of ϕ_1 and **the explicit knowledge of the eigenfunctions ϕ_1 and ϕ_3**.

These choices of p_T are not appropriate for the fractional case for at least two main reasons.

1. We cannot ensure that with such p_T the solution of the adjoint equation remains positive in ω.
2. For the eigenfunctions of the Dirichlet fractional Laplacian we do not have an explicit expression. Therefore, to perform explicit estimates is very difficult.

The exterior controllability problem

Exterior controllability of the fractional heat equation

Fractional heat equation with exterior control

$$\begin{cases} y_t + (-d_x^2)^s y = 0 & \text{in } (-1,1)\times(0,T) \\ y = g\chi_\omega & \text{in } (-1,1)^c \times (0,T) \\ y(\cdot,0) = y_0 & \text{in } (-1,1) \end{cases} \tag{11}$$

Controllability results - approximate controllability

Theorem

Let $\omega \subset \Omega^c$ be any nonempty and open subset of Ω^c and $s \in (0,1)$. For any $T > 0$, $y_0, y_T \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control function $g \in \mathcal{D}(\Omega^c \times (0,T))$ such that the unique solution y of (11) satisfies $\|y(\cdot,T) - y_T\|_{L^2(\Omega)} \leq \varepsilon$.

M. Warma, *Approximate controllability from the exterior of space-time fractional diffusive equations*, SICON, 2019.

PROOF: direct consequence of the following unique continuation principle.

Let $\omega \subset \Omega^c$ be an arbitrary nonempty open set. Let $\lambda > 0$, and let $\phi \in D((-\Delta)^s)$ satisfy

$$\begin{cases} (-\Delta)^s \phi = \lambda \phi, & \text{in } \Omega \\ \mathcal{N}_s \phi = 0, & \text{in } \Omega^c \end{cases}$$

Then $\phi = 0$ in $\mathbb{R}^N$.

Controllability results - L^2 null controllability

Theorem

Let $\omega \subset (-1,1)^c$ be any nonempty and open subset of $(-1,1)^c$. For any $T > 0$ and $y_0 \in L^2(-1,1)$, there exists a control function $g \in L^2(0,T;H^s((-1,1)^c))$ such that the unique solution y of (11) satisfies $y(\cdot,T) = 0$ a.e. in $(-1,1)$, if and only if $s \in (1/2,1)$.

M. Warma and S. Zamorano, *Null controllability from the exterior of a one-dimensional nonlocal heat equation*, Control Cybern., 2020.

PROOF: direct consequence of the following observability inequality

$$\|p(\cdot,0)\|^2_{L^2(-1,1)} \leq \mathcal{C}\int_0^T\int_\omega |\mathcal{N}_s p|^2\,dxdt$$

that holds if and only if $s \geq 1/2$ and can be proven in the same way as the interior controllability case.

Controllability results - L^∞ null controllability

Theorem

Let $\omega \subset (-1,1)^c$ be any nonempty and open subset of $(-1,1)^c$. For any $T > 0$ and $y_0 \in L^2(-1,1)$, there exists a control function $g \in L^\infty(\omega \times (0,T))$ such that the unique solution y of (11) satisfies $y(\cdot,T) = 0$ a.e. in $(-1,1)$, if and only if $s \in (1/2,1)$.

H. Antil, U. Biccari, R. Ponce, M. Warma and S. Zamorano, *Controllability properties from the exterior under positivity constraints for a 1-d fractional heat equation*, 2020.

PROOF: direct consequence of the following observability inequality

$$\|p(\cdot,0)\|^2_{L^2(-1,1)} \leq \mathcal{C}\left(\int_0^T\int_\omega |\mathcal{N}_s p|\,dxdt\right)^2$$

that holds if and only if $s \geq 1/2$ and can be proven in the same way as the interior controllability case.

Constrained controllability

Theorem

Let $s > 1/2$, $y_0 \in L^2(-1,1)$ and let $\widehat{y}$ be a positive trajectory, i.e., a solution of (11) with initial datum $0 < \widehat{y}_0 \in L^2(-1,1)$ and exterior datum $\widehat{g} \in L^\infty(\omega \times (0,T))$. Assume that there exists $\nu > 0$ such that $\widehat{g} \geq \nu$ a.e in $\omega \times (0,T)$. Then, the following assertions hold.

1. There exist $T > 0$ and a non-negative control $g \in L^\infty(\omega \times (0,T))$ such that the corresponding solution y of (11) satisfies $y(x,T) = \widehat{y}(x,T)$ a.e. in $(-1,1)$. Moreover, if $y_0 \geq 0$, $y(x,t) \geq 0$ a.e. in $(-1,1) \times (0,T)$.

2. Define the minimal controllability time by

$$T_{min}(y_0, \widehat{y}) := \inf \Big\{ T > 0 : \exists \;\; 0 \leq g \in L^\infty(\omega \times (0,T)) \; s.\ t. \\ y(\cdot, 0) = y_0 \; and \; y(\cdot, T) = \widehat{y}(\cdot, T) \Big\}.$$

   For $T = T_{min}$, there exists a non-negative control $g \in \mathcal{M}(\omega \times (0, T_{min}))$, the space of Radon measures on $\omega \times (0, T_{min})$, such that the corresponding solution of (11) satisfies $y(x,T) = \widehat{y}(x,T)$ a.e. in $(-1,1)$.

H. Antil, U. Biccari, R. Ponce, M. Warma and S. Zamorano, *Controllability properties from the exterior under positivity constraints for a 1-d fractional heat equation*, 2020.

THANK YOU FOR YOUR ATTENTION!

Funding

- European Research Council (ERC): grant agreements NO: 694126-DyCon and No.765579-ConFlex.
- MINECO (Spain): Grant PID2020-112617GB-C22 KILEARN
- Alexander von Humboldt-Professorship program
- DFG (Germany): Transregio 154 Project "Mathematical Modelling,Simulation and Optimization Using the Example of Gas Networks"
- COST Action grant CA18232, "Mathematical models for interacting dynamics on networks".

Fundación Deusto Fundazioa

Friedrich-Alexander-Universität Department Data Science | DDS

European Research Council
Established by the European Commission

European Commission

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Unterstützt von / Supported by
Alexander von Humboldt Stiftung / Foundation

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