Crossover asymptotics and a sharp confinement rate for the viscous Burgers equation

Maicon Sonego[a,*], Enrique Zuazua[b,c,d]

[a] *Instituto de Matemática e Computação, Universidade Federal de Itajubá, Itajubá-MG, Brazil*
[b] *Chair for Dynamics, Control, Machine Learning and Numerics (Alexander von Humboldt Professorship), Department of Mathematics, Friedrich–Alexander-Universität Erlangen–Nürnberg, Cauerstr. 11, 91058, Erlangen, Germany*
[c] *Chair of Computational Mathematics, University of Deusto, 48007, Bilbao, Basque Country, Spain*
[d] *Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, Spain*

Abstract

We study the one-dimensional viscous Burgers equation on $(0, L)$ with the conservative boundary conditions $u_x = -u^2$. On the half-line, solutions approach a nonlinear self-similar profile, whereas on a bounded interval they converge to a nonconstant equilibrium. We describe explicitly how the dynamics passes between these states at the critical diffusive scale $t = cL^2$. For compactly supported initial data of mass M, the rescaled solution converges as $L \to \infty$ to an explicit crossover profile Φ_c. In similarity variables, Φ_c converges to the half-line profile f_M as $c \downarrow 0$; after rescaling to domain variables, it converges to the interval equilibrium as $c \to \infty$. We also determine the sharp onset of confinement:

$$\lim_{c \downarrow 0} -c \log |\Phi_c(0) - f_M(0)| = 1 \qquad (M \neq 0).$$

Moreover, on compact sets in similarity variables, the interval and half-line solutions differ in C^k by at most $C_{k,\varepsilon} \exp(-(1-\varepsilon)L^2/t)$, uniformly in L, and the exponential constant is optimal. Thus we identify not only the transition scale L^2, but also the profile governing the crossover and the sharp rate at which the remote boundary becomes visible. We finally discuss the conclusions that persist for space-dependent diffusivity and illustrate the three asymptotic regimes numerically.

Keywords: viscous Burgers equation, crossover asymptotics, Hopf–Cole transformation, conservative boundary conditions, sharp confinement rate, variable diffusion
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1. Introduction

A fundamental feature of diffusive equations is that the large-time behavior of solutions depends crucially on the geometry of the underlying domain. In unbounded domains, solutions typically spread and converge, after suitable rescaling, to self-similar profiles that are universal up to conserved quantities. In bounded domains, confinement eventually takes over and drives the solution toward a stationary state. When the domain is large but finite, these two mechanisms compete. That the diffusive time $t \sim L^2$ separates the two regimes is classical. Understanding the nature of the transition at this scale is the central question we address.

A simple and instructive example is the linear heat equation with homogeneous Neumann boundary conditions,

$$v_t = v_{xx}, \qquad (x, t) \in (0, L) \times (0, \infty), \qquad v_x(0, t) = v_x(L, t) = 0,$$

[*]Corresponding author.
Email addresses: mcn.sonego@unifei.edu.br (Maicon Sonego), enrique.zuazua@fau.de (Enrique Zuazua)

which conserves mass: $\int_0^L v(x,t)\,dx = M$. On the half-line with Neumann condition at $x = 0$, the solution converges, after rescaling, to the Gaussian heat kernel. On the bounded interval, the solution converges exponentially to the spatial average M/L. Crucially, before the boundary is felt, the solution on $(0, L)$ is exponentially close to the half-line solution for times $t \ll L^2$; only for $t \gg L^2$ does confinement dominate. This two-scale behavior is a manifestation of the classical *principle of not feeling the boundary*, introduced by Kac [20], proved for the heat kernel by Ciesielski [7], and later quantified with an explicit rate by van den Berg [28] (see also Davies [9]): for the heat equation, the influence of the boundary is exponentially small for times $t \ll \operatorname{dist}(x, \partial\Omega)^2$.

Classical stabilization theory for expanding parabolic domains establishes qualitatively that a remote boundary is not felt and that the diffusive scale governs the competition [27, 25, 19, 17]. Here we identify the crossover profile itself, the two nonlinear states it connects, and the sharp confinement exponent.

In this work, we study the viscous Burgers equation with conservative boundary conditions,

$$u_t - u_{xx} = (u^2)_x, \qquad (x,t) \in (0, L) \times (0, \infty),$$

with

$$u_x(0,t) = -u^2(0,t), \qquad u_x(L,t) = -u^2(L,t).$$

Written conservatively the equation reads $u_t - \partial_x(u_x + u^2) = 0$, so the flux is $-(u_x + u^2)$ and these conditions are precisely the statement that it vanishes at the boundaries; mass is therefore conserved. In the literature, such conditions are sometimes referred to as *no-flux boundary conditions* in the context of the conservation-law formulation [29]; they arise naturally in applications such as the complete separation of fluids in confined geometries. Throughout this paper, we adopt the term *conservative* to highlight the mass-conservation property that is central to our analysis.

The conservative structure provides a common parameter: the total mass $M = \int_0^L u(x,t)\,dx$ is conserved in both the bounded-interval and half-line problems. As recalled in Section 2, on the half-line solutions converge in self-similar variables to a non-Gaussian profile f_M parametrized by M, while on the bounded interval they converge exponentially to a non-constant stationary profile U_M^L, also parametrized by M. The question we address is: how does the bounded-interval dynamics pass from one to the other?

Our contribution has three parts. First, we construct an explicit family $\{\Phi_c\}_{c>0}$ governing the simultaneous limit $L \to \infty$, $t = cL^2$. Second, we prove that this family connects the half-line self-similar state f_M to the finite-interval equilibrium $\mathcal{U}_M$ in their respective natural variables. Third, we determine the sharp rate at which the remote boundary becomes visible. More precisely, for compactly supported initial data, and with $M \neq 0$ for the nontrivial lower bound,

$$\lim_{c \to 0^+} -c \ln \left| \Phi_c(0) - f_M(0) \right| = 1,$$

and the corresponding comparison with the half-line problem is

$$\|v^L(\cdot,t) - v^\infty(\cdot,t)\|_{C^k([0,R])} \le C_{k,\varepsilon}\, e^{-(1-\varepsilon)/c}, \qquad t = cL^2,$$

on compact sets in similarity variables, uniformly in L. The optimal constant 1 is the Gaussian cost of the image charge at $x = 2L$. Hence the result resolves the dynamics at the diffusive scale $t \sim L^2$, rather than merely locating the scale at which confinement begins.

The Hopf–Cole transformation [18, 8] maps the equation exactly to a heat equation with constant Dirichlet data. The boundary-shadowing mechanism is therefore linear, but the states it connects are nonlinear: f_M is non-Gaussian, $\mathcal{U}_M$ is nonconstant, and Φ_c deforms one into the other. Recovering the result for $u = w_x/w$ also requires uniform control of this quotient and its derivatives on the region $x = O(\sqrt{t})$.

Two earlier works address the Burgers equation on a bounded interval and deserve explicit comparison. Kreiss and Kreiss [22] prove convergence to a unique steady state for the initial–boundary value problem and show that the rate of convergence depends on the boundary conditions and may be exponentially slow; Bertini and Ponsiglione [5] characterize the stationary solutions variationally, through a Lyapunov functional arising as the large-deviation rate of an associated interacting particle system, and analyze its behavior as the interval length diverges. Both concern the stationary regime $t \gg L^2$ and inhomogeneous Dirichlet data. The conservative boundary conditions considered here produce, after Hopf–Cole, the specific Dirichlet data $w(0,t) = 1$, $w(L,t) = e^M$, so that the mass M is conserved and parametrizes the stationary profile U_M^L; what is new here is not the stationary regime but the diffusive scale $t \sim L^2$ and the explicit family linking f_M and $\mathcal{U}_M$ in their respective natural coordinates.

The distinction from the closest results can be summarized as follows. The classical "not feeling the boundary" literature [20, 7, 28, 9] estimates linear heat kernels or solutions locally before a boundary is reached; it does not identify the nonlinear profile obtained when t/L^2 has a nonzero limit. Watanabe et al. [29] provide well-posedness and large-time information for the no-flux Burgers problem, while Biler–Karch [6] study half-line asymptotics for a Neumann problem on $(0,\infty)$, in spirit close to Section 2; what is specific to the conservative condition is the explicit form of the profile f_M. To the best of our knowledge, none of these results combines the simultaneous limit $L \to \infty$, $t = cL^2$, the explicit crossover family, and a matching sharp finite-domain estimate. These are precisely the contents of Theorem 1.1.

The transient lasts for the natural diffusive time $t \sim L^2$ and is an instance of Barenblatt's intermediate asymptotics [3, 2]. It is distinct from the small-viscosity metastability and exponentially slow shock motion studied in [21, 4, 26, 22, 24]; see Remark 3.18.

The following statement gathers the three limits and the uniform comparison estimate proved separately in Section 3.

Theorem 1.1 (Main crossover theorem). *Let $u_0 \in L^1(0,\infty)$ be supported in $[0,R_0]$, with mass M, and let u^L and u^∞ be the corresponding Hopf–Cole solutions on $(0,L)$ and $(0,\infty)$. Set*

$$v^L(\xi,t) = \sqrt{t}\, u^L(\xi\sqrt{t},t), \qquad v^\infty(\xi,t) = \sqrt{t}\, u^\infty(\xi\sqrt{t},t).$$

There is an explicit family $\{\Phi_c\}_{c>0}$, defined in (25), such that:

1. *for every fixed $c > 0$, $k \geq 0$, and $R < c^{-1/2}$,*
$$\|v^L(\cdot, cL^2) - \Phi_c\|_{C^k([0,R])} = O(L^{-1});$$
2. *for every fixed $R > 0$ and $k \geq 0$, $\Phi_c \to f_M$ in $C^k([0,R])$ as $c \downarrow 0$, and, if $M \neq 0$,*
$$\lim_{c\downarrow 0} -c \log |\Phi_c(0) - f_M(0)| = 1;$$
3. *in domain variables,*
$$\frac{1}{\sqrt{c}}\Phi_c\left(\frac{y}{\sqrt{c}}\right) \longrightarrow \mathcal{U}_M(y) \quad \textit{uniformly for } y \in [0,1] \quad (c \to \infty);$$
4. *for every $R > 0$, $k \geq 0$, and $\varepsilon \in (0,1)$, there are $C_{k,\varepsilon}, L_0 > 0$, independent of c and L, such that, whenever $0 < c < R^{-2}$, $t = cL^2$, and $L \geq L_0$,*
$$\|v^L(\cdot,t) - v^\infty(\cdot,t)\|_{C^k([0,R])} \leq C_{k,\varepsilon} e^{-(1-\varepsilon)/c}.$$

   The exponential constant 1 is optimal when $M \neq 0$.

Figure 1 illustrates the three windows and the role of Φ_c in the middle one.

Theorem 1.1 is a uniform singular-limit statement: in domain variables the datum occupies a layer of width $O(L^{-1})$, the observation time is cL^2, and the nonlinear observable is w_x/w. Section 2 is preparatory and essentially known: Theorems 2.1 and 2.2 are the L^1 version of results proved in

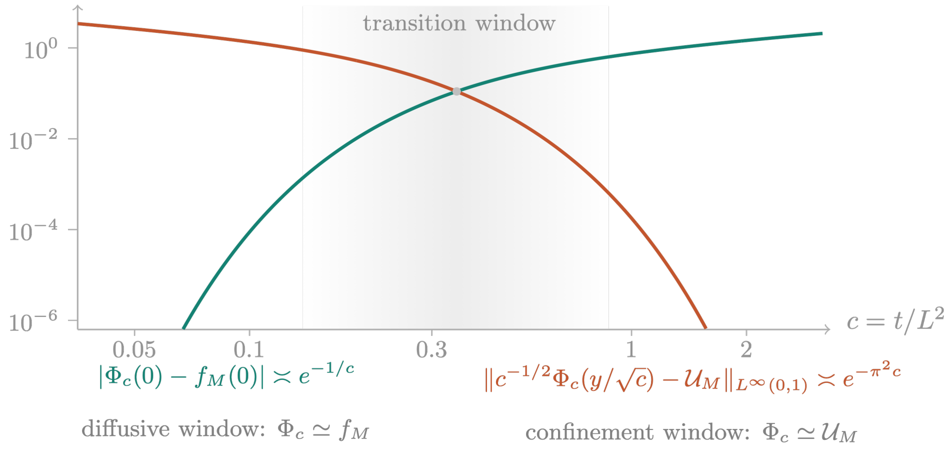

Figure 1: Illustration of Theorem 1.1 at $M = 1$: the deviation of Φ_c from each of the two states it degenerates to, evaluated from the closed form (25) and so free of discretisation error. On the left, the deviation from the self-similar profile at the origin, in similarity variables, obeying the exact asymptotics (36); on the right, the deviation from the interval equilibrium in domain variables, at the spectral rate $e^{-\pi^2 c}$ of Proposition 3.4. The two are measured in different coordinates because they are taken in different coordinates (Remark 3.5), and the uniform-in-L comparison carries the exponent of the left curve. Each deviation is small exactly on the window where the corresponding description is valid; the two cross near $c \approx 0.35$, and in between only Φ_c describes the solution. The shading marks the transition window, in which there is no threshold, the passage between regimes being continuous (Remark 3.17).

