CONTROL AND OPTIMIZATION FOR NON-LOCAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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PART IV: numerical approximation of differential equations and numerical control

LECTURE 14: numerical control of fractional PDE

Finite Element approximation of the fractional Laplacian

FE approximation of $(-\Delta)^s$

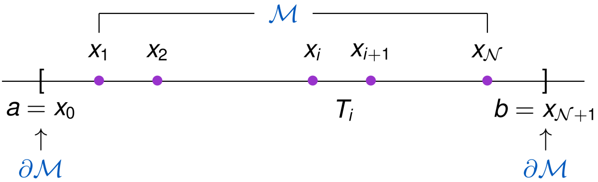
$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c \end{cases} \tag{1}$$

Let $\mathcal{M} = \{T_i\}_{i=1}^{\mathcal{N}}$ be a partition of the domain Ω (i.e. $\overline{\Omega} = \bigcup_{i=1}^{\mathcal{N}} T_i$)

- In dimension $N = 1$, we consider a uniform partition of $\Omega = (-1, 1)$

  $$-1 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{\mathcal{N}+1} = 1,$$

  with $x_{i+1} = x_i + h$, $i = 0, \ldots \mathcal{N}$, and we denote $T_i := [x_i, x_{i+1}]$. Moreover, we indicate with $\partial\mathfrak{M} = \{x_0, x_{\mathcal{N}+1}\}$ the boundary nodes.



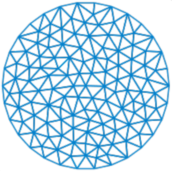
Let $\mathcal{M} = \{T_i\}_{i=1}^{\mathcal{N}}$ be a partition of the domain Ω (i.e. $\overline{\Omega} = \bigcup_{i=1}^{\mathcal{N}} T_i$)

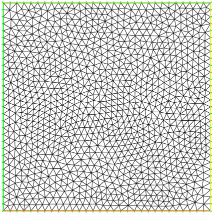
- In dimension $N = 2$, we consider triangular elements T_i, $i = 1, \ldots, \mathcal{N}$. We indicate with h_i the diameter of the element T_i and with ρ_i its inner radius, i.e. the diameter of the largest ball contained in T_i. We define

$$h = \max_{i \in \{1,\ldots,\mathcal{N}\}} h_i.$$

Moreover, we require that the triangulation satisfies the **regularity** and **local uniformity** conditions:

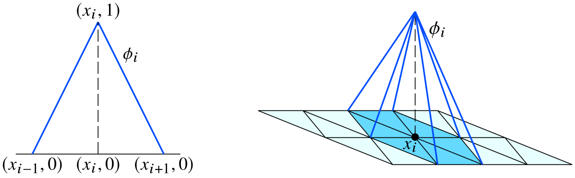
there exists $\sigma > 0$ s.t. $h_i \leq \sigma \rho_i$ for all $i = 1, \ldots, \mathcal{N}$,
there exists $\kappa > 0$ s.t. $h_i \leq \kappa h_j$ for all $i,j = 1, \ldots, \mathcal{N}$, $T_i \cap T_j = \emptyset$.





$$V_h := \left\{ v \in H_0^s(\Omega) \; : \; v|_{T_i} \in \mathcal{P}^1 \right\}.$$

We choose the usual basis functions $\{\phi_i\}_{i=1}^{\mathcal{N}}$ fulfilling $\phi_i(x_j) = \delta_{i,j}$, for all $i, j \in \{1, \ldots, \mathcal{N}\}$.



FE approximation of $(-\Delta)^s$

STEP 1: by decomposing

$$u(x) = \sum_{j=1}^{\mathcal{N}} u_j \phi_j(x), \quad f(x) = \sum_{j=1}^{\mathcal{N}} f_j \phi_j(x), \quad f_j = \int_\Omega f(x)\phi_j(x)\,dx$$

and taking $v = \phi_i$, we obtain the linear system $A_h \mathbf{u} = M_h \mathbf{f}$, where

- $\mathbf{u} = (u_1, \ldots, u_{\mathcal{N}}) \in \mathbb{R}^{\mathcal{N}}$ is an unknown vector.
- $\mathbf{f} = (f_1, \ldots, f_{\mathcal{N}}) \in \mathbb{R}^{\mathcal{N}}$.
- The **stiffness** matrix $A_h \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ has components

$$a_{i,j} = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x-y|^{N+2s}}\, dxdy$$

  for all $i,j \in \{1, \ldots, \mathcal{N}\}, \ N = 1, 2.$

- The **mass** matrix $M_h \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ has components

$$m_{i,j} = \int_\Omega \phi_i(x)\phi_j(x)\,dx, \quad \text{for all } i,j \in \{1, \ldots, \mathcal{N}\}.$$

STEP 2: the unknown vector $\mathbf{u}$ is obtained by solving $\mathbf{u} = A_h^{-1} M_h \mathbf{f}$.

STEP 3: once the vector $\mathbf{u}$ is known, the solution of the Poisson equation is approximated by

$$u(x) = \sum_{j=1}^{\mathcal{N}} u_j \phi_j(x).$$

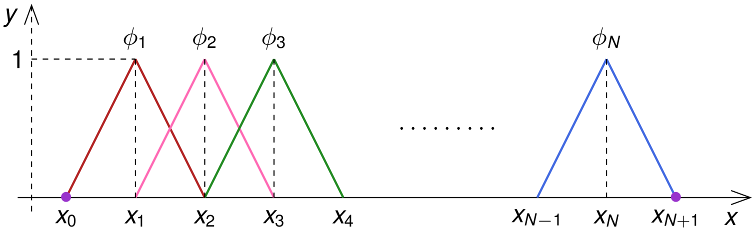
Stiffness matrix - $N = 1$

Remarks

A_h is **symmetric**. Therefore, in its construction, we only need to compute the values $a_{i,j}$ with $j \geq i$.

