Control and optimization for non-local and fractional differential equations

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PART V: complementary topics

LECTURE 15: turnpike theory

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Turnpike theory

Origins of turnpike

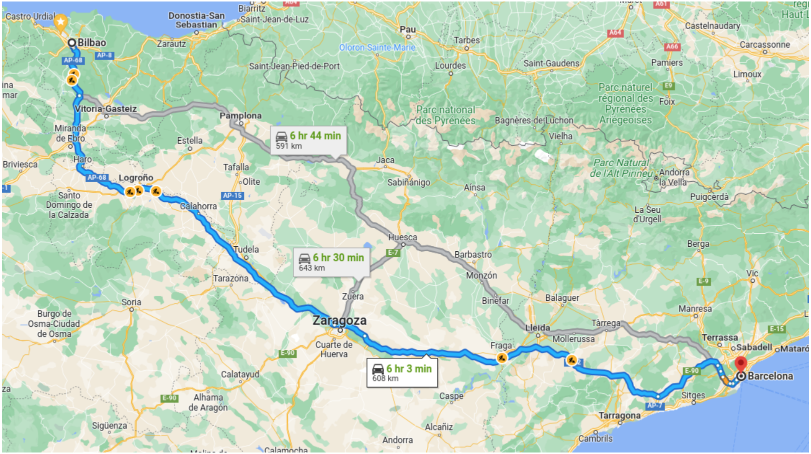
The origin of the term **turnpike** is due to R. Dorfman, P. Samuelson and R. Solow in the book *Linear Programming and Economics Analysis* (1958).

> *... There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if the origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.*

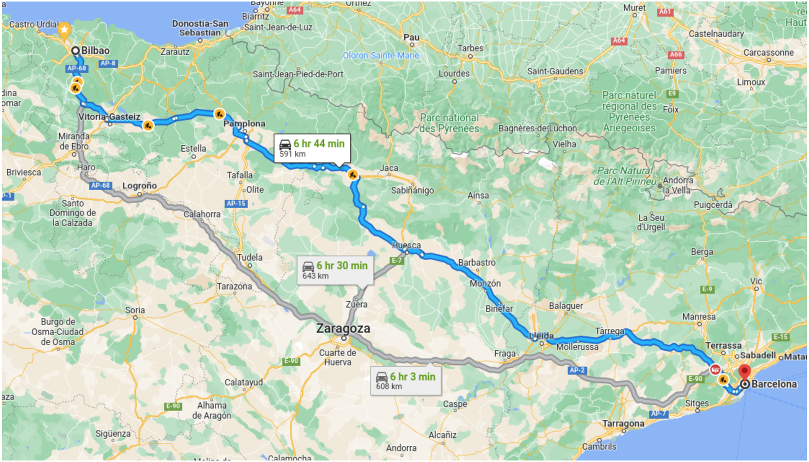
The authors interpreted this concept as: suppose we want to go from a city A to a city B by car, the best way to do this, the optimal option, is to take the highway closest to A, and exit the highway closest to B. That is, the **turpike**.

- There is always the fastest route between two points.
- If the origin and destination are close and far from the highway, the best route may be to stay off the highway.
- However, if the origin and destination are far enough apart, it will always be worthwhile to take the toll road and cover the distance at the best travel pace, even if it means adding a little mileage at each end.

Origins of turnpike

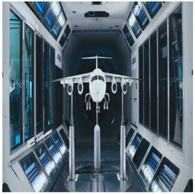


Origins of turnpike



Applications

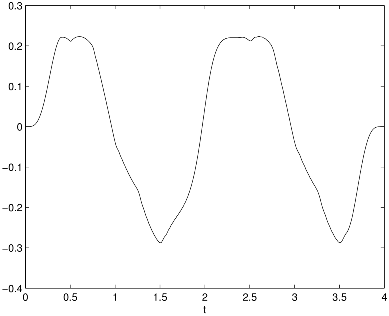
AERODYNAMICS: wind tunnel for optimal shape design.