[29] for C^2 data, and the half-line analysis is close to [6]. It is included because the whole paper is read off the Hopf–Cole reduction recalled there.

Section 3 proves Theorem 1.1. Section 4 asks how much survives when the diffusivity depends on x, $u_t = \partial_x\big(a(x)(u_x + u^2)\big)$. Hopf–Cole still linearizes and the picture separates: the stationary profile U_M^L is unchanged and the relaxation rate is governed by the principal eigenvalue $\lambda_1(a)$, but the explicit crossover is lost, the family Φ_c and the sharp constant resting on the closed-form image representation available only for constant coefficients; the half-line regime there is close in spirit to Duro and Zuazua [10]. Section 5 gives numerical illustrations, and Section 6 closes with a brief outlook.

2. Existence, uniqueness, and large-time behavior

This section collects, for completeness, the well-posedness and large-time facts used in Section 3. Nothing here is essentially new: Theorems 2.1 and 2.2 are the L^1 version of results proved in [29] for C^2 data, and Section 2.3 is close in spirit to [6], what is specific to the conservative condition being the explicit form of the profile f_M. Proofs are included because the whole paper is read off the Hopf–Cole reduction recalled here.

Throughout, a *Hopf–Cole solution* of (1) or (9) means a function $u = w_x/w$, where w is the classical solution of the associated linear problem and $u(\cdot,t) \to u_0$ in L^1 as $t \downarrow 0$. All uniqueness statements below refer to this class. We use this terminology explicitly to avoid asserting uniqueness in a larger, unspecified class of weak solutions; the transition analysis requires precisely the Hopf–Cole solution and its heat kernel estimates.

2.1. Large-time behavior in $(0, L)$

We consider the Burgers equation posed on a bounded interval:

$$\begin{cases} u_t - u_{xx} = (u^2)_x, \quad & (x,t) \in (0,L) \times (0,\infty) \\ u_x(0,t) = -u^2(0,t), \quad u_x(L,t) = -u^2(L,t), \quad & t \in (0,\infty) \\ u(x,0) = u_0(x), \quad x \in (0,L). & \end{cases} \tag{1}$$

As already mentioned, (1) reads $u_t - \partial_x(u_x + u^2) = 0$, so the flux is $-(u_x + u^2)$ and the boundary conditions are precisely the statement that it vanishes at $x = 0, L$.

Theorem 2.1. *Let $u_0 \in L^1(0,L)$ have mass M. Then problem (1) admits a unique Hopf–Cole solution, and it preserves the mass,*

$$\int_0^L u(x,t)\,dx = M, \qquad t > 0.$$

Proof. Set $v(x,t) = \int_0^x u(z,t)\,dz$, so that $u = v_x$ and $v(0,t) = 0$. Since (1) in conservative form reads $u_t = \partial_x(u_x + u^2)$, for every $t > 0$

$$\partial_t v(L,t) = \int_0^L u_t\,dx = \int_0^L \partial_x(u_x + u^2)\,dx = \big[u_x + u^2\big]_{x=0}^{x=L} = 0,$$

the last equality being exactly the boundary conditions of (1); hence $v(L,t) = v(L,0) = M$ for all $t > 0$. Differentiating $v = \int_0^x u$ and using (1) shows that v then satisfies

$$\begin{cases} v_t - v_{xx} = (v_x)^2, & (x,t) \in (0,L) \times (0,\infty), \\ v(0,t) = 0, \qquad v(L,t) = M, & t > 0, \\ v(x,0) = \displaystyle\int_0^x u_0(z)\,dz, & x \in (0,L). \end{cases} \tag{2}$$

Conversely, if v solves (2), then $v(L,\cdot) \equiv M$ gives $v_t(L,t) = 0$, and evaluating $v_t - v_{xx} = (v_x)^2$ at $x = L$ yields $v_{xx}(L,t) = -(v_x(L,t))^2$, i.e. $u_x(L,t) = -u^2(L,t)$; likewise at $x = 0$. Thus the nonlinear Neumann conditions of (1) and the Dirichlet problem (2) are equivalent, and mass conservation is a consequence rather than an assumption.

Applying the Hopf–Cole transformation $w = e^v$,

$$\begin{cases} w_t - w_{xx} = 0, & (x,t) \in (0,L) \times (0,\infty) \\ w(0,t) = 1, \quad w(L,t) = e^M, & t > 0, \\ w(x,0) = w_0(x) := \exp\Big(\displaystyle\int_0^x u_0(z)\,dz\Big), & x \in (0,L). \end{cases} \tag{3}$$

Since $u_0 \in L^1(0,L)$, the datum $v(\cdot,0)$ is absolutely continuous, so w_0 is continuous on $[0,L]$, strictly positive, and compatible with the boundary data: $w_0(0) = 1$ and $w_0(L) = e^M$. Subtracting the affine steady state $W(x) = 1 + (e^M - 1)x/L$ reduces (3) to the heat equation with homogeneous Dirichlet conditions and continuous datum $w_0 - W$ vanishing at both endpoints, for which existence and uniqueness of a classical solution, continuous up to $t = 0$, are standard. The maximum principle gives $w > 0$; more precisely $0 < m_0 \le w \le M_0 < \infty$ on $[0,L] \times [0,\infty)$ with the explicit constants of (7) below, and the same bounds hold for w_0. In particular $u := w_x/w$ is well defined and smooth for $t > 0$.

It remains to identify the initial trace. Since $u_0 \in L^1(0,L)$, the datum w_0 is absolutely continuous with $w_0' = w_0 u_0 \in L^1(0,L)$. Let $h(\cdot,t)$ solve the heat equation on $(0,L)$ with homogeneous Neumann conditions and datum w_0'. Then $\tilde{w}(x,t) := 1 + \int_0^x h(z,t)\,dz$ satisfies $\tilde{w}_t = h_x = \tilde{w}_{xx}$, has $\tilde{w}(0,t) = 1$ and, the Neumann semigroup preserving the integral, $\tilde{w}(L,t) = w_0(L) = e^M$, with $\tilde{w}(\cdot,0) = w_0$; so $\tilde{w}$ solves (3) and, by the uniqueness just recalled, $w = \tilde{w}$ and $w_x = h$. That semigroup being strongly continuous on $L^1(0,L)$, one has $w_x(\cdot,t) \to w_0 u_0$ in $L^1(0,L)$ and $w(\cdot,t) \to w_0$ uniformly on $[0,L]$. Writing $u(\cdot,t) - u_0 = (w_x - w_0 u_0)/w + w_0 u_0\big(w^{-1} - w_0^{-1}\big)$ and using $w \ge m_0 > 0$, both terms tend to 0 in L^1 as $t \to 0^+$, so $u(\cdot,t) \to u_0$ in $L^1(0,L)$.

This is the unique solution in the class fixed above, since the correspondence $u \leftrightarrow v \leftrightarrow w$ is one-to-one and w is unique.

Mass conservation is immediate from (2): for every $t > 0$,

$$\int_0^L u(x,t)\,dx = v(L,t) - v(0,t) = M.$$

□

When the interval length varies, we write the stationary solution of (1) with mass M as

$$U_M^L(x) = \frac{e^M - 1}{x(e^M - 1) + L}. \tag{4}$$

Its unit-interval rescaling is

$$\mathcal{U}_M(y) := L U_M^L(Ly) = \frac{e^M - 1}{1 + (e^M - 1)y}, \qquad 0 \le y \le 1. \tag{5}$$

The stationary solution is unique, and the uniqueness is transparent at the linear level: a stationary w solves $w_{xx} = 0$ with $w(0) = 1$, $w(L) = e^M$, whose only solution is the affine profile $W(x) = 1 + (e^M - 1)x/L > 0$. Through the bijection $u \leftrightarrow v \leftrightarrow w$ this determines the stationary states at the other two levels uniquely, $V = \log W$ and

$$U_M^L(x) = \frac{W_x}{W} = \frac{e^M - 1}{L + (e^M - 1)x},$$

which is (4). In particular U_M^L is the only possible equilibrium of (1) in the mass class M: since the mass is conserved along the evolution, every trajectory issuing from a datum of mass M can relax to no steady state other than U_M^L, which is what makes the long-time limit unambiguous.

Theorem 2.2. *Let $u_0 \in L^1(0, L)$ have mass M, let u be the corresponding solution of* (1)*, and let $\lambda_1 = \pi^2/L^2$ be the first Dirichlet eigenvalue of* $(0, L)$*. Then, for every $1 \le p \le \infty$ there is $C = C(\|u_0\|_{L^1}, L, p) > 0$ such that*

$$\|u(\cdot, t) - U_M^L\|_{L^p(0,L)} \le C\left(1 + t^{-3/4}\right) e^{-\lambda_1 t}, \qquad t > 0. \tag{6}$$

Remark 2.3. *The prefactor cannot be dispensed with: for $u_0(x) = x^{-1/2}$ on $(0, 1)$, which is admissible ($\|u_0\|_{L^1} = 2$, $w_0 = e^{2\sqrt{x}}$), one has $w_0 \notin H^1$, hence $\|u(\cdot, t)\|_{L^2} \to \infty$ as $t \to 0^+$ while $U_M^L \in L^\infty$, so no bound $Ce^{-\lambda_1 t}$ can hold up to $t = 0$ for $p > 1$. Note also that $\|U_M^L\|_{L^\infty} \sim 1/L$ and $\|u(\cdot, t)\|_{L^\infty} \lesssim t^{-1/2}$, so at $t \sim L^2$ both sides of* (6) *are $O(1/L)$; the estimate is informative in a relative sense only for $t \gg L^2$, which is the regime in which it is used (Remark 3.17(3)).*

Proof. With v and w as in Theorem 2.1, w solves (3) and $u = w_x/w$; the stationary profile is $U_M^L = W_x/W$ with $W(x) = 1 + (e^M - 1)x/L$. Set $z := w - W$, which solves the heat equation on $(0, L)$ with homogeneous Dirichlet conditions and datum $z(\cdot, 0) = w_0 - W$. By (3) and the maximum principle,

$$0 < m_0 := \min\{1, e^M\} \wedge e^{-\|u_0\|_{L^1}} \le w \le e^{\|u_0\|_{L^1}} \vee \max\{1, e^M\}, \tag{7}$$

and the same bounds hold for W. Since $z(\cdot, 0) \in L^\infty(0, L) \subset L^2(0, L)$, the smoothing of the Dirichlet heat semigroup together with its spectral gap λ_1 gives, for $j = 0, 1$,

$$\|\partial_x^j z(\cdot, t)\|_{L^\infty(0,L)} \le C\left(1 + t^{-(2j+1)/4}\right) e^{-\lambda_1 t}\, \|z(\cdot, 0)\|_{L^2(0,L)}, \qquad t > 0, \tag{8}$$

with $C = C(L)$. The two regimes are treated separately: for $0 < t \le 1$ the factor $t^{-(2j+1)/4}$ is the standard $L^2 \to W^{j,\infty}$ smoothing of the Dirichlet semigroup, while for $t \ge 1$ one first evolves for one unit of time and then applies the spectral gap on $[1, t]$, which gives the bound with a constant prefactor. A factor $t^{-(2j+1)/4}$ alone would fail for large t, as the eigenfunction $z(\cdot, 0) = \sin(\pi \cdot /L)$ shows. Finally,

$$u - U_M^L = \frac{w_x}{w} - \frac{W_x}{W} = \frac{z_x}{w} - \frac{W_x}{wW}\, z,$$

so by (7) and $\|W\|_{C^1} \le C(L)$,

$$|u - U_M^L| \le C\big(|z_x| + |z|\big).$$

Combining with (8) and $\|\cdot\|_{L^p(0,L)} \le L^{1/p}\|\cdot\|_{L^\infty(0,L)}$ gives (6) for every $p \in [1, \infty]$; here the exponent $3/4$ is the value furnished by the $j = 1$ smoothing estimate in (8). ☐

2.2. Large-time behavior in $(0,\infty)$

We now consider the same problem on the half-line:

$$\begin{cases} u_t - u_{xx} = (u^2)_x, & (x,t) \in (0,\infty)\times(0,\infty) \\ u_x(0,t) = -u^2(0,t), & t \in (0,\infty) \\ u(x,0) = u_0(x) \in L^1(0,\infty). \end{cases} \tag{9}$$

The results of this subsection extend to the half-line the classical theory of large-time behavior for convection-diffusion equations on the whole space [11, 12, 13]; as previously mentioned, the half-line case with a Neumann condition was studied by Biler and Karch [6]. The systematic study of convergence to steady states for parabolic equations was initiated by Friedman [15, 16].