Due to the non-local nature of the problem, the matrix A_h is **full**.

The basis functions satisfy the zero Dirichlet B.C. This is important in the case $s > 1/2$.



Stiffness matrix - $N = 1$

In space dimension $N = 1$, the entries of the stiffness matrix A_h can be computed explicitly.

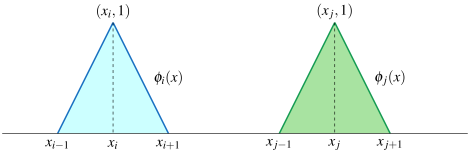
The matrix has the following structure

$$
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots\cdots & \cdots\cdots & a_{1,N} \\
 & a_{2,2} & a_{2,3} & a_{2,4} & \cdots\cdots & a_{2,N} \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & \ddots & \ddots & a_{N-2,N} \\
 & & & & a_{N-1,N-1} & a_{N-1,N} \\
 & & & & & a_{N,N}
\end{pmatrix}
$$

Computations for $j \geq i + 2$

In this case we have $supp(\phi_i) \cap supp(\phi_j) = \emptyset$. Hence, we only have to compute the integral

$$a_{i,j} = -2 \int_{x_{j-1}}^{x_{j+1}} \int_{x_{i-1}}^{x_{i+1}} \frac{\phi_i(x)\phi_j(y)}{|x-y|^{1+2s}} \, dxdy.$$



In this case we have $supp(\phi_i) \cap supp(\phi_j) = \emptyset$. Hence, we only have to compute the integral

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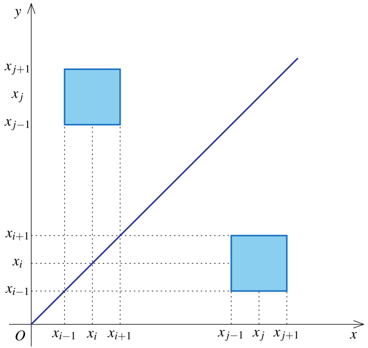
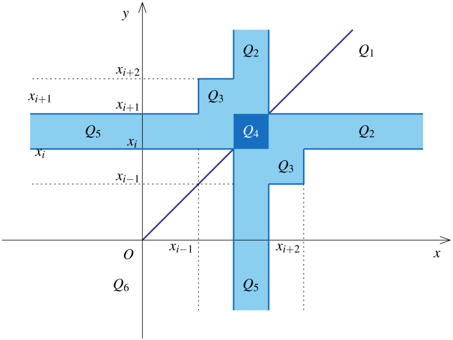


Computations for $j = i + 1$

This is the most cumbersome case, since it is the one with the most interactions between the basis functions.

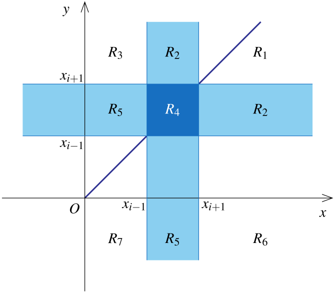
$$a_{i,j} = \sum_{\ell=1}^{6} Q_\ell.$$



Computations for $j = i$

In this case, we have

$$a_{i,j} = \sum_{\ell=1}^{7} R_\ell.$$



Entries of the stiffness matrix A_h - $s \neq 1/2$

$$a_{i,j} = -h^{1-2s} \begin{cases} \dfrac{4(k+1)^{3-2s} + 4(k-1)^{3-2s}}{2s(1-2s)(1-s)(3-2s)} \\[2ex] -\dfrac{6k^{3-2s} + (k+2)^{3-2s} + (k-2)^{3-2s}}{2s(1-2s)(1-s)(3-2s)}, & k = j-i,\ k \geq 2 \\[2ex] \dfrac{3^{3-2s} - 2^{5-2s} + 7}{2s(1-2s)(1-s)(3-2s)}, & j = i+1 \\[2ex] \dfrac{2^{3-2s} - 4}{s(1-2s)(1-s)(3-2s)}, & j = i. \end{cases}$$

Entries of the stiffness matrix A_h - $s = 1/2$

$$
a_{i,j} = \begin{cases} -4(k+1)^2\log(k+1) - 4(k-1)^2\log(k-1) & \\ \quad +6k^2\log(k) + (k+2)^2\log(k+2) & \\ \quad +(k-2)^2\log(k-2), & k = j-i,\ k > 2 \\ 56\ln(2) - 36\ln(3), & j = i+2. \\ 9\ln 3 - 16\ln 2, & j = i+1 \\ 8\ln 2, & j = i. \end{cases}
$$

Final remarks on the approximation A_h

Remarks

Each entry $a_{i,j}$ of the matrix only depend on i, j, s and h.

The matrix $\mathcal{A}_h^s$ has the structure of a N-diagonal matrix. This is analogous to the tridiagonal matrix approximating the classical Laplace operator

$$\mathcal{A}_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$A_h \to \mathcal{A}_h$ as $s \to 1^-$, which is in accordance to the behavior of the continuous operator.