PDE examples of lack of turnpike

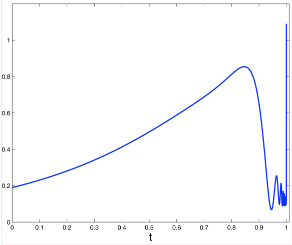
The wave equation

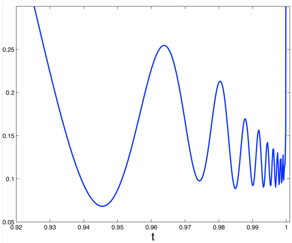
Typical controls for the wave equation exhibit an **oscillatory behavior**, and this independently of the length of the control time-horizon. But nobody would be surprised about this fact. It looks like intrinsically linked to the oscillatory (even periodic in some particular cases) nature of the wave equation solutions.



The heat equation

Typical controls for the heat equation exhibit unexpected **oscillatory and concentration effects**. This was observed by R. Glowinski and J. L. Lions in the 80's in their works in the numerical analysis of controllability problems for heat and wave equations.





The heat equation

Why this phenomenon?

Optimal controls are normally characterized as traces of solutions of the adjoint problem through the **optimality system** or the **Pontryagin Maximum Principle**, and solutions of the adjoint system of the adjoint heat equation

$$-p_t - \Delta p = 0$$

look precisely this way: large and oscillatory near $t = T$, they decay and get smoother when t gets down to $t = 0$. And this is independent of the time control horizon $[0, T]$.

First conclusion

Typical control problems for wave and heat equations do not seem to exhibit the turnpike property.

Heat and wave equations revisted

The control problem

Let $N \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^N$ with smooth boundary Γ, $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$.

Controlled heat equation

$$\begin{cases} y_t - \Delta y = f\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

We assume that $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (1) admits a unique solution

$$y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

Well-known results

The system is null-controllable in any time T and from any open non-empty subset ω of Ω.

The control of minimal L^2-norm can be found by minimizing

$$J(p_T) = \frac{1}{2}\int_0^T \int_\omega |p|^2 \, dxdt + \int_\Omega p(0) y_0 \, dx \qquad (2)$$

over the space of solutions of the adjoint system:

$$\begin{cases} -p_t - \Delta p = 0\chi_\omega & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ p(x,T) = p_T(x) & \text{in } \Omega. \end{cases} \qquad (3)$$

The functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$ and coercive because of the observability estimate

$$\|p(0)\|_{L^2(\Omega)} \leq C \int_0^T \int_\omega |p|^2 \, dxdt, \qquad \text{for all } p_T \in L^2(\Omega). \qquad (4)$$

Well-known results

If $\hat{p}_T$ is the minimizer of the functional J, the needed control is given by $f = \hat{p}$, where $\hat{p}$ is the solution of the adjoint heat equation corresponding to $\hat{p}_T$.

Because of this, we observe the tendency of the control to concentrate all the action in the final time instant $t = T$.

Well-known results

But this is so for the control of minimal L^2-norm for which the Optimality System (OS) reads

$$\begin{cases} y_t - \Delta y = p\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x) & \text{in } \Omega \\ y(x,T) = 0 & \text{in } \Omega \\ -p_t - \Delta p = 0 & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ p(x,T) = p_T(x) & \text{in } \Omega \end{cases}$$

The fact that the adjoint state p appears isolated as the solution of the adjoint equation induces this unexpected behavior and the tendency to concentrate action at $t = T$.

Better balanced controls

Let us now consider the control f minimizing a compromise between the norm of the state and the control among the class of admissible controls:

$$\min \left(\frac{1}{2} \int_0^T \int_\Omega |y|^2 \, dxdt + \frac{1}{2} \int_0^T \int_\omega |f|^2 \, dxdt \right).$$

Then the Optimality System reads

$$\begin{cases} y_t - \Delta y = -p\chi_\omega & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x) & \text{in } \Omega \\ y(x,T) = 0 & \text{in } \Omega \\ -p_t - \Delta p = y & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ p(x,T) = p_T(x) & \text{in } \Omega \end{cases}$$

We now observe a coupling between p and y on the adjoint state equation!