Theorem 2.4. *Let $u_0 \in L^1(0,\infty)$ have mass M. Then problem (9) admits a unique Hopf–Cole solution, it preserves the mass, and for every $p \geq 1$,*

$$\|u(\cdot,t)\|_{L^p} \leq C\, t^{-\frac{1}{2}(1-\frac{1}{p})}, \qquad t > 0.$$

Proof. As in Theorem 2.1, set $w = \exp\left(\int_0^x u\right)$, so that $u = w_x/w$ and w satisfies

$$\begin{cases} w_t - w_{xx} = 0, & (x,t) \in (0,\infty)\times(0,\infty), \\ w(0,t) = 1, & t > 0, \\ w(x,0) = w_0(x) := \exp\left(\displaystyle\int_0^x u_0(z)\,dz\right), & x > 0. \end{cases}$$

The datum w_0 is continuous, strictly positive, bounded, and $w_0(0) = 1$; existence, uniqueness and positivity follow as before [14, 23], and the initial trace $u(\cdot,t) \to u_0$ in $L^1(0,\infty)$ is obtained exactly as in Theorem 2.1, with the Neumann semigroup of $(0,L)$ replaced by its half-line counterpart.

For the mass, set $h = w - 1$, which has homogeneous Dirichlet boundary data. Then $h(x,t) = \int_0^\infty G_D(t,x,y)(w_0(y)-1)\,dy$. Equivalently,

$$w(x,t) - e^M = \int_0^\infty G_D(t,x,y)(w_0(y) - e^M)\,dy + (e^M - 1)\left(\int_0^\infty G_D(t,x,y)\,dy - 1\right).$$

Both terms tend to zero as $x \to \infty$. Indeed, for any $\varepsilon > 0$ choose A with $|w_0 - e^M| \leq \varepsilon$ on $[A,\infty)$; for $x \geq 2A$ the contribution of $[0,A]$ to $\int G_D(t,x,y)(w_0(y) - e^M)\,dy$ is $O(e^{-x^2/(16t)})$ and the rest is $\leq \varepsilon$, whence $\lim_{x\to\infty} w(x,t) = e^M$ for every $t > 0$; the boundary term tends to zero by the same kernel formula. Since $w(0,t) = 1$,

$$\int_0^\infty u(x,t)\,dx = \ln w(\infty,t) - \ln w(0,t) = M, \qquad t > 0.$$

For the decay, set $h := w_x$. Differentiating the Dirichlet condition $w(0,t) = 1$ in t and using the equation gives $w_{xx}(0,t) = 0$, that is $h_x(0,t) = 0$; equivalently, since $h_x = w(u_x + u^2)$, this is the conservative condition $u_x(0,t) = -u^2(0,t)$. Thus h solves the heat equation on $(0,\infty)$ with a homogeneous Neumann condition at $x = 0$. Extending h evenly across $x = 0$ yields a solution on $\mathbb{R}$, to which the standard heat semigroup estimates apply:

$$\|h(\cdot,t)\|_{L^p(0,\infty)} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}\|h(\cdot,0)\|_{L^1(0,\infty)}, \qquad 1 \leq p \leq \infty.$$

Since $h(x,0) = w_0(x)u_0(x)$ and $e^{-\|u_0\|_{L^1}} \leq w_0 \leq e^{\|u_0\|_{L^1}}$, we get $\|h(\cdot,0)\|_{L^1} \leq C\|u_0\|_{L^1}$; note also $\int_0^\infty h(\cdot,0) = w_0(\infty) - w_0(0) = e^M - 1$. The maximum principle gives $0 < c_0 \leq w \leq C_0$ at positive times, so $\|u(\cdot,t)\|_{L^p} \leq C\|h(\cdot,t)\|_{L^p}$, which is the claim. □

2.3. Self-similar asymptotic behavior

Equation (9) is invariant under $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, and solutions invariant under this scaling have the form $u(x,t) = t^{-1/2} f(x/\sqrt{t})$. We characterize the profiles f explicitly.

Proposition 2.5. *For every $M \in \mathbb{R}$ there is $f_M \in H^1(0,\infty)$ with $\int_0^\infty f_M = M$ such that*

$$u_M(x,t) = t^{-1/2} f_M\left(\frac{x}{\sqrt{t}}\right)$$

solves

$$u_t - u_{xx} = (u^2)_x \quad \text{in } (0,\infty)\times(0,\infty), \qquad u_x(0,t) = -u^2(0,t), \quad t>0. \tag{10}$$

Explicitly,

$$f_M(\xi) = \frac{(e^M - 1)\, b(\xi)}{1 + (e^M - 1)\displaystyle\int_0^\xi b}, \qquad b(\xi) = \pi^{-1/2} e^{-\xi^2/4},$$

with $f_0 \equiv 0$. Moreover $u_M(\cdot,t) \rightharpoonup M\delta_0$ weakly-$*$ as $t \to 0^+$.

Proof. Substituting $u(x,t) = t^{-1/2} f(\xi)$, $\xi = x/\sqrt{t}$, the profile must satisfy

$$-f'' - \frac{\xi}{2} f' - \frac{1}{2} f = (f^2)', \qquad \xi > 0, \qquad \int_0^\infty f = M. \tag{11}$$

Set $g(\xi) = \exp\bigl(\int_0^\xi f\bigr)$, so $g>0$, $g(0)=1$, $g(\infty) = e^M$ and $f = g'/g$. A direct computation gives

$$-g'' - \frac{\xi}{2} g' = -\Bigl(f' + f^2 + \frac{\xi}{2} f\Bigr) g,$$

while integrating (11) once yields $f' + f^2 + \frac{\xi}{2} f = \kappa$ with $\kappa = f'(0) + f(0)^2$, so that

$$-g'' - \frac{\xi}{2} g' = -\kappa g.$$

Here $\kappa = 0$: it is exactly the conservative condition $f'(0) = -f(0)^2$, and it is forced at infinity. Integrating $(e^{\xi^2/4} g')' = \kappa\, e^{\xi^2/4} g$ gives $e^{\xi^2/4} g'(\xi) = g'(0) + \kappa \int_0^\xi e^{s^2/4} g$. If $\kappa \neq 0$ and g were bounded with $g \to e^M$, the right-hand side would give $g'(\xi) \sim 2\kappa e^M/\xi$, hence $g(\xi) \sim 2\kappa e^M \log\xi \to \pm\infty$, contradicting $g(\infty) = e^M$. Thus $\kappa = 0$.

Hence $-g'' - \frac{\xi}{2} g' = 0$ with $g(0)=1$, $g(\infty) = e^M$. Its solution is

$$g(\xi) = 1 + (e^M - 1)\int_0^\xi b, \qquad b(\xi) = \pi^{-1/2} e^{-\xi^2/4},$$

since $\int_0^\infty b = 1$; here b is the unit-mass self-similar profile of the heat equation, characterized by $-b'' - \frac{\xi}{2} b' - \frac{1}{2} b = 0$. Conversely, defining g by this formula and $f_M := g'/g$, one checks that $g>0$: as $\int_0^\xi b$ increases from 0 to 1, g ranges monotonically between 1 and e^M, so $g \geq \min(1, e^M) > 0$ for every real M. Reversing the computation above shows that f_M solves (11), and $\int_0^\infty f_M = \ln g(\infty) - \ln g(0) = M$. For $M=0$ the formula gives $g \equiv 1$ and $f_0 \equiv 0$, consistently.

Finally, for $\varphi \in C_c([0,\infty))$, the change of variable $x = \xi\sqrt{t}$ gives $\int_0^\infty u_M(x,t)\varphi(x)\,dx = \int_0^\infty f_M(\xi)\varphi(\xi\sqrt{t})\,d\xi \to \varphi(0)\int_0^\infty f_M = M\varphi(0)$ as $t \to 0^+$, by dominated convergence, since $f_M \in L^1$. $\square$

Theorem 2.6. *Let $u_0 \in L^1(0,\infty)$ with mass M, let u solve (9) and let u_M be as in Proposition 2.5. Then*

$$t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \|u(\cdot,t) - u_M(\cdot,t)\|_{L^p(0,\infty)} \longrightarrow 0, \qquad t \to \infty, \tag{12}$$

for all $1 \leq p \leq \infty$.

Proof. Let w be the Hopf–Cole transform of u and set $w_M(x,t) = 1 + (e^M - 1)\int_0^{x/\sqrt{t}} b$, so that $(w_M)_x = (e^M - 1)t^{-1/2}b(x/\sqrt{t})$ and $u_M = (w_M)_x/w_M$. As in the proof of Theorem 2.4, $h := w_x$ solves the heat equation on $(0,\infty)$ with $h_x(0,t) = 0$ and $\int_0^\infty h(\cdot,0) = e^M - 1$; its even extension across $x = 0$ solves the heat equation on $\mathbb{R}$ with mass $2(e^M - 1)$, so the classical whole-line asymptotics give convergence, after rescaling, to $2(e^M - 1)$ times the unit Gaussian, which restricted to $x > 0$ is exactly $(w_M)_x$. Hence $t^{\frac{1}{2}(1-\frac{1}{p})}\|w_x(\cdot,t) - (w_M)_x(\cdot,t)\|_{L^p(0,\infty)} \to 0$ for $1 \le p \le \infty$. By the maximum principle $0 < c \le w, w_M \le C$ uniformly, and

$$u - u_M = \frac{w_x - (w_M)_x}{w} - \frac{(w_M)_x}{w\,w_M}\,(w - w_M).$$

The first term is handled by the estimate just established. For the second, $w - w_M = \int_0^x (w_y - (w_M)_y)$ gives $\|w(\cdot,t) - w_M(\cdot,t)\|_{L^\infty} \le \|w_x - (w_M)_x\|_{L^1} \to 0$, while $t^{\frac{1}{2}(1-\frac{1}{p})}(w_M)_x$ is bounded in L^p. This is (12). □

Theorem 2.6 gives no rate. For compactly supported data the next two results first establish the $t^{-1/2}$ rate in C^0 and then propagate the same rate to every local C^k norm, as required in Section 3.