Stiffness matrix - $N = 2$

In this case, we have

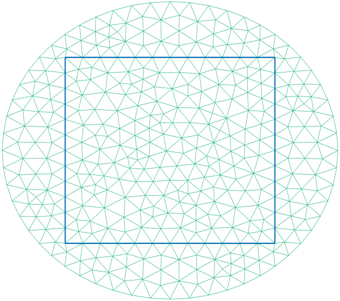
$$a_{i,j} = \frac{C_{2,s}}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x-y|^{2+2s}}\, dxdy.$$

Since $\phi_i = 0$ in Ω^c, the integral on $\mathbb{R}^2 \times \mathbb{R}^2$ is reduced to integrals on the set $(\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega)$ and, taking into account that the interactions in $\Omega \times \Omega^c$ and $\Omega^c \times \Omega$ are symmetric with respect to x and y, we get

$$\begin{aligned} a_{i,j} = \frac{C_{2,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x-y|^{2+2s}}\, dxdy \\ + C_{2,s} \int_{\Omega} \int_{\Omega^c} \frac{\phi_i(x)\phi_j(x)}{|x-y|^{2+2s}}\, dxdy. \end{aligned} \tag{2}$$

Stiffness matrix - $N = 2$

Notice that the second integral in (2) has to be computed over the unbounded domain Ω^c. To do that, it is convenient to introduce a ball B containing Ω, since this allows to employ polar coordinates and exploit symmetry properties.



Stiffness matrix - $N = 2$

$\mathcal{N}_B$: number of elements on the triangulation of B. Then, recalling (2), the coefficients $a_{i,j}$ are given by the expression

$$a_{i,j} = \frac{C_{2,s}}{2} \sum_{\ell=1}^{\mathcal{N}_B} \left(\sum_{m=1}^{\mathcal{N}_B} \mathcal{I}_{\ell,m}^{i,j} + 2\mathcal{J}_{\ell}^{i,j} \right)$$

$$\mathcal{I}_{\ell,m}^{i,j} := \int_{T_\ell} \int_{T_m} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x-y|^{2+2s}} \, dxdy$$

$$\mathcal{J}_{\ell}^{i,j} := \int_{T_\ell} \int_{B^c} \frac{\phi_i(x)\phi_j(x)}{|x-y|^{2+2s}} \, dxdy$$

The computations of the above integrals are challenging for different reasons: they involve a singular integrand if $\overline{T}_\ell \cap \overline{T}_m \neq \emptyset$, or they needs to be calculated on an unbounded domain.

G. Acosta, F. M. Bersetche and J. P. Borthagaray, *A short FE implementation for a 2d homogeneous Dirichlet problem of a fractional Laplacian*, Comput. Math. Appl., 2017.

G. Acosta and J. P. Borthagaray, *A fractional Laplace equation: Regularity of solutions and finite element approximations*, SIAM J. Numer. Anal., 2017.

Error analysis

Theorem

Let f satisfy the following regularity assumptions:

$$\begin{array}{ll} f \in C^{\frac{1}{2}-s}(\Omega), & \text{if } 0 < s < 1/2, \\ f \in L^{\infty}(\Omega), & \text{if } s = 1/2, \\ f \in C^{\beta}(\Omega) \text{ for some } \beta > 0, & \text{if } 1/2 < s < 1, \end{array}$$

where $C^{s-\frac{1}{2}}(\Omega)$ and $C^{\beta}(\Omega)$ denote the standard Hölder spaces of order $s - 1/2$ and β, respectively. Then, for the solution u of (1) and its FE approximation u_h on a uniform mesh with size h, we have the following a priori estimates

$$\|u - u_h\|_{H_0^s(\Omega)} \leq \frac{C(s,\sigma)}{\varepsilon} h^{\frac{1}{2}-\varepsilon} \, \|f\|_{C^{\frac{1}{2}-s}(\Omega)}, \quad \forall \varepsilon > 0, \text{ if } \ s < 1/2,$$

$$\|u - u_h\|_{H_0^s(\Omega)} \leq \frac{C(\sigma)}{\varepsilon} h^{\frac{1}{2}-\varepsilon} \, \|f\|_{L^{\infty}(\Omega)}, \quad \forall \varepsilon > 0, \text{ if } \ s = 1/2$$

$$\|u - u_h\|_{H_0^s(\Omega)} \leq \frac{C(s,\beta,\sigma)}{\sqrt{\varepsilon}(2s-1)} h^{\frac{1}{2}-\varepsilon} \, \|f\|_{C^{\beta}(\Omega)}, \quad \forall \varepsilon > 0, \text{ if } \ s > 1/2, \ \beta > 0$$

where C is a positive constant not depending on h.

Error analysis

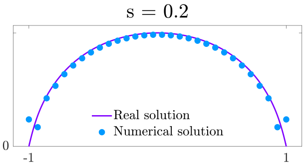
Theorem

Let $\alpha := \min\{s, 1/2 - \delta\}$, with $\delta > 0$ arbitrary small. If $f \in L^2(\Omega)$ and u is the solution to (1), for its FE approximation on a uniform mesh with size h it holds that

$$\|u - u_h\|_{L^2(\Omega)} \leq C(s, \alpha) h^{2\alpha} \|f\|_{L^2(\Omega)}.$$

To test the efficiency of the FE scheme we have proposed, we consider the Dirichlet problem (1) on $\Omega = (-1, 1)$ with $f = 1$, whose explicit solution is given by

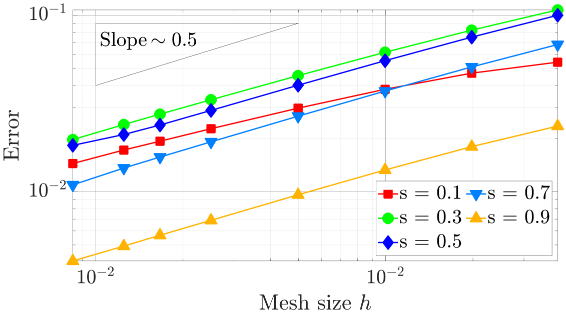
$$u(x) = \gamma_s \left(1 - x^2\right)^s \cdot \chi_{(-1,1)}, \qquad \gamma_s = \frac{2^{-2s}\sqrt{\pi}}{\Gamma\left(\frac{1+2s}{2}\right)\Gamma(1+s)}.$$



Numerical experiments - $N = 1$

One can notice that for large $s \geq 1/2$ the FE scheme provides a good approximation. On the other hand, when $s < 1/2$, the computed solution is to a certain extent different from the exact one, as there is a discrepancy approaching the boundary.