New optimality system dynamics

What is the dynamic behavior of solutions of the new fully coupled OS?

For the sake of simplicity, assume $\omega = \Omega$.

The dynamical system now reads

$$\begin{cases} y_t - \Delta y = -p & \text{in } Q \\ -p_t - \Delta p = y & \text{in } Q \\ + \text{ boundary and initial conditions} & \end{cases}$$

This is a **forward-backward parabolic system**.

A spectral decomposition exhibits the characteristic values

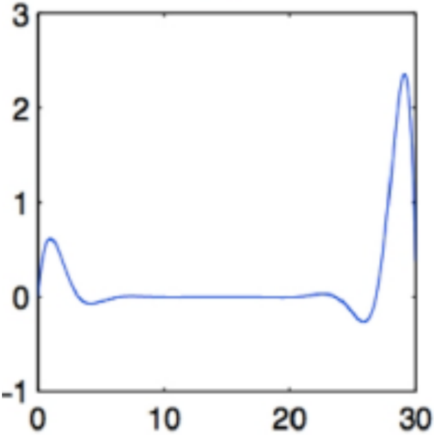
$$\mu_j^{\pm} = \pm\sqrt{1 + \lambda_j^2}$$

where $(\lambda_j)_{j \geq 1}$ are the (positive) eigenvalues of $-\Delta$.

Thus, **the system is the superposition of increasing and decreasing real exponentials**.

The turnpike property for the heat equation

This new dynamic behavior, combining exponentially stable and unstable branches, is compatible with the turnpike behavior. Controls and trajectories exhibit the expected dynamics.



The turnpike property for the wave equation

This relevant fact, that modifying the optimality criterion for the choice of the control ensures the turnpike property, **is not intrinsic to the heat equation**.

The same applies for the **wave equation**: the control and controlled trajectories are close to the steady state ones during most of the time interval of control when $T >> 1$.

M. Gugat, E. Trélat and E. Zuazua, Syst. Control Lett., 2016

What is behind?

General theory

Linear abstract theory

A. Porretta and E. Zuazua, *Remarks on long time versus steady state optimal control*, SICON, 2013.

Consider the finite dimensional dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0 \in \mathbb{R}^N \end{cases} \tag{5}$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times M}$, with control $u \in L^2(0,T;\mathbb{R}^M)$.

Given a matrix $C \in \mathbb{R}^{N\times N}$, and some $x^* \in \mathbb{R}^N$, consider the optimal control problem

$$\min_u J^T(u) = \frac{1}{2}\int_0^T \left(|u(t)|^2 + |C(x(t)-x^*)|^2\right) dt.$$

There exists a unique optimal control $u(t)$ in $L^2(0,T;\mathbb{R}^M)$, characterized by the optimality condition

$$u = -B^*p\,, \qquad \begin{cases} -p_t + A^*p = C^*C(x-x^*) \\ p(T) = 0 \end{cases} \tag{6}$$

The steady state control problem

The same problem can be formulated for the steady-state model

$$Ax = Bu.$$

Then there exists a unique minimum $\bar{u}$, and a unique optimal state $\bar{x}$ of the stationary control problem

$$\min_u J_s(u) = \frac{1}{2}\int_0^T \left(|u|^2 + |C(x - x^*)|^2\right) dt, \quad Ax = Bu.$$

The optimal control $\bar{u}$ and state $\bar{x}$ satisfy

$$A\bar{x} = B\bar{u}\,, \quad \bar{u} = -B^*\bar{p}\,, \qquad \text{and} \quad A^*\bar{p} = C^*C(\bar{x} - x^*).$$

Controllability/observability assumption

We assume that **the pair (A, B) is controllable** or, equivalently, that the matrices A, B satisfy the Kalman rank condition

$$Rank \left[B \; AB \; A^2 B \dots A^{N-1} B \right] = N. \tag{7}$$

Concerning the cost functional, we assume that the matrix C is such that (void assumption when $C = Id$) **the pair (A, C) is observable**, which means that the following algebraic condition holds:

$$Rank \left[C \; CA \; CA^2 \dots CA^{N-1} \right] = N. \tag{8}$$

Turnpike property

Under the previous controllability and observability assumptions, we have the following result.

