Control and optimization for non-local and fractional differential equations

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PART V: complementary topics

LECTURE 16: PDE with non-local integral kernels

Null controllability for non-local heat equations with integral kernel

Non-local heat equation

$$\begin{cases} y_t - \Delta y + \displaystyle\int_\Omega K(x,\theta,t)y(\theta,t)\,d\theta = v\mathbf{1}_{\mathcal{O}}, & (x,t)\in Q \\ y=0, & (x,t)\in\Sigma \\ y(x,0)=y_0(x), & x\in\Omega. \end{cases}$$

- $\Omega\subset\mathbb{R}^d$ bounded domain of class C^2.
- $Q := \Omega\times(0,T)$, $T>0$.
- $\Sigma := \partial\Omega\times(0,T)$.
- $K = K(x,\theta,t)\in L^\infty(\Omega\times\Omega\times(0,T))$.
- $y_0\in L^2(\Omega)$, $v\in L^2(\mathcal{O}\times(0,T))$.

There exists a unique solution $y\in L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega))$, which satisfies classical energy estimates.

Non-local heat equation

We are interested in proving the null controllability of the problem under analysis.

EXISTING RESULTS: it is known that the system is null controllable at least in the following situations cases.

- Under analyticity assumptions on the non-local potential, one can exploit unique continuation properties and use compactness-uniqueness arguments.

  E. Fernández-Cara, Q. Lü and E. Zuazua, *Null controllability of linear heat and wave equations with nonlocal spatial terms*, SICON, 2016. In this framework, also coupled systems have been treated

  P. Lissy and E. Zuazua, *Internal controllability for parabolic systems involving analytic nonlocal terms*, Chin. Ann. Math. Ser. B, 2018.
- When the problem is one-dimensional and the kernel is time-independent and in separated variables, the controllability follows employing spectral analysis techniques.

  S. Micu and T. Takahashi, *Local controllability to stationary trajectories of a Burgers equation with nonlocal viscosity*, JDE, 2018.
- Under suitable time-decay assumptions for the integral kernel.

  U. Biccari and V. Hernández-Santamaría, *Null controllability of linear and semilinear nonlocal heat equations with an additive integral kernel*, SICON, 2018
- When the kernel is constant, using the so-called **shadow model**.

  V. Hernández-Santamaría and K. Le Balc'h, *Local null-controllability of a nonlocal semilinear heat equation*, Appl. Math. Optim., 2021.

Main result

By means of a **Carleman approach**, we can extend the above mentioned results by considering a problem in any space dimension and by weakening the assumptions on the kernel.

Theorem

Suppose that the kernel $K = K(x, \theta, t) \in L^\infty(\Omega \times \Omega \times (0, T))$ satisfies

$$\mathcal{K} =: \sup_{(x,t)\in Q} \left[\exp\left(\frac{\varepsilon \mathcal{A}}{t(T-t)} \right) \int_\Omega |K(x,\theta,t)| \, d\theta \right] < +\infty, \qquad (\mathcal{H})$$

for any $\varepsilon > 0$, where $\mathcal{A}$ is a positive constant. Then, given $y_0 \in L^2(\Omega)$ and $T > 0$, there exists a control function $v \in L^2(\mathcal{O} \times (0, T))$ such that $y(x, T) = 0$.

U. Biccari and V. Hernández-Santamaría, *Null controllability of linear and semilinear nonlocal heat equations with an additive integral kernel*, SICON, 2018

Observability inequality

Theorem

There exist two positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$, depending only on the domain Ω, such that

$$\|\varphi(x,0)\|_{L^2(\Omega)} \leq \frac{\mathcal{C}_1}{T} \exp\left[\mathcal{C}_2\left(1+\mathcal{K}^{\frac{2}{3}}+\frac{1}{T}\right)\right] \int_0^T \int_{\mathcal{O}} |\varphi|^2 \, dx \, dt$$

holds for any solution of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + \displaystyle\int_\Omega K(x,\theta,t)\varphi(\theta,t)\,d\theta = 0, & (x,t) \in Q \\ \varphi = 0, & (x,t) \in \Sigma \\ \varphi(x,T) = \varphi_T(x), & x \in \Omega. \end{cases}$$

with $\varphi_T \in L^2(\Omega)$ and K satisfying $(\mathcal{H})$.

Carleman estimate

Lemma

Let $\mathcal{O} \subset\subset \Omega$ be a nonempty open set. Then, there exists $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$ and $|\nabla \eta^0| > 0$ in $\overline{\Omega \setminus \mathcal{O}}$.