Theorem 2.7. *Let $u_0 \in L^1(0,\infty)$ with mass M and $\operatorname{supp} u_0 \subset [0,R_0]$, and let $v^\infty(x,t) = t^{1/2}u(x\sqrt{t},t)$ be the rescaled solution of* (9)*. Then, for every $R > 0$ there is $C = C(\|u_0\|_{L^1}, R_0, R) > 0$ such that*

$$\|v^\infty(\cdot,t) - f_M\|_{L^\infty(0,R)} \le C\,t^{-1/2}, \qquad t \ge 1. \tag{13}$$

Proof. Let w and w_M be the Hopf–Cole transforms of u and of the self-similar solution u_M, and set $d := w - w_M$. Then d solves the homogeneous Dirichlet heat equation with initial datum $d_0(x) = w_0(x) - e^M$ on $x > 0$, which is compactly supported. Its odd extension belongs to $L^1(\mathbb{R})$. Standard heat-semigroup derivative estimates applied to that extension give

$$\|w_x(\cdot,t) - (w_M)_x(\cdot,t)\|_{L^1(0,\infty)} = O(t^{-1/2}), \qquad \|w_x(\cdot,t) - (w_M)_x(\cdot,t)\|_{L^\infty(0,\infty)} = O(t^{-1}),$$

the constants depending only on $\|u_0\|_{L^1}$ and R_0. As in the proof of Theorem 2.6, $0 < c \le w, w_M \le C$ uniformly, and

$$u - u_M = \frac{w_x - (w_M)_x}{w} - \frac{(w_M)_x}{w\,w_M}\,(w - w_M),$$

where $\|w(\cdot,t) - w_M(\cdot,t)\|_{L^\infty} \le \|w_x - (w_M)_x\|_{L^1} = O(t^{-1/2})$ and $|(w_M)_x(x,t)| \le Ct^{-1/2}$. Hence $|u - u_M| \le Ct^{-1}$ for $0 \le x \le R\sqrt{t}$, and multiplying by $t^{1/2}$ gives (13). □

Remark 2.8. *The exponent in* (13) *is not claimed to be optimal; numerically one observes a faster rate on compact sets in similarity variables. Theorem 2.9 below shows that the same $t^{-1/2}$ bound holds in every local C^k norm, which is the quantitative form used later.*

Theorem 2.9. *Under the hypotheses of Theorem 2.7, for every $k \in \mathbb{N}_0$ and every $R > 0$, there is $C_k > 0$ such that*

$$\|v^\infty(\cdot,t) - f_M\|_{C^k([0,R])} \le C_k t^{-1/2}, \qquad t \ge 1.$$

Proof. Use the notation of Theorem 2.7 and put $d = w - w_M$. Its initial datum $d_0 = w_0 - e^M$ is supported in $[0,R_0]$; its odd extension belongs to $L^1(\mathbb{R})$. Direct differentiation of the whole-line heat kernel therefore gives, for every $m \ge 0$,

$$\|\partial_x^m d(\cdot,t)\|_{L^\infty(0,\infty)} \le C_m t^{-(m+1)/2}, \qquad t \ge 1.$$

The same kernel calculation gives $|\partial_x^m w| + |\partial_x^m w_M| \le C_m t^{-m/2}$ for $m \ge 1$, while w, w_M and their reciprocals are uniformly bounded. Since

$$u - u_M = \frac{d_x}{w} - \frac{(w_M)_x}{ww_M}\,d,$$

the Leibniz rule and the derivative formula for a reciprocal show

$$\|\partial_x^k(u-u_M)(\cdot,t)\|_{L^\infty(0,\infty)} \le C_k t^{-(k+2)/2}.$$

Finally, $\partial_\xi^k(v^\infty - f_M) = t^{(k+1)/2}\partial_x^k(u-u_M)$ at $x=\xi\sqrt{t}$. Multiplication by $t^{(k+1)/2}$ proves the asserted $t^{-1/2}$ rate, uniformly on $[0,R]$ (indeed, on the entire half-line). □

3. The diffusive scale: transition profiles and sharp rates

Throughout this section u^L and u^∞ denote the solutions of (1) and (9) with the same initial datum $u_0 \in L^1(0,\infty)$, assumed to satisfy

$$\operatorname{supp} u_0 \subset [0,R_0], \qquad M := \int_0^\infty u_0(x)\,dx, \tag{14}$$

and $w^L = \exp\big(\int_0^x u^L\big)$, $w^\infty = \exp\big(\int_0^x u^\infty\big)$ are their Hopf–Cole transforms. Thus w^L solves the heat equation on $(0,L)$ with $w^L(0,t)=1$ and $w^L(L,t)=e^M$, while w^∞ solves the heat equation on $(0,\infty)$ with $w^\infty(0,t)=1$; both have initial datum $w_0=\exp\big(\int_0^x u_0\big)$, which equals e^M on $[R_0,\infty)$ and satisfies

$$e^{-\|u_0\|_{L^1}} \le w_0 \le e^{\|u_0\|_{L^1}}, \qquad |M| \le \|u_0\|_{L^1}. \tag{15}$$

Unless stated otherwise, constants in this section may depend on $\|u_0\|_{L^1}$, R_0, the observation radius R, the derivative order k, and an indicated loss ε, but never on L, t, or $c=t/L^2$. Dependence on a fixed value of c is displayed explicitly in the transition limit of Subsection 3.1. This convention is important only for the estimates; all limiting statements specify which parameters are held fixed.

At diffusive times $t=cL^2$ we use the similarity variable $\xi = x/\sqrt{t}$ and the rescaled profiles

$$v^L(\xi,t)=\sqrt{t}\,u^L(\xi\sqrt{t},t), \qquad v^\infty(\xi,t)=\sqrt{t}\,u^\infty(\xi\sqrt{t},t), \tag{16}$$

the first being defined for $0\le\xi\le L/\sqrt{t}=c^{-1/2}$.

Section 3.1 produces an explicit one-parameter family $\{\Phi_c\}_{c>0}$, the $L\to\infty$ limit of $v^L(\cdot,cL^2)$ for each fixed $c>0$. In similarity coordinates it degenerates to the self-similar profile f_M as $c\to0$, while after the domain-scale rescaling $y=\xi\sqrt{c}$ it degenerates to the unit-interval stationary profile $\mathcal{U}_M$ as $c\to\infty$. Section 3.2 determines the sharp exponential scale of the first degeneration, with a matching lower bound. Section 3.3 establishes the corresponding uniform-in-L comparison of v^L with v^∞ in C^k, with the same exponent. Section 3.4 describes the dynamics at all time scales.

3.1. The transition family

Fix $c>0$ and set

$$y=\frac{x}{L}, \qquad \tau=\frac{t}{L^2}, \qquad \hat{w}^L(y,\tau) := w^L(yL,\tau L^2), \tag{17}$$

so that $\hat{w}^L$ solves

$$\begin{cases} \hat{w}_\tau = \hat{w}_{yy}, & (y,\tau)\in(0,1)\times(0,\infty),\\ \hat{w}(0,\tau)=1, \quad \hat{w}(1,\tau)=e^M, & \tau>0,\\ \hat{w}(y,0)=w_0(yL), & y\in(0,1). \end{cases} \tag{18}$$

By (14) the initial datum of (18) equals e^M on $[R_0/L,1]$; as $L\to\infty$ it converges to the constant e^M, the discrepancy being confined to a layer of width R_0/L at the origin. Let φ solve the limit problem

$$\begin{cases} \varphi_\tau = \varphi_{yy}, & (y,\tau)\in(0,1)\times(0,\infty),\\ \varphi(0,\tau)=0, \quad \varphi(1,\tau)=1, & \tau>0,\\ \varphi(y,0)=1, & y\in(0,1), \end{cases} \tag{19}$$

and set

$$\Psi(y,\tau) := 1 + (e^M - 1)\varphi(y,\tau), \tag{20}$$

which solves the heat equation on $(0,1)$ with $\Psi(0,\tau) = 1$, $\Psi(1,\tau) = e^M$ and $\Psi(\cdot,0) \equiv e^M$. Since $\varphi - y$ solves the heat equation with homogeneous Dirichlet conditions and initial datum $1 - y$, whose coefficients in the basis $\{\sin(n\pi y)\}_{n\geq 1}$ are $2/(n\pi)$,

$$\varphi(y,\tau) = y + \frac{2}{\pi}\sum_{n\geq 1}\frac{\sin(n\pi y)}{n}\, e^{-n^2\pi^2\tau}, \qquad \varphi_y(y,\tau) = 1 + 2\sum_{n\geq 1}\cos(n\pi y)\, e^{-n^2\pi^2\tau}, \tag{21}$$

the series converging in $C^\infty([0,1])$ for every $\tau > 0$. By the maximum principle $0 \leq \varphi \leq 1$, whence

$$0 < \min(1, e^M) \leq \Psi \leq \max(1, e^M). \tag{22}$$

Lemma 3.1. *Assume* (14). *For every $j \in \mathbb{N}_0$ there is $C_j = C_j(\|u_0\|_{L^1}, R_0, j)$ such that, for all $\tau > 0$ and $L \geq R_0$,*

$$\big\|\partial_y^j\big(\hat{w}^L(\cdot,\tau) - \Psi(\cdot,\tau)\big)\big\|_{L^\infty(0,1)} \leq \frac{C_j}{L}\,\min\{\tau,1\}^{-(j+1)/2}. \tag{23}$$

Proof. Set $\eta := \hat{w}^L - \Psi$. Both functions solve the heat equation on $(0,1)$ with the same boundary data, so η solves it with homogeneous Dirichlet conditions and initial datum $\eta_0(y) = w_0(yL) - e^M$. By (14) and (15), η_0 is supported in $[0, R_0/L]$ and bounded by $2e^{\|u_0\|_{L^1}}$, so

$$\|\eta_0\|_{L^1(0,1)} \leq \frac{A}{L}, \qquad A := 2e^{\|u_0\|_{L^1}} R_0. \tag{24}$$

Let G_1 be the Dirichlet heat kernel of $(0,1)$. For $0 < \tau \leq 1$ the method of images represents G_1 as an absolutely convergent signed sum of Gaussians of variance 2τ, and termwise differentiation gives $|\partial_y^j G_1(\tau,y,\zeta)| \leq C_j\tau^{-(j+1)/2}$ uniformly on $(0,1)^2$. For $\tau \geq 1$ the spectral representation $G_1(\tau,y,\zeta) = 2\sum_{n\geq 1} e^{-n^2\pi^2\tau}\sin(n\pi y)\sin(n\pi\zeta)$ gives $|\partial_y^j G_1(\tau,y,\zeta)| \leq C_j e^{-\pi^2\tau} \leq C_j$. Since $\eta(y,\tau) = \int_0^1 G_1(\tau,y,\zeta)\eta_0(\zeta)\,d\zeta$, estimate (23) follows from (24). □

Definition 3.2. *For $c > 0$ and $0 \leq \xi \leq c^{-1/2}$,*

$$\Phi_c(\xi) := \sqrt{c}\,\frac{(e^M - 1)\,\varphi_y\big(\xi\sqrt{c}, c\big)}{1 + (e^M - 1)\,\varphi\big(\xi\sqrt{c}, c\big)} = \sqrt{c}\,\frac{\Psi_y(\xi\sqrt{c}, c)}{\Psi(\xi\sqrt{c}, c)}. \tag{25}$$

By (22) the denominator is bounded away from zero for every real M, so Φ_c is well defined and smooth on the closed interval $[0, c^{-1/2}]$.

Theorem 3.3. *Assume* (14) *and fix $c > 0$, $k \in \mathbb{N}_0$ and $R \in (0, c^{-1/2})$. Then there is $C = C(\|u_0\|_{L^1}, R_0, R, k, c)$ such that, for $t = cL^2$ and all $L \geq R_0$,*

$$\big\|v^L(\cdot, cL^2) - \Phi_c\big\|_{C^k([0,R])} \leq \frac{C}{L}. \tag{26}$$

In particular $v^L(\cdot, cL^2) \to \Phi_c$ in $C^k_{\mathrm{loc}}\big([0, c^{-1/2})\big)$ as $L \to \infty$, for every fixed $c > 0$.