Despite this fact, we can that, for all $s \in (0,1)$ the H^s error of our FE approximations decreases with h at a rate $\|u - u_h\|_{H_0^s(-1,1)} \sim \sqrt{h}$, which is the expected one according to the previous theorem.



Numerical control for the fractional heat equation

Fractional heat equation - numerical control

$$\begin{cases} y_t + (-\Delta)^s y = f\chi_\omega, & \text{in } (-1,1)\times(0,T) \\ y = 0, & \text{in } (-1,1)^c \times (0,T) \\ y(0) = y_0, & \text{in } (-1,1) \end{cases} \tag{3}$$

We apply the penalized Hilbert uniqueness method to compute the control function f and discuss the controllability properties of the system analyzing the behavior of the optimization procedure.

U. Biccari, M. Warma and E. Zuazua, *Control and numerical approximation of fractional diffusion equations*, Handbook of Numerical Analysis, (2022).

Fully-discrete dynamics

Let us introduce the fully-discrete version of (3). Given a uniform $\mathcal{N}$-points mesh $\mathfrak{M}$ of size h on $(-1,1)$ and any integer $M > 0$, we set $\delta t = T/M$ and we approximate (3) through an implicit Euler method.

$$\begin{cases} \mathcal{M}_h \dfrac{y_h^{m+1} - y_h^m}{\delta t} + \mathcal{A}_h y_h^{m+1} = \mathcal{B}_h u_h^{m+1}, & \text{for all } m \in \{1, \dots, M-1\} \\ y_h^1 = y_{0,h}, \end{cases} \tag{4}$$

- $y_{0,h} \in \mathbb{R}^{\mathcal{N}}$: the projection of the initial datum $y_0 \in L^2(-1,1)$ on the mesh $\mathfrak{M}$.
- $\mathcal{A}_h$: stiffness matrix computed in FE.
- $\mathcal{M}_h$: mass matrix.
- The matrix $\mathcal{B}_h$ has entries

$$b_{i,j} = \int_\omega \phi_i(x)\phi_j(x)\,dx, \quad i,j = 1, \dots, \mathcal{N}.$$

Fully-discrete dynamics

In (4), $u_h = (u_h^m)_{m=1}^M \in \mathbb{R}^{\mathcal{N}\times M}$ is a fully-discrete control function, whose cost is given by the discrete $L^2((-1,1)\times(0,T))$-norm

$$\|u_h\|_{L^2_{h,\delta t}} := \left(\sum_{m=1}^M \delta t |u_h^m|^2_{L^2_{h,\mathcal{M}_h}}\right)^{1/2}$$

and where $|\cdot|_{L^2_{h,\mathcal{M}_h}}$ is the norm associated with the L^2-inner product on $\mathfrak{M}$ and the mass matrix $\mathcal{M}_h$

for all $v = (v_i)_{i=1}^N \in \mathbb{R}^N$ and $w = (w_i)_{i=1}^N \in \mathbb{R}^N$

$$\langle v, w\rangle_{L^2_{h,\mathcal{M}_h}} = \langle \mathcal{M}_h v, w\rangle_{L^2_h} = h\sum_{i=1}^N (\mathcal{M}_h v)_i w_i \quad \longrightarrow \quad |v|^2_{L^2_{h,\mathcal{M}_h}} = \langle v, v\rangle_{L^2_{h,\mathcal{M}_h}}$$

With this above notation, given some penalization parameter $\beta > 0$ we can introduce the fully-discrete primal and dual functionals

$$
\begin{aligned}
F_{\beta,h}(u_h) &= \frac{1}{2}\left\|u_h\right\|^2_{L^2_{h,\delta t}} + \frac{1}{2\beta}|y_h^M|^2_{L^2_{h,\mathcal{M}_h}} \\
J_{\beta,h}(p_h^M) &= \frac{1}{2}\left\|\mathcal{B}_h p_h\right\|^2_{L^2_{h,\delta t}} + \frac{\beta}{2}|p_h^M|^2_{L^2_{h,\mathcal{M}_h}} + \left\langle p_h^M, e^{\mathcal{A}_h T} y_{0,h}\right\rangle_{L^2_{h,\mathcal{M}_h}}
\end{aligned}
\tag{5}
$$

with $p_h = (p_h^n)_{n=1}^M \mathbb{R}^{N\times M}$ solution to the adjoint system

$$
\begin{cases}
\mathcal{M}_h \dfrac{p_h^m - p_h^{m+1}}{\delta t} + \mathcal{A}_h p_h^m = 0, \quad \text{for all } m \in \{1,\dots,M-1\} \\
p_h^M = p_{T,h},
\end{cases}
\tag{6}
$$

where $p_{T,h} \in \mathbb{R}^N$ is the projection of $p_T \in L^2(\Omega)$ on the mesh $\mathfrak{M}$.