For a parameter $\lambda > 0$, we define

$$\sigma(x) := e^{4\lambda \left\|\eta^0\right\|_\infty} - e^{\lambda\left(2\left\|\eta^0\right\|_\infty + \eta^0(x)\right)},$$

and we introduce the weight functions

$$\alpha(x,t) := \frac{\sigma(x)}{t(T-t)}, \qquad \xi(x,t) := \frac{e^{\lambda\left(2\left\|\eta^0\right\|_\infty + \eta^0(x)\right)}}{t(T-t)}.$$

Moreover, we use the notation

$$\sigma^+ := \max_{x\in\overline{\Omega}} \sigma = e^{4\lambda\left\|\eta^0\right\|_\infty} - e^{2\lambda\left\|\eta^0\right\|_\infty}, \quad \sigma^- := \min_{x\in\overline{\Omega}} \sigma = e^{4\lambda\left\|\eta^0\right\|_\infty} - e^{3\lambda\left\|\eta^0\right\|_\infty}$$

$$\mathcal{I}(\cdot) := s\lambda^2 \int_Q e^{-2s\alpha}\xi|\nabla \cdot|^2 \, dx\, dt + s^3\lambda^4 \int_Q e^{-2s\alpha}\xi^3 |\cdot|^2 \, dx\, dt.$$

Carleman estimate (cont.)

Proposition

There exist positive constants $\mathcal{C}$, s_1 and λ_1 such that, for any $s \geq s_1$, $\lambda \geq \lambda_1$, $F \in L^2(Q)$ and $z_T \in L^2(\Omega)$, the solution z to

$$\begin{cases} z_t + \Delta z = F, & (x,t) \in Q \\ z = 0, & (x,t) \in \Sigma \\ z(x,T) = z_T(x), & x \in \Omega \end{cases}$$

satisfies

$$\mathcal{I}(z) \leq \mathcal{C} \left[s^3 \lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha} \xi^3 |z|^2 \, dx\, dt + \int_Q e^{-2s\alpha} |F|^2 \, dx\, dt \right].$$

E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability*, SICON, 2006

Carleman estimate (cont.)

Proposition

Let $\varphi^T \in L^2(\Omega)$ and assume that the kernel K satisfies $(\mathcal{H})$. Then, there exist positive constants $\mathcal{C}$, λ_1 and ϱ_2, only depending on Ω and $\mathcal{O}$, such that the solution φ of the adjoint system corresponding to the initial datum φ^T satisfies

$$\mathcal{I}(\varphi) \leq \mathcal{C} s^3 \lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt,$$

for any $\lambda \geq \lambda_0$ and any $s \geq \varrho_2 \left(T + T^2 + \mathcal{K}^{\frac{2}{3}} T^2 \right)$.

Preliminary result

Proposition

For any fixed $\lambda > 0$ and $s > 1$ it holds

$$\exp\left(-\frac{(1+s)\sigma^-}{t(T-t)}\right) < \exp\left(-\frac{s\sigma^+}{t(T-t)}\right),$$

where, we recall, σ^- and σ^+ are defined as

$$\sigma^+ := \max_{x\in\overline{\Omega}} \sigma = e^{4\lambda\left\|\eta^0\right\|_\infty} - e^{2\lambda\left\|\eta^0\right\|_\infty}$$

$$\sigma^- := \min_{x\in\overline{\Omega}} \sigma = e^{4\lambda\left\|\eta^0\right\|_\infty} - e^{3\lambda\left\|\eta^0\right\|_\infty}.$$

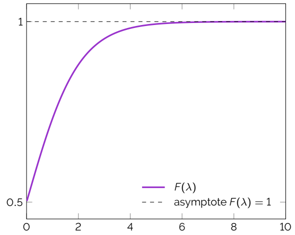
Proof

From the definitions of σ^- and σ^+ we have $\sigma^- = F(\lambda)\sigma^+$, with

$$F(\lambda) := \frac{e^{2\lambda\left\|\eta^0\right\|_\infty} - e^{\lambda\left\|\eta^0\right\|_\infty}}{e^{2\lambda\left\|\eta^0\right\|_\infty} - 1}.$$

$F(\lambda)$ is a **monotone increasing** function, verifying

$$\lim_{\lambda\to+\infty} F(\lambda) = 1 \quad \text{and} \quad \lim_{\lambda\to 0^+} F(\lambda) = 1/2$$



Moreover, since $s > 1$ we have $(1+s)F(\lambda) > 2F(\lambda) > 1$.