Proof. Since $u^L = w^L_x/w^L$ and $\partial_x = L^{-1}\partial_y$ under (17), at $t = cL^2$ and $x = \xi\sqrt{t}$, so that $y = x/L = \xi\sqrt{c}$, and using $\sqrt{t}/L = \sqrt{c}$,

$$v^L(\xi, cL^2) = \sqrt{t}\,\frac{L^{-1}\hat{w}^L_y(\xi\sqrt{c}, c)}{\hat{w}^L(\xi\sqrt{c}, c)} = \sqrt{c}\,\frac{\hat{w}^L_y(\xi\sqrt{c}, c)}{\hat{w}^L(\xi\sqrt{c}, c)}. \tag{27}$$

Comparing with (25), the claim reduces to estimating $\partial_y^j\big(\hat{w}^L_y/\hat{w}^L - \Psi_y/\Psi\big)$ at $\tau = c$ on $[0, R\sqrt{c}] \subset [0,1]$.

Put $\eta = \hat{w}^L - \Psi$. The maximum principle applied to (18), together with (15), gives

$$m_0 := e^{-\|u_0\|_{L^1}} \le \hat{w}^L \le e^{\|u_0\|_{L^1}}. \tag{28}$$

From (21), $|\partial_y^i \Psi(\cdot, c)| \le K_i(c)$ on $[0,1]$, while Lemma 3.1 gives $|\partial_y^i \eta(\cdot, c)| \le C_i(c)/L$ and hence $|\partial_y^i \hat{w}^L(\cdot, c)| \le K_i(c) + C_i(c)/L$. Since $\hat{w}^L \ge m_0 > 0$ and $\Psi \ge \min(1, e^M) > 0$, Faà di Bruno yields

$$\Big|\partial_y^b\Big(\frac{1}{\hat{w}^L}\Big)\Big| \le K_b'(c), \qquad \Big|\partial_y^b\Big(\frac{1}{\Psi}\Big)\Big| \le K_b'(c), \qquad b \le k+1,\ L \ge R_0. \tag{29}$$

Writing

$$\frac{\hat{w}^L_y}{\hat{w}^L} - \frac{\Psi_y}{\Psi} = \frac{\eta_y}{\hat{w}^L} - \frac{\Psi_y}{\hat{w}^L \Psi}\,\eta$$

and differentiating $j \le k$ times, the Leibniz rule produces finitely many terms, each carrying exactly one factor $\partial_y^a \eta$ with $a \le j+1$, the remaining factors being derivatives of Ψ, $1/\hat{w}^L$ and $1/\Psi$ of order at most $j+1$. By (29) and Lemma 3.1,

$$\Big|\partial_y^j\Big(\frac{\hat{w}^L_y}{\hat{w}^L} - \frac{\Psi_y}{\Psi}\Big)(y,c)\Big| \le \frac{C_j(c)}{L}, \qquad y \in [0,1].$$

Since $\partial_\xi^j = c^{j/2}\partial_y^j$ along $y = \xi\sqrt{c}$, (27) gives (26). $\square$

Proposition 3.4 ($c \to \infty$: the stationary profile)**.** *Uniformly for $y \in [0,1]$, and at rate $e^{-\pi^2 c}$,*

$$\frac{1}{\sqrt{c}}\,\Phi_c\Big(\frac{y}{\sqrt{c}}\Big) \longrightarrow \frac{e^M - 1}{1 + (e^M - 1)y} = \mathcal{U}_M(y) \qquad \text{as } c \to \infty. \tag{30}$$

For comparison, for fixed L and $t \to \infty$,

$$u^L(x,t) \longrightarrow \frac{e^M - 1}{L + (e^M - 1)x} = U_M^L(x) \qquad \text{uniformly on } [0,L]. \tag{31}$$

Proof. For $\tau \ge 1$ the series in (21) are dominated by $Ce^{-\pi^2\tau}$, so $\varphi(\cdot,\tau) \to y$ and $\varphi_y(\cdot,\tau) \to 1$ uniformly on $[0,1]$ at that rate, whence $\Psi(y,\tau) \to 1 + (e^M - 1)y$ and $\Psi_y(y,\tau) \to e^M - 1$. By (25), $c^{-1/2}\Phi_c(y/\sqrt{c}) = \Psi_y(y,c)/\Psi(y,c)$, and the denominator is bounded below by (22); this is (30).

For (31), fix L and let $\tau = t/L^2 \to \infty$. The difference $\hat{w}^L - [1 + (e^M - 1)y]$ solves the homogeneous Dirichlet heat equation on $(0,1)$ with bounded initial datum; parabolic smoothing on $\tau \ge 1$ together with the spectral gap gives $C^1([0,1])$ decay at the rate $e^{-\pi^2\tau}$. Thus $\hat{w}^L(\cdot,\tau) \to 1 + (e^M - 1)\cdot$ in $C^1([0,1])$, that is $w^L(x,t) \to 1 + (e^M - 1)x/L$ together with $w^L_x(x,t) \to (e^M - 1)/L$, uniformly on $[0,L]$. Since $w^L \ge m_0 > 0$ by (28),

$$u^L = \frac{w^L_x}{w^L} \longrightarrow \frac{(e^M - 1)/L}{1 + (e^M - 1)x/L} = \frac{e^M - 1}{L + (e^M - 1)x} = U_M^L(x),$$

the stationary profile (4). $\square$

Remark 3.5. *The rescaling in* (30) *is essential: Φ_c is a profile in the diffusive variable $\xi = x/\sqrt{t}$, whereas the stationary state lives on the scale of the domain, $y = x/L = \xi\sqrt{c}$. A fixed positive ξ eventually lies outside the domain of Φ_c as $c \to \infty$. The meaningful statement is $c^{-1/2}\Phi_c(y/\sqrt{c}) \to \mathcal{U}_M(y)$ for fixed $y \in [0,1]$; at $y = 0$ this also gives $\Phi_c(0) \sim \sqrt{c}\,(e^M - 1)$.*

By uniqueness of the steady state in each mass class, U_M^L is the only equilibrium a mass-M trajectory can approach. The family $\{\Phi_c\}_{c>0}$ therefore links f_M and $\mathcal{U}_M$ through limits in their respective natural coordinates; its existence identifies $t \sim L^2$ as the transition scale.

3.2. The sharp rate

Lemma 3.6. *For every $\tau > 0$ and $y \in \mathbb{R}$,*

$$1 + 2\sum_{n\ge 1} \cos(n\pi y)\, e^{-n^2\pi^2\tau} = \frac{1}{\sqrt{\pi\tau}} \sum_{k\in\mathbb{Z}} \exp\Big(-\frac{(y-2k)^2}{4\tau}\Big), \tag{32}$$

both sides converging absolutely. In particular (32) *represents $\varphi_y(y,\tau)$ for $y \in [0,1]$.*

Proof. Let $g(s) = (4\pi\tau)^{-1/2} e^{-s^2/(4\tau)}$, so that $\hat g(\lambda) = \int_{\mathbb{R}} g(s) e^{-i\lambda s}\, ds = e^{-\lambda^2\tau}$. Both g and $\hat g$ are Schwartz, so Poisson summation over the lattice $2\mathbb{Z}$ applies:

$$\sum_{k\in\mathbb{Z}} g(y-2k) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \hat g(n\pi)\, e^{in\pi y} = \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{-n^2\pi^2\tau} e^{in\pi y}.$$

Multiplying by 2 and pairing n with $-n$ gives (32). □

The right-hand side of (32) is the method-of-images representation generated by the reflections of the source across the boundaries $y \in 2\mathbb{Z}$. Its $k = 0$ term produces the half-line profile f_M; the term $k = 1$, namely $(\pi\tau)^{-1/2} e^{-(y-2)^2/(4\tau)}$, is the image charge at $y = 2$, that is at $x = 2L$, and is the leading correction. In the variables of the problem its exponent is $-(2L-x)^2/4t$: the Gaussian cost of the path from the origin out to the boundary at $x = L$ and back to x. This is the mechanism by which the boundary is felt, and it fixes the exponent.

The estimates below are needed for every order of differentiation, so we first record the elementary bound on the derivatives of a Gaussian that converts each y-derivative of an image term into a power of c.

Lemma 3.7. *Let $m \in \mathbb{N}_0$ and $\Lambda > 0$. There is $C_m = C_m(\Lambda)$ such that, for every $0 < c \le 1$, every $a \in \mathbb{R}$ and every y with $|y - a| \le \Lambda$,*

$$\Big|\partial_y^m e^{-(y-a)^2/(4c)}\Big| \;\le\; C_m\, c^{-m}\, e^{-(y-a)^2/(4c)}. \tag{33}$$

Proof. Write $s = (y-a)/(2\sqrt{c})$, so that $\partial_y = (2\sqrt{c})^{-1}\partial_s$ and $e^{-(y-a)^2/(4c)} = e^{-s^2}$. By the definition of the Hermite polynomials, $\partial_s^m e^{-s^2} = (-1)^m H_m(s) e^{-s^2}$, whence

$$\partial_y^m e^{-(y-a)^2/(4c)} = (-1)^m (2\sqrt{c})^{-m} H_m(s)\, e^{-s^2}.$$

Since H_m is a polynomial of degree m, $|H_m(s)| \le C_m(1+|s|)^m$. The hypothesis $|y-a| \le \Lambda$ gives $|s| \le \Lambda/(2\sqrt{c})$, so $(1+|s|)^m \le C_m(1+\Lambda)^m c^{-m/2}$ for $c \le 1$. Combining the two factors yields $(2\sqrt{c})^{-m}(1+|s|)^m \le C_m c^{-m}$, which is (33). □

Theorem 3.8. *Assume* (14) *with $M \ne 0$ and fix $R > 0$. There exist $c_1 = c_1(R,M) > 0$ and $\kappa_- = \kappa_-(M,R) > 0$ such that, for $0 < c \le c_1$:*

(i) for every $k \in \mathbb{N}_0$ there exists $\kappa_{+,k} = \kappa_{+,k}(M,R) > 0$ such that

$$\big\|\Phi_c - f_M\big\|_{C^k([0,R])} \le \kappa_{+,k}\, c^{-k/2} \exp\Big(-\frac{1}{c} + \frac{R}{\sqrt{c}}\Big), \tag{34}$$

and hence, for every $\varepsilon > 0$, with $C_{k,\varepsilon}$ independent of c,

$$\big\|\Phi_c - f_M\big\|_{C^k([0,R])} \le C_{k,\varepsilon}\, e^{-(1-\varepsilon)/c}; \tag{35}$$

(ii) at $\xi = 0$ the difference has the exact asymptotics

$$\big|\Phi_c(0) - f_M(0)\big| = \frac{2|e^M - 1|}{\sqrt{\pi}}\, e^{-1/c}\big(1 + O(e^{-3/c})\big), \qquad c \to 0^+; \tag{36}$$

in particular

$$\big|\Phi_c(0) - f_M(0)\big| \;\ge\; \kappa_-\, e^{-1/c}, \tag{37}$$

and

$$\lim_{c\to 0^+} -c \ln\big|\Phi_c(0) - f_M(0)\big| = 1. \tag{38}$$

The exponent $\gamma = 1$ in (35) *is therefore attained, and no larger exponent is admissible.*

Proof. Fix $c_1 \le \min\{1, 1/(4R^2)\}$, so that $y := \xi\sqrt{c} \in [0, \frac{1}{2}]$ for $\xi \in [0, R]$.