Analyzing the behavior of the penalized problem with respect to the parameter $\beta > 0$, we can discuss controllability properties for our system.

Theorem

Let $\hat{p}_{T,h}$ be the unique minimizer of $J_{\beta,h}$ and $\hat{u}_h$ be the corresponding control. The system is

- **NULL CONTROLLABLE** at time T if and only if the **optimal energy**

$$\mathcal{E}_{y_0,h} := F_{\beta,h}(\hat{u}_h)$$

  is bounded. In this case

$$\|\hat{u}_h\|_{L^2_{h,\delta t}} \leq \sqrt{\mathcal{E}_{y_0,h}} \quad \text{and} \quad |y_h^M|_{L^2_{h,\mathcal{M}_h}} \leq \sqrt{\beta \mathcal{E}_{y_0,h}}$$

- **APPROXIMATELY CONTROLLABLE** at time T if and only if $|y_h^M|_{L^2_{h,\mathcal{M}_h}} \to 0$ as $\beta \to 0^+$.

Practical considerations

We apply the conjugate gradient method to minimize the functional $J_{\beta,h}$ and discuss the control properties of (4) from the viewpoint of the previous theorem.

In the context of fully-discrete problems as (4), this may be a delicate issue.

In general, we cannot expect for a given bounded family of initial data that the fully-discrete controls are uniformly bounded when h, δt and β tend to zero independently. Instead, **we expect to obtain uniform bounds by taking $\beta = \phi(h)$ that tends to zero in connection with the mesh size not too fast**. It is then crucial to choose properly the penalization parameter β.

A reasonable practical rule is to take $\beta = \phi(h) \sim h^{2p}$, where p is the order of accuracy in space of the numerical method employed for the discretization of $(-\Delta)^s$.

The effectiveness of this choice has only been demonstrated numerically and is not supported by rigorous mathematical results.

F. Boyer, *On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems*, ESAIM: Proc., 2013.

Practical considerations

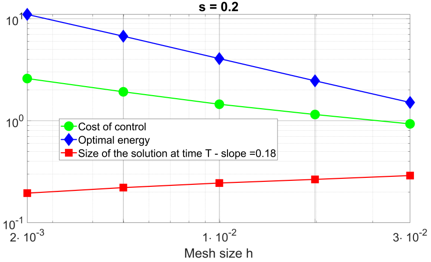
To select the correct value for p, let us recall that the solution y to (3) with $y_0 \in L^2(\Omega)$ and $u \in L^2(\omega \times (0,T))$ belongs to $L^2(0,T;H_0^s(\Omega)) \cap C([0,T];L^2(\Omega))$. In particular, we have that $y(\cdot,T) \in L^2(\Omega)$.

Therefore, we shall choose the value of p as the convergence rate in the L^2-norm for the approximation of the elliptic problem. Recalling the convergence theorems for the FE approximation of $(-\Delta)^s$, the appropriate value of p that we shall employ is

$$p = \begin{cases} 2s, & \text{for } s < \frac{1}{2} \\ 1-2\delta, & \text{for } s \geq \frac{1}{2} \end{cases} \quad \longrightarrow \quad \beta = h^{2p} = \begin{cases} h^{4s}, & \text{for } s < \frac{1}{2} \\ h^{2-4\delta}, & \text{for } s \geq \frac{1}{2}, \end{cases}$$

with $\delta > 0$ arbitrary small.

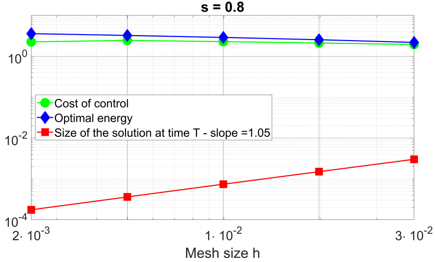
Numerical experiments - $s = 0.2$



We observe that the discrete L^2 norm of $\hat{y}^M$ tends to zero as $h \to 0$, **confirming computationally the approximate controllability of the equation**.

On the other hand, we see that the cost of the control and the optimal energy increase as $h \to 0$. Hence, **numerical evidence indicates that the null controllability is not fulfilled**.

Numerical experiments - $s = 0.8$

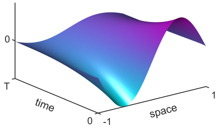


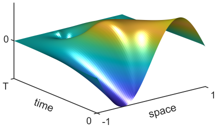
We can observe that this time the control cost and the optimal energy remain bounded as $h \to 0$. Furthermore, we also see that

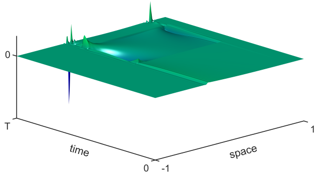
$$|\hat{y}^M|_{L^2_{h,\mathcal{M}_h}} \sim h = \sqrt{\beta},$$

which is the expected convergence rate for the discrete L^2 norm of $y(\cdot, T)$. All these facts **provide numerical evidence indicating that the null controllability is fulfilled when $s = 0.8$**.

Numerical experiments - $s = 0.8$



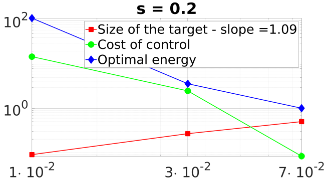


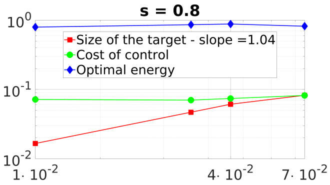


The control is almost inactive for a large part of the time interval, and then experiences large oscillations in the proximity of the final time.