Proof (cont.)

Multiplying by σ^+, and since $[t(T-t)]^{-1} > 0$, we have

$$\sigma^+[t(T-t)]^{-1}\Big(1-(1+s)F(\lambda)\Big) < 0.$$

Hence

$$\exp\left(\frac{\sigma^+}{t(T-t)}\Big(1-(1+s)F(\lambda)\Big)\right) < 1$$

We therefore conclude

$$\exp\left(\frac{s\sigma^+}{t(T-t)}\right) < \exp\left(\frac{(1+s)F(\lambda)\sigma^+}{t(T-t)}\right) = \exp\left(\frac{(1+s)\sigma^-}{t(T-t)}\right).$$

which immediately gives

$$\exp\left(-\frac{(1+s)\sigma^-}{t(T-t)}\right) < \exp\left(-\frac{s\sigma^+}{t(T-t)}\right).$$

Proof of the Carleman estimate

We begin by applying the Carleman estimate of Fernández-Cara and Guerrero to φ, obtaining, for any $\lambda \geq \lambda_1$ and any $s \geq \varrho_1 \left(T + T^2\right)$,

$$\mathcal{I}(\varphi) \leq \mathcal{C}\left[s^3\lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \int_Q e^{-2s\alpha}\left|\int_\Omega K(\theta,x,t)\varphi(\theta,t)\,d\theta\right|^2 dx\,dt\right].$$

Set the parameter $\lambda = \lambda_1$ to a fixed value sufficiently large. We have

$$\begin{aligned}\left|\int_\Omega K(\theta,x,t)\varphi(\theta,t)\,d\theta\right| &= \left|\int_\Omega e^{\frac{\sigma^-}{t(T-t)}}K(\theta,x,t)e^{-\frac{\sigma^-}{t(T-t)}}\varphi(\theta,t)\,d\theta\right| \\ &\leq \left[\left(\int_\Omega e^{\frac{2\sigma^-}{t(T-t)}}|K(\theta,x,t)|^2\,d\theta\right)\left(\int_\Omega e^{-\frac{2\sigma^-}{t(T-t)}}|\varphi(x,\theta)|^2\,d\theta\right)\right]^{\frac{1}{2}}.\end{aligned}$$

Since λ has been fixed, σ^- (and therefore σ^+) is a constant **depending only on Ω and $\mathcal{O}$**. Replacing into the first inequality we get

$$\begin{aligned}\mathcal{I}(\varphi) \leq \mathcal{C}\Bigg[&s^3\lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\ &+ \mathcal{K}^2\int_Q e^{-2s\alpha(x,t)}\left(\int_\Omega e^{-\frac{2\sigma^-}{t(T-t)}}|\varphi(\theta,t)|^2\,d\theta\right)dx\,dt\Bigg].\end{aligned}$$

Proof of the Carleman estimate (cont.)

Using **Fubini's Theorem** we get

$$\int_Q e^{-2s\alpha(x,t)} \left(\int_\Omega e^{-\frac{2\sigma^-}{t(T-t)}} |\varphi(\theta,t)|^2 \, d\theta \right) dxdt = \int_Q e^{-\frac{2\sigma^-}{t(T-t)}} |\varphi(\theta,t)|^2 \left(\int_\Omega e^{-2s\alpha(x,t)} \, dx \right) d\theta dt.$$

Moreover, we have

$$\int_\Omega e^{-2s\alpha(x,t)} \, dx \leq |\Omega| e^{-\frac{2s\sigma^-}{t(T-t)}}.$$

Hence, we can compute

$$\begin{aligned} \int_Q e^{-\frac{2\sigma^-}{t(T-t)}} |\varphi(\theta,t)|^2 \left(\int_\Omega e^{-2s\alpha(x,t)} \, dx \right) d\theta \, dt &\leq \mathcal{C} \int_Q e^{-\frac{2(1+s)\sigma^-}{t(T-t)}} |\varphi(\theta,t)|^2 \, d\theta \, dt \\ &\leq \mathcal{C} \int_Q e^{-\frac{2s\sigma^+}{t(T-t)}} |\varphi(\theta,t)|^2 \, d\theta \, dt \\ &\leq \mathcal{C} \int_Q e^{-2s\alpha(\theta,t)} |\varphi(\theta,t)|^2 \, d\theta \, dt, \end{aligned}$$