(i) By Lemma 3.6,

$$\varphi_y(y,c) = \frac{1}{\sqrt{\pi c}} e^{-y^2/(4c)} + E_1(c,y), \qquad E_1(c,y) = \frac{1}{\sqrt{\pi c}} \sum_{\ell \neq 0} e^{-(y-2\ell)^2/(4c)}. \tag{39}$$

For $0 \le y \le \frac{1}{2}$ and $\ell \neq 0$ we have $|y - 2\ell| \ge 2|\ell| - y$, so $(y-2\ell)^2 \ge (2|\ell| - y)^2$, and the smallest of these values is $(2-y)^2$, attained at $\ell = 1$. Moreover, for $|\ell| \ge 1$ and $y \le \frac{1}{2}$,

$$(2|\ell| - y)^2 - (2-y)^2 = 4(|\ell| - 1)\big(|\ell| + 1 - y\big) \ge 2(|\ell| - 1),$$

so the terms of E_1 are dominated by $e^{-(2-y)^2/(4c)}$ times a summable geometric series of ratio $e^{-1/(2c)}$. Hence

$$0 < E_1(c,y) \le \frac{C}{\sqrt{c}}\, e^{-(2-y)^2/(4c)} = \frac{C}{\sqrt{c}} \exp\Big(-\frac{1}{c} + \frac{\xi}{\sqrt{c}} - \frac{\xi^2}{4} \Big), \tag{40}$$

the last equality expanding $(2-y)^2/(4c) = 1/c - y/c + y^2/(4c)$ and substituting $y = \xi\sqrt{c}$. Integrating (39) from 0, using $\varphi(0,c) = 0$ and $\int_0^y (\pi c)^{-1/2} e^{-s^2/(4c)}\, ds = \operatorname{erf}\big(y/(2\sqrt{c})\big)$,

$$\varphi(y,c) = \operatorname{erf}\Big(\frac{y}{2\sqrt{c}}\Big) + E_0(c,y), \qquad E_0(c,y) = \int_0^y E_1(c,s)\, ds. \tag{41}$$

By (40) the integrand is positive and, on $[0, \frac{1}{2}]$, bounded by its value at $s = y$, since $s \mapsto e^{-(2-s)^2/(4c)}$ is increasing there; therefore

$$0 < E_0(c,y) \le y \cdot \frac{C}{\sqrt{c}} e^{-(2-y)^2/(4c)} \le C \exp\Big(-\frac{1}{c} + \frac{\xi}{\sqrt{c}} - \frac{\xi^2}{4} \Big), \tag{42}$$

absorbing $y/\sqrt{c} = \xi \le R$ into the constant.

Substituting $y = \xi\sqrt{c}$ into (39) and (41), and using $\operatorname{erf}(\xi/2) = \int_0^\xi b$,

$$\sqrt{c}\, \varphi_y(\xi\sqrt{c}, c) = \frac{1}{\sqrt{\pi}} e^{-\xi^2/4} + \sqrt{c}\, E_1, \qquad \varphi(\xi\sqrt{c}, c) = \int_0^\xi b + E_0.$$

Shrinking c_1 so that $|e^M - 1| \sup_{[0,R]} |E_0| \le \frac{1}{2}\min(1, e^M)$, both denominators in $\Phi_c(\xi)$ and $f_M(\xi)$ are bounded below by $\frac{1}{2}\min(1, e^M) > 0$, and the quotient rule gives

$$|\Phi_c(\xi) - f_M(\xi)| \le C\big(\sqrt{c}\, |E_1| + |E_0|\big) \le \kappa_+ \exp\Big(-\frac{1}{c} + \frac{R}{\sqrt{c}} \Big),$$

which is (34) for $k = 0$.

For $k \ge 1$ we first estimate the derivatives of the two remainders. Fix $m \in \mathbb{N}_0$. Differentiating (39) termwise and applying Lemma 3.7 with $a = 2\ell$ and $\Lambda = 2|\ell| + \frac{1}{2}$, which is legitimate since $0 \le y \le \frac{1}{2}$,

$$\big|\partial_y^m E_1(c,y)\big| \le \frac{C_m}{\sqrt{\pi c}}\, c^{-m} \sum_{\ell \neq 0} (1 + 2|\ell|)^m\, e^{-(y-2\ell)^2/(4c)}.$$

By the two inequalities established above, $(y - 2\ell)^2 \ge (2-y)^2 + 2(|\ell| - 1)$ for $\ell \neq 0$ and $y \le \frac{1}{2}$, so the sum is bounded by $e^{-(2-y)^2/(4c)} \sum_{\ell \neq 0} (1 + 2|\ell|)^m e^{-(|\ell|-1)/(2c)}$, and the last series converges to a constant depending only on m, uniformly for $c \le c_1$. Hence

$$\big|\partial_y^m E_1(c,y)\big| \le C_m\, c^{-m-1/2}\, e^{-(2-y)^2/(4c)}, \qquad 0 \le y \le \tfrac{1}{2}. \tag{43}$$

Since $E_0(c,\cdot)$ is the primitive of $E_1(c,\cdot)$ vanishing at $y=0$, one has $\partial_y^m E_0 = \partial_y^{m-1} E_1$ for $m \ge 1$, so (43) and (42) give

$$\bigl|\partial_y^m E_0(c,y)\bigr| \le C_m\, c^{-m+1/2}\, e^{-(2-y)^2/(4c)}, \qquad m \ge 1. \tag{44}$$

Along $y = \xi\sqrt{c}$ one has $\partial_\xi^m = c^{m/2}\partial_y^m$, so (43), (44) and (42) become

$$\bigl|\partial_\xi^m\bigl(\sqrt{c}\,E_1\bigr)\bigr| \le C_m\, c^{-m/2}\,\mathcal{E}(c,\xi), \qquad \bigl|\partial_\xi^m E_0\bigr| \le C_m\, c^{-m/2}\,\mathcal{E}(c,\xi), \tag{45}$$

for every $m \in \mathbb{N}_0$, where $\mathcal{E}(c,\xi) := \exp\bigl(-\frac{1}{c} + \frac{\xi}{\sqrt{c}} - \frac{\xi^2}{4}\bigr)$ is the common exponential factor of (40) and (42).

It remains to differentiate the quotient. Write $A = e^M - 1$, $b(\xi) = \pi^{-1/2}e^{-\xi^2/4}$ and $B(\xi) = \int_0^\xi b$, so that $f_M = Ab/(1+AB)$ while, by the display following (42), $\Phi_c = A\,(b+\sqrt{c}\,E_1)/\bigl(1+A(B+E_0)\bigr)$. Subtracting,

$$\Phi_c - f_M = \frac{A\,\sqrt{c}\,E_1}{1+A(B+E_0)} - \frac{A^2\,b\,E_0}{(1+AB)\bigl(1+A(B+E_0)\bigr)}. \tag{46}$$

Both denominators are bounded below by $\frac{1}{2}\min(1,e^M) > 0$, by the choice of c_1 made above; the derivatives of b and B are bounded on $[0,R]$ independently of c; and by (45) together with Faà di Bruno the derivatives of $1/(1+A(B+E_0))$ up to order k are bounded independently of c as well, since every factor $\partial_\xi^m E_0$ carries $\mathcal{E}(c,\xi) \le e^{-1/c+R/\sqrt{c}}$, which tends to 0 as $c \to 0^+$. Differentiating (46) $m \le k$ times, the Leibniz rule therefore produces finitely many terms, each carrying exactly one factor $\partial_\xi^a(\sqrt{c}\,E_1)$ or $\partial_\xi^a E_0$ with $a \le m$, the remaining factors being bounded uniformly in c. By (45) each such term is bounded by $C_k c^{-a/2}\mathcal{E}(c,\xi) \le C_k c^{-k/2}\mathcal{E}(c,\xi)$, and since $\mathcal{E}(c,\xi) \le \exp(-1/c + R/\sqrt{c})$ on $[0,R]$, summing gives (34). Finally (35) follows from $c^{-k/2}e^{R/\sqrt{c}} \le C_{k,\varepsilon}e^{\varepsilon/c}$.

(ii) At $\xi = 0$, i.e. $y = 0$, we have $\varphi(0,c) = 0$ and $\operatorname{erf}(0) = 0$, so $E_0(c,0) = 0$ and the denominators of $\Phi_c(0)$ and $f_M(0)$ both equal 1. Consequently

$$\Phi_c(0) - f_M(0) = (e^M - 1)\Bigl(\sqrt{c}\,\varphi_y(0,c) - \frac{1}{\sqrt{\pi}}\Bigr) = (e^M - 1)\,\sqrt{c}\,E_1(c,0),$$

and, by (39),

$$\sqrt{c}\,E_1(c,0) = \frac{2}{\sqrt{\pi}}\sum_{\ell\ge 1} e^{-\ell^2/c} = \frac{2}{\sqrt{\pi}}\,e^{-1/c}\Bigl(1 + \sum_{\ell\ge 2} e^{-(\ell^2-1)/c}\Bigr) = \frac{2}{\sqrt{\pi}}\,e^{-1/c}\bigl(1 + O(e^{-3/c})\bigr).$$

All terms are positive, so the sum is bounded below by its first term and no cancellation occurs. Hence, for $M \ne 0$ and $c \le c_1$,

$$|\Phi_c(0) - f_M(0)| = \frac{2|e^M - 1|}{\sqrt{\pi}}\,e^{-1/c}\bigl(1 + O(e^{-3/c})\bigr) \;\ge\; \kappa_- e^{-1/c},$$

which is (36) and (37); taking $-c\ln$ gives (38). ∎

Corollary 3.9 ($c \to 0$: the self-similar profile)**.** *For every $R > 0$ and $k \in \mathbb{N}_0$, $\|\Phi_c - f_M\|_{C^k([0,R])} \to 0$ as $c \to 0^+$, where f_M is the self-similar profile of Proposition 2.5.*

Proof. If $M \ne 0$, this is immediate from Theorem 3.8(i), which gives the quantitative bound (35). If $M = 0$, then $e^M - 1 = 0$, so (25) and the formula of Proposition 2.5 give $\Phi_c \equiv f_M \equiv 0$ for every $c > 0$. ∎

Remark 3.10. *The estimate* (34) *contains the subexponential factor* $\exp(R/\sqrt{c})$ *(and the leading image term contains* $\exp(-1/c + R/\sqrt{c} - R^2/4)$*). Thus* $\ln\|\Phi_c - f_M\|_{C^0([0,R])}$ *is not affine in* $1/c$*: the subexponential factor* $e^{R/\sqrt{c}}$ *flattens the apparent slope over any bounded window of* $1/c$*. A one-parameter fit of* $\ln E$ *against* $1/c$ *therefore returns an effective slope strictly between* 0 *and* 1*, depending on the window and not on the problem. The two-parameter form* $-a/c + bR/\sqrt{c}$ *should be fitted instead, or the range restricted to* $R\sqrt{c} \ll 1$*; both are consistent with* $a = 1$*, in accordance with* (38)*.*

3.3. Uniform comparison with the half-line problem

Lemma 3.11. *Assume* (14). *There is $C = C(\|u_0\|_{L^1}, R_0) > 0$ such that*

$$\big|e^M - w^\infty(L,t)\big| \le C \exp\Big(-\frac{(L-R_0)^2}{4t}\Big) \tag{47}$$

for all $L > R_0$ and $t > 0$.