Theorem

For some $\gamma > 0$ for $T > 0$ large enough we have

$$\int_{aT}^{bT} \left(|u - \bar{u}|^2 + |x - \bar{x}|^2 \right) ds \leq K \left(e^{-\gamma a T} + e^{-\gamma(1-b)T} \right)$$

for every $a, b \in [0, 1]$.

Proof

STEP 1: a dissipativity identity

We have

$$[(x-\bar{x})(p-\bar{p})]_t = -[B^*(p-\bar{p})|^2 + |C(x-\bar{x})|^2]$$

as a direct consequence of

$$\begin{cases} (x-\bar{x})_t + A(x-\bar{x}) = B(u-\bar{u}) \\ u-\bar{u} = -B^*(p-\bar{p}) \\ -(p-\bar{p})_t + A^*(p-\bar{p}) = C^*C(x-\bar{x}). \end{cases}$$

Proof

STEP 2: decay for correlations

If B^* and C are coercive (these conditions can be relaxed under the controllability-observability conditions above) we also have

$$|B^*(p-\bar{p})|^2 + |C(x-\bar{x})|^2 \geq \gamma \left(|p-\bar{p}|^2 + |x-\bar{x}|^2\right).$$

Hence

$$[(x-\bar{x})(p-\bar{p})]_t = -|B^*(p-\bar{p})|^2 - |C(x-\bar{x})|^2 \leq -\gamma|(x-\bar{x})(p-\bar{p})|,$$

for some $\gamma > 0$.

Consequently,

$$-Ke^{-\gamma(T-t)} \leq [(x-\bar{x})(p-\bar{p})](t) \leq Ke^{-\gamma t}$$

if $(x-\bar{x})(p-\bar{p})$ is bounded at $t=0$ and $t=T$.

P. Cardaliaguet, J.-M. Lasry, P.-L. Lions and A. Porretta, *Long time average of Mean Field Games*, Network Heter. Media, 2012.

Proof

STEP 3: convergence of averages

In fact, the bounds on the extremal values at $t = 0$ and $T = T$ immediately yields the turnpike property in an averaged sense. Indeed, as a consequence of the identity,

$$\int_0^T \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2 \right) dt = (x_0 - \bar{x})(p(0) - \bar{p}) - (x(T) - \bar{x})\bar{p}$$

and the bounds at the extremal values $t = 0$ and $t = T$ we then have

$$\int_0^T \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2 \right) dt \leq C \tag{9}$$

with C independent of T and

$$\frac{1}{T} \int_0^T \left(|u - \bar{u}|^2 + |C(x - \bar{x})|^2 \right) dt \leq \frac{C}{T} \to 0.$$

This, of course, also implies the convergence of the averaged minima to the stationary minimum.

Proof

STEP 4: bounds on the extremal values

Using the observability inequality of the pair (A^*, B^*) we have

$$|(p(0)-\bar{p})| \leq c\left[\left(\int_0^T |C(x-\bar{x})|^2\,dt\right)^{\frac{1}{2}} + \left(\int_0^T |B^*(p-\bar{p})|^2\,dt\right)^{\frac{1}{2}} + |\bar{p}|\right]. \tag{10}$$

Similarly, in the equation of $x-\bar{x}$ we use the observability inequality for (A, C) which is ensured by (8):

$$|x(T)-\bar{x}| \leq c\left(\int_0^T |u-\bar{u}|^2 dt + \int_0^T |C(x(t)-\bar{x})|^2 dt + |x_0-\bar{x}|^2\right)^{\frac{1}{2}}. \tag{11}$$

This, together with the identity

$$\int_0^T \left(|u-\bar{u}|^2 + |C(x-\bar{x})|^2\right) dt = (x_0-\bar{x})(p(0)-\bar{p}) - (x(T)-\bar{x})\bar{p}$$

yields the needed bounds.