Putting all together, we get

$$\mathcal{I}(\varphi) \leq \mathcal{C} \left[s^3 \lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt + \mathcal{K}^2 \int_Q e^{-2s\alpha(x,t)} |\varphi(x,t)|^2 \, dx \, dt \right].$$

Proof of the Carleman estimate (conclusion)

Recalling the definition of $\mathcal{I}(\varphi)$, we find

$$s\lambda^2 \int_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\ - \mathcal{C}\mathcal{K}^2 \int_Q e^{-2s\alpha}|\varphi|^2\,dx\,dt \leq \mathcal{C}s^3\lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt.$$

We have the estimate

$$\xi(t)^{-1} \leq \mathcal{C}T^2,$$

which yields

$$s\lambda^2 \int_Q e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\ \leq \mathcal{C}s^3\lambda^4 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt,$$

for all $s > \mathcal{C}\mathcal{K}^{\frac{2}{3}}T^2$.

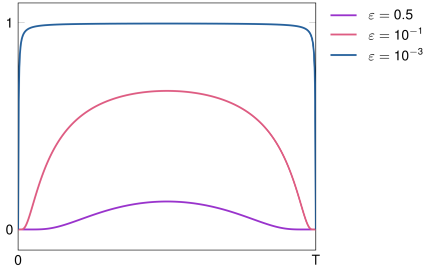
A remark

Remark

According to hypothesis ($\mathcal{H}$), the kernel K should behave like

$$K(\cdot,\cdot,t) \sim e^{-\frac{\mathcal{A}(s,\lambda)\varepsilon}{t(T-t)}},$$

i.e. it should decay exponentially as t goes to 0^+ and T^-. This is the minimum decay that we shall ask for the kernel.



Proof of the observability inequality

From the Carleman estimate:

$$s^3 \int_Q e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \leq \mathcal{C}s^3 \int_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt.$$

Due to the definition of the weight function α, if we choose $s \geq \mathcal{C}T^2$, we have:

- $s^3e^{-2s\alpha}\xi^3 \leq \mathcal{C}s^3T^{-6}e^{-\frac{\mathcal{C}s}{T^2}} \leq \mathcal{C}(T)$
- $s^3e^{-2s\alpha}\xi^3 \geq \mathcal{C}e^{-\frac{\mathcal{C}s}{T^2}}, \quad \text{if} \;\; t \in \left[\frac{T}{4}, \frac{3}{4}T\right].$

Therefore, we obtain

$$\int_{\frac{T}{4}}^{\frac{3}{4}T}\int_\Omega |\varphi|^2\,dx\,dt \leq \mathcal{C}e^{\frac{\mathcal{C}s}{T^2}} \int_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt.$$

We multiply the adjoint equation by φ and integrate by parts. From the fact that $K \in L^\infty(\Omega \times \Omega \times (0,T))$ and using Fubini's Theorem, we have

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi(x,t)|^2dx + \int_\Omega |\nabla\varphi(x,t)|^2dx \leq \mathcal{C}\left(\int_\Omega \varphi(x,t)\,dx\right)^2.$$

Using Jensen's inequality and Gronwall's Lemma, we deduce

$$\int_\Omega |\varphi(x,0)|^2dx \leq \widetilde{\mathcal{C}}\int_\Omega |\varphi(x,t)|^2, \quad t \in [0,T].$$

Hence: $\widetilde{\mathcal{C}}\int_{\frac{T}{4}}^{\frac{3}{4}T}\int_\Omega |\varphi(x,t)|^2\,dx\,dt \geq \int_{\frac{T}{4}}^{\frac{3}{4}T}\int_\Omega |\varphi(x,0)|^2\,dx\,dt = \frac{T}{2}\,\|\varphi(x,0)\|^2_{L^2(\Omega)}.$

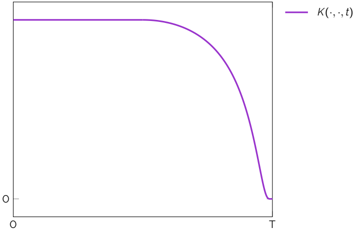
Removing the decay assumption at $t = 0$

Remark

We are allowed to consider kernels K which does not decay at $t = 0$:

$$\mathcal{M} := \sup_{(x,t)\in\overline{Q}} \left[\exp\left(\frac{\mathcal{B}}{T-t}\right) \int_\Omega |K(x,\theta,t)|\, d\theta \right] < +\infty.$$