Proof. Set $h := w^\infty - 1$, which solves the heat equation on $(0,\infty)$ with $h(0,t) = 0$ and $h(\cdot,0) = h_0 := w_0 - 1$, so that

$$h(x,t) = \int_0^\infty G_D(t,x,y)h_0(y)\,dy, \qquad G_D(t,x,y) = \frac{1}{\sqrt{4\pi t}}\Big(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\Big),$$

G_D being the Dirichlet heat kernel of the half-line. By (14), $h_0 = e^M - 1$ on $[R_0,\infty)$. Since $e^M - w^\infty(L,t) = (e^M-1) - h(L,t)$ and $(e^M-1) - h_0(y) = e^M - w_0(y)$ vanishes for $y \ge R_0$,

$$e^M - w^\infty(L,t) = \int_0^{R_0} G_D(t,L,y)\big(e^M - w_0(y)\big)\,dy + (e^M-1)\Big(1 - \int_0^\infty G_D(t,L,y)\,dy\Big). \tag{48}$$

For the first term, (15) gives $|e^M - w_0(y)| \le e^{|M|} + e^{\|u_0\|_{L^1}} \le 2e^{\|u_0\|_{L^1}}$, and $0 \le G_D(t,L,y) \le (4\pi t)^{-1/2}e^{-(L-y)^2/(4t)}$; with $z = L-y$,

$$\Big|\int_0^{R_0} G_D(t,L,y)\big(e^M - w_0(y)\big)\,dy\Big| \le C\int_{L-R_0}^\infty t^{-1/2}e^{-\frac{z^2}{4t}}\,dz.$$

Substituting $z = 2\sqrt{t}\,s$,

$$\int_a^\infty t^{-1/2}e^{-\frac{z^2}{4t}}\,dz = 2\int_{a/(2\sqrt{t})}^\infty e^{-s^2}\,ds = \sqrt{\pi}\,\operatorname{erfc}\Big(\frac{a}{2\sqrt{t}}\Big) \le \sqrt{\pi}\,e^{-\frac{a^2}{4t}}, \qquad a \ge 0, \tag{49}$$

by the elementary bound $\operatorname{erfc}(\lambda) \le e^{-\lambda^2}$, $\lambda \ge 0$. Taking $a = L - R_0$ bounds this term by $Ce^{-(L-R_0)^2/(4t)}$. For the second term of (48), $\int_0^\infty G_D(t,L,y)\,dy = \operatorname{erf}\big(L/(2\sqrt{t})\big)$, so it equals $|e^M - 1|\operatorname{erfc}\big(L/(2\sqrt{t})\big) \le Ce^{-L^2/(4t)} \le Ce^{-(L-R_0)^2/(4t)}$. Adding the two bounds gives (47). □

The next lemma is the quantitative form of the image-charge mechanism identified after Lemma 3.6: it estimates $z = w^L - w^\infty$ at interior points with the exponent $(2L-x)^2/4t$, rather than the exponent $L^2/4t$ that the maximum principle alone would give. The gain is the Gaussian decay accumulated on the return path from $x = L$ to $x = O(\sqrt{t})$, and the underlying identity is the elementary optimization (57) below.

Lemma 3.12. *Assume* (14) *and let $z := w^L - w^\infty$ on $(0,L)\times(0,\infty)$. For every $\theta \in (0,1)$ there is $C_\theta = C_\theta(\|u_0\|_{L^1}, R_0) > 0$ such that*

$$|z(x,t)| \;\le\; C_\theta \exp\Big(-\theta\,\frac{(2L-R_0-x)^2}{4t}\Big) \tag{50}$$

for all $0 < t \le L^2$, all $L \ge 2R_0$ and all $0 \le x \le \frac{1}{2}L$.

Proof. Both w^L and w^∞ solve the heat equation on $(0,L)$ with the same initial datum w_0 and the same value 1 at $x = 0$; they differ only in the datum at $x = L$. Hence z solves

$$\begin{cases} z_s = z_{xx}, & (x,s) \in (0,L)\times(0,\infty),\\ z(0,s) = 0, \quad z(L,s) = \beta(s) := e^M - w^\infty(L,s), & s > 0,\\ z(x,0) = 0, & x \in (0,L), \end{cases} \tag{51}$$

and, the initial datum being zero, the classical boundary (Duhamel) representation gives

$$z(x,t) = \int_0^t \beta(s)\, N(t-s,x)\, ds, \qquad N(r,x) := -\,\partial_y G_{(0,L)}(r,x,y)\Big|_{y=L}, \tag{52}$$

$G_{(0,L)}$ being the Dirichlet heat kernel of $(0,L)$.

Step 1 (the boundary kernel). By the method of images,

$$G_{(0,L)}(r,x,y) = \frac{1}{\sqrt{4\pi r}} \sum_{k\in\mathbb{Z}} \Big(e^{-\frac{(x-y+2kL)^2}{4r}} - e^{-\frac{(x+y+2kL)^2}{4r}} \Big).$$

Differentiating termwise in y and evaluating at $y = L$, the arguments of the two families become $x - L + 2kL = x + (2k-1)L$ and $x + L + 2kL = x + (2k+1)L$; the second family is the first one shifted by one index, so the two merge into a single sum over the odd multiples of L translated by x. Writing

$$d_m := x + (2m-1)L, \qquad m \in \mathbb{Z},$$

one obtains

$$N(r,x) = \frac{2}{\sqrt{4\pi r}} \sum_{m\in\mathbb{Z}} \Big(-\frac{d_m}{2r} \Big) e^{-d_m^2/(4r)}. \tag{53}$$

Only an upper bound on $|N|$ is needed below, so we estimate (53) term by term. Assume $0 \le x \le \frac{1}{2}L$; then $|d_0| = L - x \in [\frac{1}{2}L, L]$.

The term $m = 1$. Here $d_1 = x + L$, so x adds to L rather than subtracting from it, and no gain over $|d_0|$ is available: at $x = 0$ one has $|d_1| = |d_0| = L$. (This is forced by $z(0,s) = 0$: the image at $m = 1$ is precisely the one cancelling $m = 0$ at the origin.) It suffices to compare the two directly. Since $(L+x)^2 \ge (L-x)^2$ and $L + x \le \frac{3}{2}L \le 3(L-x)$ for $x \le \frac{1}{2}L$,

$$\frac{|d_1|}{2r}\, e^{-d_1^2/(4r)} = \frac{L+x}{2r}\, e^{-(L+x)^2/(4r)} \le \frac{3(L-x)}{2r}\, e^{-(L-x)^2/(4r)}, \tag{54}$$

which merely triples the $m = 0$ term and is absorbed into the constant of (55) below.

The terms $m = -1$ and $|m| \ge 2$. For these, $|d_m| \ge (2|m|-1)L - x \ge (2|m| - \frac{3}{2})L$, and since $(L-x)^2 \le L^2$,

$$d_m^2 - (L-x)^2 \;\ge\; \Big(2|m| - \tfrac{3}{2}\Big)^2 L^2 - L^2 \;\ge\; \frac{L^2}{2}\Big(|m| - \tfrac{1}{2}\Big),$$

as is checked directly for $m = -1$ and for $|m| \ge 2$. Hence these terms are dominated by

$$\frac{|d_m|}{2r}\, e^{-d_m^2/(4r)} \le \frac{|d_m|}{2r}\, e^{-(L-x)^2/(4r)}\, e^{-L^2(|m|-1/2)/(8r)},$$

and, using $|d_m| \le 2(|m|+1)L$,

$$\sum_{m\ne 0,1} \frac{|d_m|}{2r}\, e^{-L^2(|m|-1/2)/(8r)} \;\le\; \frac{2L}{r} \sum_{j\ge 1} (j+1)\, e^{-\lambda(j-1/2)/8}, \qquad \lambda := \frac{L^2}{r} \ge 1,$$

the restriction $r \le L^2$ being exactly the range in which (55) is applied in Step 2, where $r = t - s \le t \le L^2$. The last series is therefore bounded by the convergent $\sum_{j\ge1}(j+1)e^{-(j-1/2)/8}$, uniformly in r and L.

Retaining the term $m = 0$, whose size is $\frac{L-x}{2r} e^{-(L-x)^2/(4r)}$, adding (54), and using $L/r \le C(L-x)/r$ for $x \le \frac{1}{2}L$ to absorb the prefactor of the remaining sum, the triangle inequality gives

$$|N(r,x)| \;\le\; \frac{C\,(L-x)}{r^{3/2}}\, \exp\Big(-\frac{(L-x)^2}{4r} \Big), \qquad 0 < r \le L^2, \quad 0 \le x \le \tfrac{1}{2}L, \tag{55}$$

with C absolute.

Step 2 (Duhamel and the optimal splitting of time). By Lemma 3.11, $|\beta(s)| \le Ce^{-(L-R_0)^2/(4s)}$. Inserting this and (55) into (52),

$$|z(x,t)| \le C(L-x)\int_0^t \frac{1}{(t-s)^{3/2}} \exp\Big(-\frac{(L-R_0)^2}{4s} - \frac{(L-x)^2}{4(t-s)}\Big)ds. \tag{56}$$

The exponent is bounded below by the Cauchy–Schwarz inequality: for $a, b \ge 0$ and $0 < s < t$,

$$\frac{a^2}{s} + \frac{b^2}{t-s} \ \ge\ \frac{(a+b)^2}{t}, \tag{57}$$

with equality at $s = ta/(a+b)$. Indeed (57) is the Cauchy–Schwarz inequality applied to the vectors $(a/\sqrt{s},\ b/\sqrt{t-s})$ and $(\sqrt{s},\ \sqrt{t-s})$. With $a = L - R_0$ and $b = L - x$,

$$\frac{(L-R_0)^2}{4s} + \frac{(L-x)^2}{4(t-s)} \ \ge\ \frac{(2L-R_0-x)^2}{4t}, \qquad 0 < s < t. \tag{58}$$

This is the assertion that the cheapest Gaussian path from the origin to x which touches $x = L$ costs $(2L - R_0 - x)^2/4t$, the times s and $t - s$ being allocated optimally between the outward and the return leg.

Fix $\theta \in (0, 1)$. Applying (58) to the fraction θ of the exponent and discarding the fraction $1 - \theta$ of its first term,

$$\exp\Big(-\frac{(L-R_0)^2}{4s} - \frac{(L-x)^2}{4(t-s)}\Big) \le \exp\Big(-\theta\,\frac{(2L-R_0-x)^2}{4t}\Big)\exp\Big(-(1-\theta)\,\frac{(L-x)^2}{4(t-s)}\Big),$$

so that, substituting $r = t - s$ and then $r = (L - x)^2\sigma$,

$$\int_0^t \frac{1}{(t-s)^{3/2}}\exp\Big(-(1-\theta)\frac{(L-x)^2}{4(t-s)}\Big)ds \ \le\ \int_0^\infty \frac{e^{-(1-\theta)(L-x)^2/(4r)}}{r^{3/2}}\,dr = \frac{c_\theta}{L-x},$$

with $c_\theta = \int_0^\infty \sigma^{-3/2}e^{-(1-\theta)/(4\sigma)}\,d\sigma < \infty$ for $\theta < 1$. Combining with (56), the factors $L - x$ cancel and (50) follows. ☐

Remark 3.13. *The exponent in* (50) *is optimal: as $L \to \infty$ with $x = \xi\sqrt{t}$ and $t = cL^2$ fixed, $(2L - R_0 - x)^2/4t \to (2 - \xi\sqrt{c})^2/4c$ which equals $1/c$ at $\xi = 0$, matching the lower bound of Theorem 3.8(ii). It is the exponent of the image charge at $x = 2L$, and* (57) *is the statement that the cheapest Gaussian path from the origin to x that touches $x = L$ costs $(2L - x)^2/4t$, the times s and $t - s$ being allocated optimally between the outward and return legs.*

Lemma 3.14. *Let $z = w^L - w^\infty$. Fix $R > 0$, and $j \in \mathbb{N}_0$. There is $C = C(R, j, \|u_0\|_{L^1}, R_0)$ such that, whenever $0 < c < R^{-2}$, $t = cL^2$, and $L > R_0$,*

$$t^{j/2}\|\partial_x^j z(\cdot,t)\|_{L^\infty(0,L)} \le C.$$

Proof. We bound w^∞ and w^L separately. Since $w^\infty - 1 = \int_0^\infty \big(\Gamma(t, x-y) - \Gamma(t, x+y)\big)(w_0(y) - 1)\,dy$ with Γ the Gaussian kernel and $|w_0 - 1| \le 2e^{\|u_0\|_{L^1}}$, the classical bound $\int_{\mathbb{R}} |\partial_x^m\Gamma(t, s)|\,ds = C_m t^{-m/2}$ gives $|\partial_x^m w^\infty(x, t)| \le C_m t^{-m/2}$ for $m \ge 1$, all $x \ge 0$ and $t > 0$.