Proof

STEP 5: the exponential turnpike estimate

The conclusion holds employing the exponential decay of the correlation term and the fact that

$$\int_{aT}^{bT} \left(|u-\bar{u}|^2 + |C(x-\bar{x})|^2\right) dt = (x(aT)-\bar{x})(p(aT)-\bar{p}) - (x(bT)-\bar{x})(p(bT)-\bar{p}).$$

To obtain the exponential turnpike estimate we exploit the correlation of optimal control with the theory of **Riccati differential equations**. For this it is necessary that **the control operator B is bounded**.

Notice that this implies that the previous result **is not immediately extendable to the case of Dirichlet boundary controls**. Indeed, in this case, the control is the trace on the boundary of some given function, and **the Dirichlet trace is an unbounded operator**.

What is the reason behind turnpike?

It is a direct consequence of the hyperbolicity of the underlying dynamics, whose steady state solutions are characterized by the system

$$\begin{aligned} A\bar{x} + BB^*\bar{p} &= 0 \\ -A^*\bar{p} + C^*C\bar{x} &= C^*Cx^* \end{aligned}$$

generated by the operator matrix

$$\tilde{A} = \begin{pmatrix} A & BB^* \\ C^*C & -A^* \end{pmatrix}$$

Note however that the hyperbolicity of this matrix operator needs of controllability conditions.

What is the reason behind turnpike?

In other words, the fact that the spectrum of the operator matrix $\tilde{A}$ is symmetric to the left and right half complex plane, ensures the stability+unstability pattern.

Summarizing, two key ingredients are needed for the turnpike property to arise for the optimal control problem.

- The cost criterion for the optimal control needs to penalize **both state and control**.
- The system **needs to be controllable**.

In particular, it is worth underlying that controllability is needed for the turnpike property to hold !!!

Remark

The controllability assumption might be dropped out if we impose some further hypothesis on the matrix A (for instance, coercivity, $\langle Ax, x\rangle \geq \gamma|x|^2$). In this case, the observation operator C could be taken to be the identity: $C = I$.

Extensions

- Extension of this linear finite-dimensional theory to a **linear abstract setting of infinite-dimensional semigroups**, including wave and heat equations.

  Note that, since (null) controllability is required, turnpike holds for the heat equation with any support ω of the control, but that, for the wave equation, ω is required to fulfill the Geometric Control Condition (by Bardos-Lebeau-Rauch).

  When the GCC fails, weaker turnpike properties are achieved, with slower convergence rates (not exponential ones).
  A. Porretta and E. Zuazua, *Remarks on long time versus steady state optimal control*, SICON, 2013.

- Nonlinear finite-dimensional systems.
  E. Trélat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, JDE, 2015.

- Hamilton-Jacobi-Bellman equation.
  C. Esteve, H. Kouhkouh, D. Pighin and E. Zuazua, *The turnpike property and the longtime behavior of the Hamilton-Jacobi-Bellman equation for finite-dimensional LQ control problems*, Math. Control Sign. Syst., 2022.

- Neural networks.
  B. Geshkovski and E. Zuazua, *Turnpike in optimal control of PDEs, ResNets, and beyond*, Acta Num., 2022.

- Fractional heat equation.
  M. Warma and S. Zamorano, *Exponential turnpike for fractional parabolic equations with non-zero exterior data*, ESAIM:COCV, 2021.

Turnpike for the fractional heat equation

Turnpike for the fractional heat equation

Turnpike results have been obtained also in the context of the fractional heat equation.