This requires a modification in the Carleman weight



$$\alpha \mapsto \beta := \frac{e^{4\lambda\left\|\eta^0\right\|_\infty} - e^{\lambda\left(2\left\|\eta^0\right\|_\infty + \eta^0\right)}}{\ell(t)}$$

$$\xi \mapsto \gamma := \frac{e^{\lambda\left(2\left\|\eta^0\right\|_\infty + \eta^0\right)}}{\ell(t)}$$

$$\ell(t) := \begin{cases} T^2/4, & t \in [0, T/2] \\ t(T-t), & t \in [T/2, T] \end{cases}$$

Removing the decay assumption at $t = 0$

Theorem

Let $T > 0$ and assume that K satisfies

$$\mathcal{M} := \sup_{(x,t)\in\overline{Q}} \left[\exp\left(\frac{\mathcal{B}}{T-t}\right) \int_\Omega |K(x,\theta,t)|\, d\theta \right] < +\infty.$$

Then, for any $y_0 \in L^2(\Omega)$, there exists a control function

$$v \in L^2(\mathcal{O} \times (0,T))$$

such that the associated solution y satisfies $y(x,T) = 0$.

Proof (preliminaries)

Proposition

There exist a positive constants $\mathcal{C}$, depending on T, s and λ, such that, for all $F \in L^2(Q)$ and $z_T \in L^2(\Omega)$, the solution z to

$$\begin{cases} z_t + \Delta z = F, & (x,t) \in Q \\ z = 0, & (x,t) \in \Sigma \\ z(x,T) = z_T(x), & x \in \Omega \end{cases}$$

satisfies

$$\begin{aligned} &\|z(x,0)\|^2_{L^2(\Omega)} + \int_Q e^{-2s\beta}\gamma^3|z|^2\,dxdt \\ &\qquad \leq \mathcal{C}\left[\int_{\mathcal{O}\times(0,T)} e^{-2s\beta}\gamma^3|z|^2\,dx\,dt + \int_Q e^{-2s\beta}|F|^2\,dx\,dt\right]. \end{aligned}$$

E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov and J.-P. Puel, *Local exact controllability of the Navier-Stokes system*, J. Math. Pures Appl., 2004.

Proof (preliminaries)

Proposition

Let $T > 0$ and suppose that $e^{s\beta} f \in L^2(Q)$. Then, for any $y_0 \in L^2(\Omega)$ there exists a control function $v \in L^2(\mathcal{O} \times (0,T))$ such that the associated solution to

$$\begin{cases} y_t - \Delta y = f + v\mathbf{1}_{\mathcal{O}}, & (x,t) \in Q \\ y = 0, & (x,t) \in \Sigma \\ y(x,0) = y_0(x), & x \in \Omega \end{cases}$$

is in the space $\mathcal{E} := \left\{ y : e^{s\beta} y \in L^2(Q) \right\}$. Moreover, there exists a positive constant $\mathcal{C} = \mathcal{C}(T, s, \lambda)$ such that the following estimate holds

$$\int_{\mathcal{O}\times(0,T)} e^{2s\beta} \gamma^{-3} |v|^2 \, dx \, dt + \int_Q e^{2s\beta} |y|^2 \, dx \, dt \le \mathcal{C} \left(\|y_0\|^2_{L^2(\Omega)} + \left\| e^{s\beta} f \right\|^2_{L^2(Q)} \right).$$

- $y \in \mathcal{E} \quad \Rightarrow \quad \displaystyle\int_Q e^{2s\beta} |y|^2 \, dx \, dt < +\infty.$
  Since the weight β blows up as $t \to T^-$, this yields $y(x,T) = 0$.

Proof of the theorem

We employ a **fixed point** strategy. For $R > 0$, we define

$$\mathcal{E}_R := \left\{ w \in \mathcal{E} \, : \, \left\| e^{s\beta} w \right\|_{L^2(Q)} \leq R \right\},$$

which is a **bounded**, **closed** and **convex** subset of $L^2(Q)$.

For any $w \in \mathcal{E}_R$, we us consider the control problem

$$\begin{cases} y_t - \Delta y + \int_\Omega K(x,\theta,t) w(\theta,t)\, d\theta = v \mathbf{1}_{\mathcal{O}}, & (x,t) \in Q \\ y = 0, & (x,t) \in \Sigma \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases}$$

We have that

$$\int_Q \left(e^{s\beta} \int_\Omega K(x,\theta,t) w(\theta,t)\, d\theta \right)^2 dxdt \leq \mathcal{M}^2 \int_Q e^{2s\beta} w^2 e^{-2s\beta}\, dxdt \leq \mathcal{M}^2 R^2.$$

Therefore, the above system is **null controllable at time T**.