This fact, related with the characterization of the control as restrictions of solutions of the adjoint system, is in accordance with the **lazy behavior** observed by Glowinski and Lions of controls for the local heat equation which, at the very beginning, leave the solution evolve under the dissipative effect of the heat semi-group and, only when approaching the final controllability time, inject energy into the system in order to match the desired configuration.

Two-dimensional problem





We observe analogous behavior as in the one-dimensional case.

$s = 0.2$: the discrete L^2 norm of the solution at time T decreases with h, thus confirming the approximate controllability. However, the control cost and the optimal energy both increase as $h \to 0$, thus suggesting the failure of the null controllability property.

$s = 0.8$: the control cost and the optimal energy remain bounded as $h \to 0$, while the discrete L^2 norm of $y(T)$ decreases with rate h. This suggests that the fractional heat equation in 2-D is null-controllable at time T.

Numerical exterior control for the fractional heat equation

$$
\begin{cases}
y_t + (-\Delta)^s y = 0, & \text{in } (-1,1) \times (0,T) \\
y = g\chi_{\mathcal{O}}, & \text{in } (-1,1)^c \times (0,T) \\
y(0) = y_0, & \text{in } (-1,1)
\end{cases}
\tag{7}
$$

We apply the penalized Hilbert uniqueness method to compute the control g by minimizing the functional

$$
F_\beta^{ext}(g) := \frac{1}{2}\int_0^T \|g\|^2_{H^s(\mathcal{O})}\,dt + \frac{1}{2\beta}\,\|y(\cdot,T)\|^2_{L^2(-1,1)}\,.
$$

and discuss the controllability properties of the system analyzing the behavior of the optimization procedure.

Discretization of the dynamics

To discretize the dynamics, we first approximate (7) through the following exterior Robin problem

$$\begin{cases} y^n_t + (-\Delta)^s y^n = 0 & \text{in } (-1,1)\times(0,T), \\ \mathcal{N}_s y^n + n\kappa y^n = n\kappa g & \text{in } (-1,1)^c\times(0,T), \\ y^n(\cdot,0) = y_0 & \text{in } (-1,1), \end{cases} \tag{8}$$

where $n \in \mathbb{N}$ is a fixed natural number and $\kappa \in L^1((-1,1)^c) \cap L^\infty((-1,1)^c)$ is a non-negative function.

H. Antil, R. Khatri and M. Warma, *External optimal control of nonlocal PDEs*, Inv. Problems, 2019.

H. Antil, D. Verma and M. Warma, *External optimal control of fractional parabolic PDEs*, ESAIM: Control Optim, Calc. Var., 2020.

Theorem

Let $y_0 \in L^2(-1,1)$, $g \in L^2((0,T);H^s(-1,1)^c)$, $\kappa \in L^1((-1,1)^c) \cap L^\infty((-1,1)^c)$ non-negative and

$$y^n \in L^2((0,T);H^s_\kappa(-1,1)) \cap H^1((0,T);H^{-s}_\kappa(-1,1))$$

be the weak solution of (8). Let $y \in L^2((0,T);H^s(\mathbb{R}))$ be the weak solution of (7). There is a constant $\mathcal{C} > 0$, independent of n, such that

$$\left\| y - y^n \right\|_{L^2(\mathbb{R}\times(0,T))} \leq \frac{\mathcal{C}}{n} \left\| y \right\|_{L^2((0,T);H^s(\mathbb{R}))} . \tag{9}$$

In particular, y^n converges strongly to y in $L^2((-1,1) \times (0,T))$ as $n \to +\infty$.

Optimal control problem

The controllability of (8) can be characterized through a dual argument.

For any $p^n_T \in L^2(-1,1)$ we denote with p^n the solution of the following adjoint problem with Robin exterior conditions

$$\begin{cases} -p^n_t + (-\Delta)^s p^n = 0 & \text{in } (-1,1)\times(0,T), \\ \mathcal{N}_s p^n + n\kappa p^n = 0 & \text{in } (-1,1)^c\times(0,T), \\ p^n(\cdot,T) = p^n_T & \text{in } (-1,1). \end{cases}$$

Then

$$y^n(\cdot,T) = 0 \text{ a.e. in } (-1,1) \;\text{ if and only if }\; \int_{-1}^{1} y_0 p^n(\cdot,0)\,dx + n\int_0^T\int_{\mathcal{O}} p^n\kappa g\,dx = 0.$$

Hence, the controllability of (8), is equivalent to the observability inequality

$$\|p^n(\cdot,0)\|^2_{L^2(-1,1)} \leq C\int_0^T \|p^n(t)\|^2_{L^2(\mathcal{O})}\,dt,$$

and the exterior control can be obtained from the following optimal control problem:

$$g_\beta = \operatorname*{argmin}_{g\in L^2(\mathcal{O}\times(0,T))} G^{ext}_\beta(g)$$

$$G^{ext}_\beta(g) := \frac{1}{2}\int_0^T \|g\|^2_{L^2(\mathcal{O})}\,dt + \frac{1}{2\beta}\|y^n(\cdot,T)\|^2_{L^2(-1,1)}.$$

To approximate (8), we consider the interval $\mathcal{I} = (-2,2) \supset (-1,1)$ and assume that the control function g is supported in $\mathcal{O} \subset ((-2,2) \setminus (-1,1))$.