M. Warma and S. Zamorano, *Exponential turnpike for fractional parabolic equations with non-zero exterior data*, ESAIM:COCV, 2021.

Preliminary remarks

Remark

The fractional heat operator is coercive in appropriate functional spaces.

H. Antil, R. Khatri and M. Warma, *External optimal control of nonlocal PDEs*, Inv. Problems, 2020.

Hence, the turnpike property for this operator does not require the controllability assumption.

Notice that the controllability of fractional heat-like equations is known only in one space dimension!!

Preliminary remarks

Remark

In the case of interior control

$$\begin{cases} u_t + (-\Delta)^s u = f\chi_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \Omega^c \times (0,T), \\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

the turnpike property is a direct consequence of the general theory of Porretta and Zuazua, since the fractional Laplacian **is a self-adjoint operator**.

Remark

What about the exterior control problem?

The Robin case

Consider the following exterior optimal control problem:

$$\min_{g\in U} J^T(g) := \frac{1}{2}\int_0^T \|u-u^d\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^T \|g(\cdot,t)\|^2_{L^2(\Omega^c,\mu)}dt, \tag{12}$$

s.t. u solves

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{in } \Omega\times(0,T),\\ \mathcal{N}_s u + \beta u = \beta g & \text{in } \Omega^c\times(0,T),\\ u(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{13}$$

where $\beta \in L^1(\Omega^c)$ is a given nonnegative function.

$U := L^2((0,T); L^2(\Omega^c,\mu))$, where the measure μ on Ω^c is defined by $d\mu := \beta dx$ with dx the usual N-dimensional Lebesgue measure.

Well-posedness

For $\beta \in L^1(\Omega^c)$ a nonnegative function, we denote by $H^s_{\Omega,\beta}$ the space

$$H^s_{\Omega,\beta} := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } \|u\|_{H^s_{\Omega,\beta}} < +\infty \right\},$$

where

$$\|u\|_{H^s_{\Omega,\beta}} := \left(\|u\|^2_{L^2(\Omega)} + \|\beta^{1/2} u\|^2_{L^2(\Omega^c)} + \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}},$$

and

$$\mathbb{R}^{2N} \setminus (\mathbb{R}^{2N} \setminus \Omega)^2 = (\Omega \times \Omega) \cup ((\Omega^c) \times \Omega) \cup (\Omega \times (\Omega^c)).$$

Let μ be the measure in Ω^c given by $d\mu = \beta dx$. Then, the norm can be written as

$$\|u\|_{H^s_{\Omega,\beta}} = \left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega^c,\mu)} + \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

Well-posedness

Definition

Let $g \in L^2((0,T); L^2(\Omega^c, \mu))$. We say that a function $u \in L^2((0,T); H^s_{\Omega,\beta}) \cap H^1((0,T); (H^s_{\Omega,\beta})^*)$ is a weak solution of (13), if the identity

$$\langle u_t, v \rangle_{(H^s_{\Omega,\beta})^*, H^s_{\Omega,\beta}} + \mathcal{E}(u,v) = \int_{\Omega^c} g v d\mu,$$

holds for every $v \in H^s_{\Omega,\beta}$ and almost every $t \in (0,T)$.

Here

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, dxdy + \int_{\Omega^c} \beta uv \, dx,$$

for $u, v \in H^s_{\Omega,\beta}$.

Optimality conditions

Theorem

There exist a unique optimal control $g^T \in L^2((0,T); L^2(\Omega^c, \mu))$ and a state $u^{\in} L^2((0,T); H^s_{\Omega,\beta}) \cap H^1((0,T); (H^s_{\Omega,\beta})^*)$ such that the functional J^T attains its minimum at g^T. In addition, there exists $\psi^T \in L^2((0,T); D(-\Delta)^s_R) \cap H^1((0,T); L^2(\Omega))$ solution of

$$\begin{cases} -\psi_t^T + (-\Delta)^s \psi^T = u^T - u^d & \text{in } \Omega \times (0,T), \\ \mathcal{N}_s \psi^T + \beta \psi^T = 0 & \text{in } (\Omega^c) \times (0,T), \\ \psi(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{14}$$