Proof of the theorem (cont.)

In order to obtain the same controllability result for $w = y$, we apply **Kakutani's fixed point theorem**.

For any $w \in \mathcal{E}_R$, we define the multi-valued map $\Lambda : \mathcal{E}_R \mapsto 2^{\mathcal{E}}$ such that

$$\Lambda(w) = \left\{ y \in \mathcal{E} : \exists v \; s.t. \int_{\mathcal{O}\times(0,T)} e^{2s\beta}\gamma^{-3}|v|^2 \, dx \, dt \leq C \left(R^2 + \|y_0\|^2_{L^2(\Omega)} \right) . \right\}$$

$\Lambda(w)$ is a **nonempty**, **closed** and **convex** subset of $L^2(Q)$. Moreover:

$$\begin{aligned} &\int_{\mathcal{O}\times(0,T)} e^{2s\beta}\gamma^{-3}|v|^2 \, dx \, dt + \int_Q e^{2s\beta}|y|^2 \, dx \, dt \\ &\qquad \leq C \left[\|y_0\|^2_{L^2(\Omega)} + \int_Q e^{2s\beta} \left(\int_\Omega K(x,\theta,t) y(\theta,t) \, d\theta \right)^2 dxdt \right] \\ &\qquad \leq C \left(\mathcal{M}^2 R^2 + \|y_0\|^2_{L^2(\Omega)} \right) \leq CR^2, \end{aligned}$$

for R large enough. Hence, up to a multiplicative constant we have $\Lambda(\mathcal{E}_R) \subset \mathcal{E}_R$.

Proof of the theorem (cont.)

Let $\{w_k\}$ be a sequence in $\mathcal{E}_R$. Then the corresponding solutions $\{y_k\}$ are bounded in $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and $\Lambda(\mathcal{E}_R)$ is compact in $L^2(Q)$ by **Aubin-Lions' Theorem**.

For any $w \in \mathcal{E}_R$, we have at least one control v such that the corresponding solution y belongs to $\mathcal{E}_R$. Hence, for the sequence $\{w_k\}$ we can find a sequence of controls $\{v_k\}$ such that the corresponding solutions $\{y_k\}$ is in $L^2(Q)$.

Let $w_k \to w$ in $\mathcal{E}_R$ and $y_k \in \Lambda(w_k)$, $y_k \to y$ in $L^2(Q)$. By the regularity of the solutions it follows that

$$\begin{aligned} v_k &\rightharpoonup v \quad \textbf{weakly } \textit{in } L^2(\mathcal{O}\times(0,T)),\\ y_k &\rightharpoonup y \quad \textbf{weakly } \textit{in } L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega)),\\ y_k &\to y \quad \textbf{strongly } \textit{in } L^2(Q). \end{aligned}$$

We obtain $y \in L^2(Q)$ and, letting $k \to +\infty$ in the system

$$\begin{cases} (y_k)_t - \Delta y_k + \displaystyle\int_\Omega K(x,\theta,t)w_k(\theta,t)\,d\theta = v_k \mathbf{1}_{\mathcal{O}}, & (x,t)\in Q\\ y_k = 0, & (x,t)\in\Sigma\\ y_k(x,0) = y_0(x), & x\in\Omega. \end{cases}$$

we can conclude that $\Lambda(w) = y$. Thus the map Λ is **upper hemicontinuous**.

Proof of the theorem (conclusion)

Theorem

Let S be a non-empty, compact and convex subset of the Euclidean space $\mathbb{R}^N$. Let $\phi : S \to 2^S$ be an upper hemicontinuous set-valued function on S with the property that $\phi(x)$ is non-empty, closed, and convex for all $x \in S$. Then ϕ has a fixed point.

S. Kakutani, *A generalization of Brouwers fixed point theorem*, Duke Math. J., 1941.

According from the previous discussion, all the assumptions of Kakutani's fixed point theorem are fulfilled and there is at least one $y \in \mathcal{E}_R$ such that $y = \Lambda(y)$.

By the definition of Λ, there exists at least one pair (u, y) satisfying the conditions of the Theorem.

We have $y(x, T) = 0$ in Ω due to the definition of $\mathcal{E}$ and the weight function β.