Notice that, in this case, the regularity required in the previous approximation theorem for the function κ, namely $\kappa \in L^1((-2,2) \setminus (-1,1)) \cap L^\infty((-2,2) \setminus (-1,1))$, simply reduces to $\kappa \in L^\infty((-2,2) \setminus (-1,1))$ and is fulfilled by considering κ to be constant. For simplicity, we take $\kappa = 1$.

$$\begin{cases} y_t^n + (-\Delta)^s y^n = 0 & \text{in } (-1,1) \times (0,T), \\ \mathcal{N}_s y^n + n y^n = n g \chi_{\mathcal{O} \times (0,T)} & \text{in } ((-2,2) \setminus (-1,1)) \times (0,T), \\ y^n(\cdot, 0) = y_0 & \text{in } (-1,1). \end{cases} \tag{10}$$

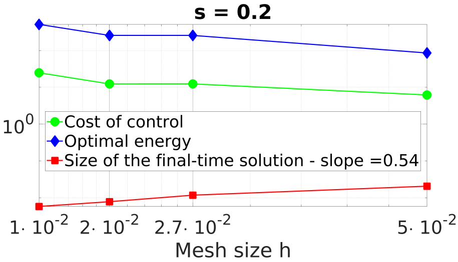
To discretize (10):

- We use FE scheme in space based on the variational formulation

$$n \int_{\mathcal{O}} g v \, dxdt = \int_{-1}^{1} y_t^n v \, dxdt + \dashv(y^n, v) + n \int_{(-1,1)^c} y^n v \, dxdt, \quad \text{for all } v \in H_0^s(-1,1).$$

- We use implicit Euler in time.

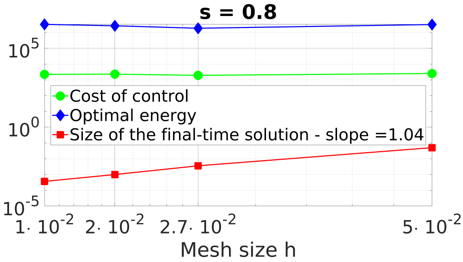
Numerical experiments - $s = 0.2$



We observe that the discrete L^2 norm of the solution at time T tends to zero as $h \to 0$, **confirming computationally the approximate controllability of the equation**.

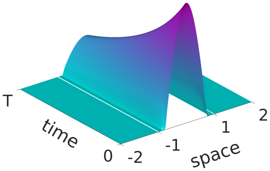
On the other hand, we see that the cost of the control and the optimal energy increase as $h \to 0$. Hence, **numerical evidence indicates that the null controllability is not fulfilled**.

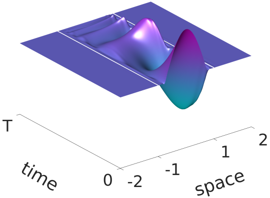
Numerical experiments - $s = 0.8$

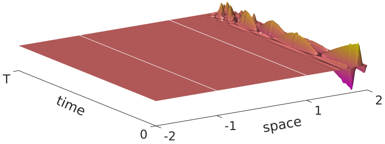


We observe that the control cost and the optimal energy remain bounded as $h \to 0$. Furthermore, we also see that the discrete L^2 norm of the solution at time T behaves as h, which is the expected convergence rate. All these facts **provide numerical evidence indicating that the null controllability is fulfilled when $s = 0.8$**.

Numerical experiments - $s = 0.8$







Acting from the exterior of the domain, the control g is capable to steer the state y to zero at time T.

Alternative discretization of the dynamics

The exterior control problem for the fractional heat equation can be discretized without introducing the Robin approximation.

The exterior control can be computed minimizing a dual functional in the spirit of Fenchel and Rockafellar.

U. Biccari, S. Zamorano and E. Zuazua, *Adjoint formulation for the fractional exterior control problem*, in preparation.

FE for the elliptic exterior problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = g & \text{in } \Omega^c \end{cases} \tag{11}$$

Define the bilinear and linear forms

$$a : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R} \qquad a(u,v) := \frac{\mathcal{C}_{N,s}}{2} \int_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$

$$\mathcal{Q} := (\Omega \times \mathbb{R}) \cup (\mathbb{R} \times \Omega)$$

$$b : H^s(\mathbb{R}^N) \times H^{-s}(\Omega^c) \to \mathbb{R} \qquad b(u,\lambda) := \int_{\Omega^c} u(x)\lambda(x)\, dx$$

$$F : H^s(\mathbb{R}^N) \to \mathbb{R} \qquad F(u) := \int_{\Omega} f(x)u(x)\, dx$$

$$G : H^{-s}(\Omega^c) \to \mathbb{R} \qquad G(\lambda) := \int_{\Omega^c} g(x)\lambda(x)\, dx$$

We have the following variational formulation for (11): find $(u,\lambda) \in H^s(\mathbb{R}^N) \times H^{-s}(\Omega^c)$ such that

$$a(u,v) - b(v,\lambda) = F(v) \qquad \text{for all } v \in H^s(\mathbb{R}^N)$$

$$b(u,\mu) = G(\mu) \qquad \text{for all } \mu \in H^{-s}(\Omega^c)$$

G. Acosta, J. P. Borthagaray and N. Heuer, *Finite element approximations of the nonhomogeneous fractional Dirichlet problem*, IMA J. Numer. Anal., 2019.

U. Biccari, S. Zamorano and E. Zuazua, *Adjoint formulation for the fractional exterior control problem*, in preparation.

Numerical constrained control for the fractional heat equation

Theorem

Let $s > 1/2$, $y_0 \in L^2(-1,1)$ and let $\widehat{y}$ be a positive trajectory, i.e., a solution of (3) with initial datum $0 < \widehat{y}_0 \in L^2(-1,1)$ and right hand side $\widehat{f} \in L^\infty(\omega \times (0,T))$. Assume that there exists $\nu > 0$ such that $\widehat{f} \geq \nu$ a.e in $\omega \times (0,T)$. Then, the following assertions hold.