Moreover,

$$g^T = -\psi^T \Big|_{(\Omega^c)\times(0,T)}.$$

H. Antil, D. Verma and M. Warma, ESAIM-COCV, 2020.

Stationary problem

We also consider the corresponding stationary problem

$$\min_{g \in \mathcal{U}} J(g) := \frac{1}{2}\|u - u^d\|^2_{L^2(\Omega)} + \frac{1}{2}\|g\|^2_{L^2(\Omega^c,\mu)}, \tag{15}$$

s.t. u is the solution of

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega \\ \mathcal{N}_s u + \beta u = \beta g & \text{in } \Omega^c. \end{cases} \tag{16}$$

Stationary problem

Theorem

There exist a unique optimal control $\overline{g} \in L^2(\Omega^c, \mu)$ and $\overline{u} \in H^s_{\Omega,\beta}$, such that the functional J in (15) attains its minimum at $\overline{g}$. In addition, there exists $\overline{\psi} \in H^s_{\Omega,\beta}$ solution of

$$\begin{cases} (-\Delta)^s \overline{\psi} = \overline{u} - u^d & \text{in } \Omega \\ \mathcal{N}_s \overline{\psi} + \beta \overline{\psi} = 0 & \text{in } \Omega^c. \end{cases}$$

Moreover,

$$\overline{g} = -\overline{\psi}\Big|_{\Omega^c}.$$

H. Antil, R. Khatri and M. Warma, Inv. Problems, 2019.

Turnpike for the Robin problem

Theorem

Let (u^T, g^T, ψ^T) be the solution of the optimality system and $(\overline{u}, \overline{g}, \overline{\psi})$ solution of the stationary optimal system. Then,

$$\frac{1}{T}\int_0^T g^T\, dt \quad \longrightarrow \quad \overline{g}, \quad \text{strongly in } L^2(\Omega^c, \mu) \text{ as } T \to +\infty,$$

and

$$\frac{1}{T}\int_0^T u^T\, dt \quad \longrightarrow \quad \overline{u}, \quad \text{strongly in } L^2(\Omega) \text{ as } T \to +\infty.$$

M. Warma and S. Zamorano, ESAIM-COCV, 2021.

Turnpike for the Robin problem

Theorem

Let $\gamma \geq 0$ be a real number. There is a constant $C = C(\gamma) > 0$ (independent of T) such that for every $t \in [0, T]$ we have the following estimate

$$\begin{aligned} &\|u^T(\cdot,t) - \overline{u}\|_{L^2(\Omega)} + \|\psi^T(\cdot,t) - \overline{\psi}\|_{L^2(\Omega)} \\ &\qquad \leq C\Big(e^{-\gamma t} + e^{-\gamma(T-t)}\Big)\Big(\|\overline{u}\|_{L^2(\Omega)} + \|\overline{\psi}\|_{L^2(\Omega)}\Big). \end{aligned}$$

M. Warma and S. Zamorano, ESAIM-COCV, 2021.

Turnpike for the Robin problem

Remark

From these results, we can also obtain an estimate for the control. Indeed, from the previous estimate, we can deduce that there is a constant $C > 0$ (independent of T) such that

$$\left\| \frac{1}{e^{-\gamma t} + e^{-\gamma(T-t)}} (g^T - \overline{g}) \right\|_{L^2((0,T);L^2(\Omega^c,\mu))} \leq C \Big(\|\overline{u}\|_{L^2(\Omega)} + \|\overline{\psi}\|_{L^2(\Omega)} \Big).$$