UPPER HEMICONTINUITY:

A map $\Lambda : A \to B$ is said to be upper hemicontinuous at a point x if for any open neighborhood V of $\Lambda(x)$ there exists a neighborhood U of x such that, for all $y \in U$, $\Lambda(y) \subset V$.

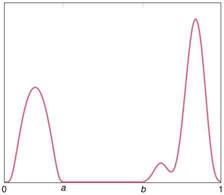
On the necessity of the hypothesis $(\mathcal{H})$

Remark

Hypotheses on the kernel K more general that just being bounded are necessary. Otherwise, it is possible to provide counterexamples to the unique continuation for the solution of the adjoint equation.

Consider a function u with the following properties:

- $u \in C_0^\infty(0,1)$;
- $u(x) = 0$ for $x \in (a,b) \subset (0,1)$;
- $u \not\equiv 0$ in $(0,1)$.



On the necessity of the hypothesis $(\mathcal{H})$

Decomposition in the eigenfunctions of the Laplacian

$$u \in L^2(0,1) \;\Rightarrow\; u(x) = \sum_{k \geq 1} c_k \phi_k(x)$$

with $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ and $c_k = \langle u, \phi_k \rangle_{L^2(0,1)}$

Moreover, for $0 < \lambda < \pi$ and up to a change of variables of the type $u \mapsto \sigma u$, $\sigma > 0$, we have

$$\sum_{k \geq 1} \left(k^2\pi^2 - \lambda^2\right) c_k^2 = 1.$$

$$p(x) = \sum_{k \geq 1} \left(k^2\pi^2 - \lambda^2\right) c_k \phi_k(x).$$

- $-u_{xx} - \lambda^2 u = p$ is verified in the sense of distributions;
- $p \in C_0^\infty(0,1)$ with $p(x) = 0$ in (a,b) (since u has these properties);
- $\int_0^1 pu\,dx = 1$.

Therefore, u satisfies the non-local elliptic problem

$$\begin{cases} -u_{xx} + \displaystyle\int_0^1 K(x,\theta)u(\theta)\,d\theta = \lambda^2 u, & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

with $K(x,\theta) = p(x)p(\theta)$. Furthermore, by assumption $u(x) = 0$ for $x \in (a,b) \subset (0,1)$ but $u \not\equiv 0$ elsewhere.

Extension to semilinear problems

The approach previously presented can be combined with a by now standard methodology to deduce similar controllability results for the semilinear heat equation with globally Lipschitz nonlinearity

$$\begin{cases} y_t - \Delta y + \displaystyle\int_\Omega K(x,\theta,t)y(\theta,t)\,d\theta = f(y) + v\chi\mathbf{1}_{\mathcal{O}}, & (x,t)\in Q \\ y = 0, & (x,t)\in\Sigma \\ y(x,0) = y_0(x), & x\in\Omega. \end{cases} \tag{1}$$

In fact, it is possible to prove the following result.

Theorem

Assume $f \in C^1(\mathbb{R})$ is globally Lipschitz with $f(0) = 0$. Then, given any $y_0 \in L^2(\Omega)$ and $T > 0$ there exists a control function $v \in L^2(\mathcal{O}\times(0,T))$ such that the solution to (1) satisfies $y(x,T) = 0$.

Proof (sketch)

The proof of this result is by now standard and uses well-known results on the controllability of nonlinear systems

C. Fabre, J.-P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A, 1995.

L. A. Fernández and E. Zuazua, *Approximate controllability for the semilinear heat equation involving gradient terms*, J. Optim. Theory Appl., 1999.

E. Fernández-Cara, *Null controllability of the semilinear heat equation*, ESAIM: COCV, 1997.

Proof (sketch)

Since $f \in C^1(\mathbb{R})$, we can introduce the function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(s) := \begin{cases} \dfrac{f(s)}{s}, & \text{if } s \neq 0 \\ f'(0), & \text{if } s = 0. \end{cases}$$

Then, for all $\eta \in L^2(Q)$ we can consider the following linearized version of (1)

$$\begin{cases} y_t - \Delta y + \displaystyle\int_\Omega K(x,\theta,t)y(\theta,t)\,d\theta = g(\eta)y + v\mathbf{1}_{\mathcal{O}}, & (x,t) \in Q \\ y = 0, & (x,t) \in \Sigma \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{2}$$

Proof (sketch)

The continuity of f and the density of $C_0^\infty(Q)$ in $L^2(Q)$ allows to see that $g(\eta) \in L^\infty(Q)$ for all $\eta \in L^2(Q)$. Therefore, arguing as before, we can obtain the following observability estimate

$$\|\varphi(x,0)\|_{L^2(\Omega)} \leq C_1 \exp\left[C_2\left(1+\frac{1}{T}+T\|g\|_\infty+\|g\|_\infty^{\frac{2}{3}}+\mathcal{K}^{\frac{2}{3}}\right)\right]\iint_{\mathcal{O}\times(0,T)}|\varphi|^2\,dxdt,$$

where φ is the solution to the adjoint system associated to (2).