1. There exist $T > 0$ and a non-negative control $f \in L^\infty(\omega \times (0,T))$ such that the corresponding solution y of (3) satisfies $y(x,T) = \widehat{y}(x,T)$ a.e. in $(-1,1)$. Moreover, if $y_0 \geq 0$, we also have $y(x,t) \geq 0$ for every $(x,t) \in (-1,1) \times (0,T)$.
2. Define the minimal controllability time by

$$T_{min}(y_0, \widehat{y}) := \inf \Big\{ T > 0 : \exists \;\; 0 \leq f \in L^\infty(\omega \times (0,T)) \; s.\; t. \\ y(\cdot, 0) = y_0 \; and \; y(\cdot, T) = \widehat{y}(\cdot, T) \Big\}.$$

For $T = T_{min}$, there exists a non-negative control $f \in \mathcal{M}(\omega \times (0, T_{min}))$, the space of Radon measures on $\omega \times (0, T_{min})$, such that the corresponding solution of (3) satisfies $y(x,T) = \widehat{y}(x,T)$ a.e. in $(-1,1)$.

Numerical simulations

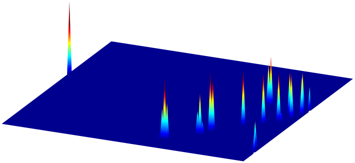
- We consider the problem of steering the initial datum $y_0(x) = \frac{1}{2}\cos\left(\frac{\pi}{2}x\right)$ to the target trajectory $\widehat{y}$ solution of 3 with initial datum $\widehat{y}_0(x) = 6\cos\left(\frac{\pi}{2}x\right)$ and right-hand side $\widehat{f} \equiv 1$.
- We choose $s = 0.8$ and $\omega = (-0.3, 0.8) \subset (-1, 1)$ as the control region.
- The approximation of the minimal controllability time is obtained by solving the following constrained minimization problem:

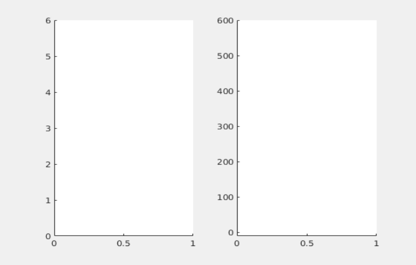
minimize T

$$\begin{cases} T > 0 \\ y_t + (-\Delta)^s y = f\chi_\omega, & a.e. \ in \ (-1,1) \times (0,T) \\ y(\cdot, 0) = y_0 \geq 0, & a.e. \ in \ (-1,1) \\ y \geq 0, & a.e. \ in \ (-1,1) \times (0,T) \\ f \geq 0, & a.e. \ in \ \omega \times (0,T). \end{cases}$$

We obtain the minimal time $T_{min} \simeq 0,2101$.

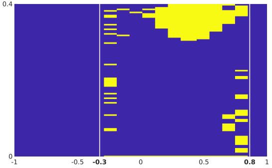
In this time horizon, the fractional heat equation is controllable from the initial datum y_0 to the desired trajectory $\widehat{y}(\cdot, T)$ by maintaining the positivity of the solution.

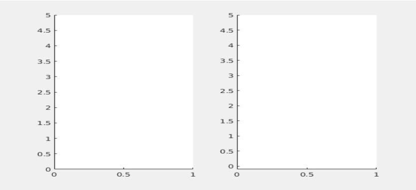




Numerical simulations

The impulsional behavior of the control is lost when extending the time horizon beyond T_{min}.



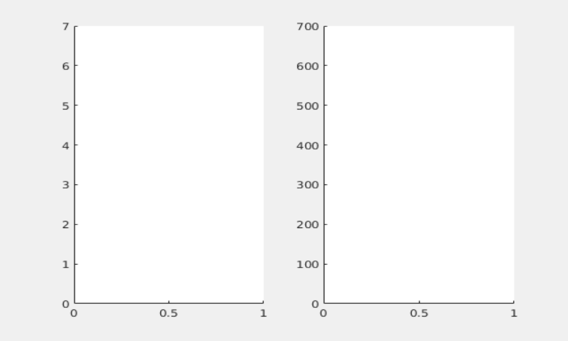


This control has been computed by solving the minimization problem:

$$\min \; \|y(\cdot,T) - \widehat{y}(\cdot,T)\|_{L^2(-1,1)}$$
$$\begin{cases} T > 0 \\ y_t + (-\Delta)^s y = f\chi_\omega, & a.e. \ in \ (-1,1)\times(0,T) \\ y(\cdot,0) = y_0 \geq 0, & a.e. \ in \ (-1,1) \\ y \geq 0, & a.e. \ in \ (-1,1)\times(0,T) \\ f \geq 0, & a.e. \ in \ \omega\times(0,T). \end{cases}$$

Numerical simulations

When considering a time horizon $T < T_{min}$, constrained controllability fails.



THANK YOU FOR YOUR ATTENTION!

Funding

- European Research Council (ERC): grant agreements NO: 694126-DyCon and No.765579-ConFlex.
- MINECO (Spain): Grant PID2020-112617GB-C22 KILEARN
- Alexander von Humboldt-Professorship program
- DFG (Germany): Transregio 154 Project "Mathematical Modelling,Simulation and Optimization Using the Example of Gas Networks"
- COST Action grant CA18232, "Mathematical models for interacting dynamics on networks".

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