The Dirichlet case

Let $U = L^2(0,T;L^2(\Omega^c))$ and consider the following exterior optimal control problem

$$\min_{g\in U} J^T(g) := \frac{1}{2}\int_0^T \|u-u^d\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_0^T \|g(\cdot,t)\|^2_{L^2(\Omega^c)}dt, \tag{17}$$

s.t. u solves

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{in } \Omega\times(0,T) \\ u = g & \text{in } (\Omega^c)\times(0,T) \\ u(\cdot,0) = 0 & \text{in } \Omega \end{cases} \tag{18}$$

Transposition solutions

Let $g \in L^2((0,T);L^2(\Omega^c))$. We say that $u \in L^2(\Omega \times (0,T))$ is a solution by transposition of (18) if the identity

$$\int_0^T \int_\Omega u\Big(- v_t + (-\Delta)^s v\Big)dxdt = -\int_0^T \int_{\Omega^c} g\mathcal{N}_s v dxdt,$$

holds for every $v \in L^2((0,T);D((-\Delta)^s_D)) \cap H^1((0,T);L^2(\Omega))$ with $v(\cdot,T) = 0$.

Optimality condition

Theorem

There exists an optimal pair (g^T, u^T) to the problem (17)-(18). Moreover,

$$g^T = \mathcal{N}_s \lambda^T,$$

where $\lambda^T \in L^2((0,T); D((-\Delta)^s_D) \cap H^1((0,T); L^2(\Omega))$ solves

$$\begin{cases} -\lambda^T_t + (-\Delta)^s \lambda^T = u^T - u^d & \text{in } Q \\ \lambda^T = 0 & \text{in } \Sigma \\ \lambda^T(\cdot, T) = 0 & \text{in } \Omega \end{cases}$$

H. Antil, D. Verma, M. Warma, ESAIM-COCV, 2020.

Turnpike for the Dirichlet problem

Theorem

Let $\gamma \geq 0$ be a real number. Let (u^T, g^T, λ^T) be the solution of the optimality system and $(\overline{u}, \overline{g}, \overline{\lambda})$ the corresponding solution of the stationary optimality system. Then, there is a constant $C = C(\gamma) > 0$ (independent of T) such that for every $t \in [0, T]$ we have

$$\begin{aligned}&\|u^T(\cdot,t) - \overline{u}\|_{L^2(\Omega)} + \|\lambda^T(\cdot,t) - \overline{\lambda}\|_{L^2(\Omega)} \\ &\leq C\Big(e^{-\gamma t} + e^{-\gamma(T-t)}\Big)\Big(\|\overline{u}\|_{L^2(\Omega)} + \|\overline{\lambda}\|_{L^2(\Omega)}\Big).\end{aligned}$$

Moreover,

$$\begin{aligned}&\left\|\frac{u^T - \overline{u}}{e^{-\gamma t} + e^{-\gamma(T-t)}}\right\|_{L^2((0,T);H_0^s(\Omega))} + \left\|\frac{g^T - \overline{g}}{e^{-\gamma t} + e^{-\gamma(T-t)}}\right\|_{L^2((0,T);L^2(\Omega^c))} \\ &+ \left\|\frac{\lambda^T - \overline{\lambda}}{e^{-\gamma t} + e^{-\gamma(T-t)}}\right\|_{L^2((0,T);H_0^s(\Omega))} \leq C\Big(\|\overline{u}\|_{L^2(\Omega)} + \|\overline{\lambda}\|_{L^2(\Omega)}\Big).\end{aligned}$$

M. Warma and S. Zamorano, ESAIM-COCV, 2021.

Proofs (general ideas)

ROBIN CASE: the proof of the convergence of the averages follows the same steps as in the work of Porretta and Zuazua.

To obtain the exponential turnpike estimate, the main ingredient is to prove that the solution operator of the optimality system is bounded **uniformly with respect to the time horizon** T.

Notice that this is not necessarily true for general dynamics.

DIRICHLET CASE: for the Dirichlet problem, the situation is much more delicate, due to the **low regularity of the solutions**. It is required the employment of the abstract control theory of Tucsnak and Weiss, based on the concepts of **admissible control** and **admissible observation** operators.

M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser, 2009.

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