This in particular implies that (2) is null-controllable in time $T>0$ with a control $v_\eta \in L^2(\mathcal{O}\times(0,T))$ satisfying

$$\|v_\eta\|_{L^2(\mathcal{O}\times(0,T))} \leq \sqrt{\mathcal{C}}\,\|y_0\|_{L^2(\Omega)}, \quad \forall\,\eta \in L^2(Q),$$

where with $\mathcal{C}$ we indicate the constant in the above inequality.

Proof (sketch)

Consider the map $\Lambda : L^2(Q) \to L^2(Q)$ defined by $\Lambda\eta = y_\eta$, where y_η is the solution to (2) corresponding to the control v_η.

Due to the regularity of the solution y, we deduce that Λ maps $L^2(Q)$ into a bounded set of $L^2(0, T, H_0^1(\Omega)) \cap H^1(0, T, H^{-1}(\Omega))$.

This space being compactly embedded in $L^2(Q)$, there exists a fixed compact set W such that $\Lambda(L^2(Q)) \subset W$.

Moreover, it can be readily verified that Λ is also continuous from $L^2(Q)$ into $L^2(Q)$.

In view of that, applying the **Schauder fixed point theorem** and proceeding as before the result follows immediately.

Proof (sketch)

It is possible to address also more general versions of (1) in which the nonlinearity is included in the non-local term, that is, problems in the form

$$\begin{cases} y_t - \Delta y + \displaystyle\int_\Omega K(x,\theta,t) f\left[y(\theta,t)\right] d\theta = v\mathbf{1}_{\mathcal{O}}, & (x,t) \in Q \\ y = 0, & (x,t) \in \Sigma \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{3}$$

Proof (sketch)

For doing that, we just need to prove a Carleman estimate for the linearized adjoint system corresponding to (3), which reads as

$$\begin{cases} -\varphi_t - \Delta\varphi + g(\eta) \int_\Omega K(\theta, x, t)\varphi(\theta, t)\, d\theta = 0, & (x,t) \in Q \\ \varphi = 0, & (x,t) \in \Sigma \\ \varphi(x, 0) = \varphi_T(x), & x \in \Omega, \end{cases} \tag{4}$$

to obtain from there an observability inequality for (4) and conclude by following the same argument as before.

Proof (sketch)

The proof of such Carleman inequality is a straightforward adaptation of our previous results. Indeed, it is sufficient to notice that, in this case,

$$\begin{aligned}\mathcal{I}(\varphi) &\leq C\left[s^3\lambda^4\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \iint_Q e^{-2s\alpha}\left|g(\eta)\int_\Omega K(\theta,x,t)\varphi(\theta,t)\,d\theta\right|^2 dx\,dt\right]\\ &\leq C\left[s^3\lambda^4\iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt + \|g\|_\infty^2\iint_Q e^{-2s\alpha}\left|\int_\Omega K(\theta,x,t)\varphi(\theta,t)\,d\theta\right|^2 dx\,dt\right],\end{aligned}$$

since $g(\eta)\in L^\infty(Q)$. From here, the remaining of the proof is the same as we did before, with the only change that we now have to choose

$$s \geq \varrho_3\left[T + T^2 + \left(\mathcal{K}\,\|g\|_\infty\right)^{\frac{2}{3}} T^2\right],$$

with ϱ_3 a positive constant only depending on Ω and $\mathcal{O}$. In view of that, the observability estimate that we obtain is in the form

$$\|\varphi(x,0)\|_{L^2(\Omega)} \leq C_1\exp\left[C_2\left(1+\frac{1}{T}+T\left(\mathcal{K}\,\|g\|_\infty\right)^{\frac{2}{3}}\right)\right]\iint_{\mathcal{O}\times(0,T)}|\varphi|^2\,dxdt. \quad (5)$$

From (5), the null controllability in time $T>0$ for (4) follows immediately by means of a classical argument.

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