For w^L, pass to domain variables $\hat{w}^L(y, \tau) = w^L(Ly, L^2\tau)$ and let $W(y) = 1 + (e^M - 1)y$. Then $\hat{w}^L - W$ solves the homogeneous Dirichlet heat equation on $(0, 1)$ with initial datum bounded by $2e^{\|u_0\|_{L^1}}$ uniformly in L, so the kernel bounds $\int_0^1 |\partial_y^j G_1(\tau, y, \zeta)|\,d\zeta \le C_j \min\{\tau, 1\}^{-j/2}$ (images for $\tau \le 1$, the spectral series for $\tau \ge 1$) yield $\|\partial_y^j \hat{w}^L(\cdot, \tau)\|_{L^\infty(0,1)} \le C_j \min\{\tau, 1\}^{-j/2}$, with C_j independent of L. Since $\partial_x^j w^L = L^{-j}\partial_y^j \hat{w}^L$ and $t^{j/2}L^{-j} = c^{j/2}$, this gives

$$t^{j/2}\|\partial_x^j w^L(\cdot,t)\|_{L^\infty(0,L)} \le C_j\, c^{j/2}\min\{c, 1\}^{-j/2} \le C_j \max\{1, R^{-j}\}, \qquad 0 < c < R^{-2}, \tag{59}$$

The claim for $j \ge 1$ follows by the triangle inequality; for $j = 0$ it is the maximum principle. ☐

Theorem 3.15. *Assume* (14)*. Fix $R > 0$, $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$, and let $t = cL^2$ with*

$$0 < c < \frac{1}{R^2}.$$

Then there exist $C_k = C_k(\|u_0\|_{L^1}, R_0, R, k, \varepsilon) > 0$, independent of c and L, and $L_0 = L_0(R_0, R, \varepsilon)$, such that

$$\left\|v^L(\cdot,t) - v^\infty(\cdot,t)\right\|_{C^k([0,R])} \le C_k\, e^{-(1-\varepsilon)/c}, \qquad L \ge L_0. \tag{60}$$

If $M \neq 0$, the exponential constant 1 is optimal in the following uniform sense: no estimate of this form can hold with $1-\varepsilon$ replaced by a fixed number larger than 1.

Proof. Let $z = w^L - w^\infty$.

Step 1 (L^∞ bound). Let $\varepsilon \in (0,1)$ and choose $\theta = \theta(\varepsilon) \in (0,1)$ and $c_2 = c_2(R,\varepsilon) > 0$ as follows. Take L_0 so large that $R_0/L \le \varepsilon/8$ for $L \ge L_0$, and restrict first to $c \le c_2 := \min\{1, \varepsilon^2/[256(1+R)^2]\}$, so that $R\sqrt{c} \le \varepsilon/8$. Then, for $0 \le x \le R\sqrt{t}$ and $t = cL^2 \le L^2$,

$$\frac{(2L - R_0 - x)^2}{4t} \ge \frac{L^2\big(2 - \varepsilon/8 - \varepsilon/8\big)^2}{4t} \ge \frac{(1-\varepsilon/4)}{c},$$

using $(2-\varepsilon/4)^2/4 \ge 1 - \varepsilon/4$. Choosing $\theta := (1-\varepsilon)/(1-\varepsilon/4) \in (0,1)$, Lemma 3.12 applies, since $t \le L^2$ and $x \le R\sqrt{t} \le \frac{1}{2}L$ for $L \ge L_0$, and gives

$$\sup_{0 \le x \le R\sqrt{t}} |z(x,t)| \le C_\theta\, e^{-(1-\varepsilon)/c}, \tag{61}$$

with C_θ independent of c and L. For $c \in [c_2, 1/R^2)$, Lemma 3.14 gives every derivative bound needed below, uniformly up to both endpoints. Since $e^{-(1-\varepsilon)/c}$ is bounded below by $e^{-(1-\varepsilon)/c_2}$ in this range, those bounds are absorbed into the constant in (60).

Step 2 (derivative estimates). Fix $0 \le x \le R\sqrt{t}$ and set

$$\rho := \delta\sqrt{t}, \qquad \delta := \min\left\{\frac{1}{4}, \frac{1 - R\sqrt{c}}{2\sqrt{c}}\right\} > 0,$$

which is positive precisely because $c < 1/R^2$. Using $\sqrt{t} = \sqrt{c}\,L$,

$$L - R\sqrt{t} - \rho = L\big(1 - R\sqrt{c} - \delta\sqrt{c}\big) \ge L\Big(1 - R\sqrt{c} - \frac{1 - R\sqrt{c}}{2}\Big) = \frac{1 - R\sqrt{c}}{2} L > 0, \tag{62}$$

and $t - \rho^2 \ge \frac{15}{16}t > 0$. Let $Q_\rho(x,t) := (t - \rho^2, t) \times (x - \rho, x + \rho)$. By (62) the parabolic cylinder $Q_\rho(x,t) \cap \big((0,L) \times (0,t)\big)$ lies above $\{s = 0\}$ and its points satisfy $x' \le R\sqrt{t} + \rho \le \frac{1}{2}L$ in the small-c regime. We apply Lemma 3.12 directly at every (x', s) in this cylinder, instead of invoking a similarity window scaled with a different time. Decrease c_2, and increase L_0, so that

$$q := \frac{R_0}{L} + (R + \tfrac{1}{4})\sqrt{c} \le \frac{\varepsilon}{4} \qquad (c \le c_2,\ L \ge L_0).$$

Since $s \le t = cL^2$ and $x'/L \le (R + \frac{1}{4})\sqrt{c}$,

$$\frac{(2L - R_0 - x')^2}{4s} \ge \frac{(2-q)^2}{4c} = \frac{1 - q + q^2/4}{c} \ge \frac{1 - \varepsilon/4}{c}.$$

With the choice $\theta = (1-\varepsilon)/(1-\varepsilon/4)$ already made in Step 1, this gives

$$\theta \frac{(2L - R_0 - x')^2}{4s} \ge \frac{1-\varepsilon}{c}.$$

All choices are independent of c and L. Hence $\|z\|_{L^\infty(Q_\rho \cap ((0,L)\times(0,t)))} \le Ce^{-(1-\varepsilon)/c}$. Since z solves the heat equation exactly, the classical interior derivative estimates for the heat kernel give

$$|\partial_x^j z(x,t)| \le C_j\, \rho^{-j}\, \|z\|_{L^\infty(Q_\rho \cap ((0,L)\times(0,t)))} \le C_j\, t^{-j/2}\, e^{-(1-\varepsilon)/c}, \qquad 0 \le x \le R\sqrt{t}, \tag{63}$$

where δ^{-j} has been absorbed into C_j. This is legitimate with C_j independent of c: for $c \le c_2$ one has $R\sqrt{c} \le \varepsilon/8$, hence $1 - R\sqrt{c} \ge 1 - \varepsilon/8 \ge \frac{1}{2}$ and therefore $\delta = \frac{1}{4}$, so that $\delta^{-j} \le 4^j$ is bounded independently of c and L; for $c \in [c_2, 1/R^2)$ the derivative estimate follows from Lemma 3.14. For x within distance ρ of $x = 0$ the same bound follows by applying (63) to the odd extension $\tilde{z}(x,s) := -z(-x,s)$, $x < 0$, which solves the heat equation across $x = 0$ and has the same L^∞ size, precisely because $z(0,s) = 0$ for all s.

Step 3 (bounds on w^L, w^∞). By (15) the initial and boundary data of w^L and w^∞ lie in $[e^{-\|u_0\|_{L^1}}, e^{\|u_0\|_{L^1}}]$, so the maximum principle gives

$$0 < m_0 := e^{-\|u_0\|_{L^1}} \le w^L,\ w^\infty \le e^{\|u_0\|_{L^1}}. \tag{64}$$

The derivative bounds established in the proof of Lemma 3.14 — the Gaussian-kernel bound for w^∞ and (59) for w^L, the latter uniform in $0 < c < R^{-2}$ — give

$$|\partial_x^m w^\infty(x,t)| + |\partial_x^m w^L(x,t)| \le C_m t^{-m/2}, \qquad 0 \le x \le L,\ m \ge 1, \tag{65}$$

with C_m independent of c and L. Consequently, by Faà di Bruno together with (64) and (65),

$$\left|\partial_x^b\Big(\frac{1}{w^L}\Big)\right| \le C_b t^{-b/2}, \qquad \left|\partial_x^b\Big(\frac{1}{w^\infty}\Big)\right| \le C_b t^{-b/2}, \qquad b \ge 0. \tag{66}$$

Step 4 (transfer to u). Since $u = w_x/w$ and $\frac{1}{w^L} - \frac{1}{w^\infty} = \frac{w^\infty - w^L}{w^L w^\infty} = -\frac{z}{w^L w^\infty}$,

$$u^L - u^\infty = \frac{z_x}{w^L} - \frac{w_x^\infty}{w^L w^\infty}\, z. \tag{67}$$

Differentiating (67) k times and expanding by Leibniz, one obtains finitely many terms, each carrying exactly one factor $\partial_x^a z$ with $0 \le a \le k+1$, the remaining $k+1-a$ derivatives being distributed among factors w_x^∞, $1/w^L$ and $1/w^\infty$. By (63), (65) and (66) each such term is bounded by

$$C\, t^{-a/2} e^{-(1-\varepsilon)/c} \cdot t^{-(k+1-a)/2} = C\, t^{-(k+1)/2}\, e^{-(1-\varepsilon)/c},$$

the homogeneities adding to $a + (k+1-a) = k+1$. Summing,

$$|\partial_x^k(u^L - u^\infty)(x,t)| \le C_k\, t^{-(k+1)/2}\, e^{-(1-\varepsilon)/c}, \qquad 0 \le x \le R\sqrt{t}. \tag{68}$$

Since $\partial_\xi^k v = t^{(k+1)/2}\partial_x^k u$ by (16), this is (60).

It remains to justify the sharpness assertion, since Theorem 3.8(ii) is stated for the limiting profiles. Fix $c > 0$ and take $L \to \infty$. Theorem 3.3 gives $v^L(0, cL^2) \to \Phi_c(0)$, while Theorem 2.7 gives $v^\infty(0, cL^2) \to f_M(0)$. Thus

$$\lim_{L\to\infty} |v^L(0,cL^2) - v^\infty(0,cL^2)| = |\Phi_c(0) - f_M(0)|.$$

If a uniform estimate held with exponent $\gamma > 1$, passage to this limit would give $|\Phi_c(0) - f_M(0)| \le Ce^{-\gamma/c}$, contradicting Theorem 3.8(ii) as $c \downarrow 0$. ☐

3.4. *Two-scale asymptotics and comparison of regimes*

By Theorem 2.7, for data satisfying (14),

$$\|v^\infty(\cdot,t) - f_M\|_{L^\infty(0,R)} \le C\, t^{-1/2}, \qquad t \ge 1, \tag{69}$$

with $C = C(\|u_0\|_{L^1}, R_0, R)$.

Corollary 3.16. *Assume* (14) *and fix $R > 0$, $p \in [1,\infty]$ and $\varepsilon \in (0,1)$. There is $C = C(\|u_0\|_{L^1}, R_0, R, p, \varepsilon)$ such that, for all $L \ge L_0$ and all $t \ge 1$ with $c = t/L^2 < 1/R^2$,*

$$\big\|v^L(\cdot,t) - f_M\big\|_{L^p(0,R)} \le C\, e^{-(1-\varepsilon)L^2/t} + C\, t^{-1/2}. \tag{70}$$

In particular $v^L(\cdot,t) \to f_M$ in $L^p(0,R)$ along any sequence with $t \to \infty$ and $t/L^2 \to 0$, that is throughout the window $1 \ll t \ll L^2$; the two terms balance at $t \asymp L^2/\ln L$, where the error is $O\big((\ln L)^{1/2}/L\big)$.

Proof. By the triangle inequality,

$$\|v^L - f_M\|_{L^p(0,R)} \leq \|v^L - v^\infty\|_{L^p(0,R)} + \|v^\infty - f_M\|_{L^p(0,R)}.$$

Theorem 3.15 with $k = 0$ bounds the first term by $R^{1/p}C_0 e^{-(1-\varepsilon)/c}$, and (69) bounds the second by $R^{1/p}Ct^{-1/2}$. Substituting $c = t/L^2$ gives (70), and the balance follows from $e^{-(1-\varepsilon)L^2/t} = t^{-1/2}$. $\square$

Estimate (70) is a fixed-(L, t) inequality whose two terms have distinct origins: the first is the exponentially small cost of replacing the half-line by the interval at the diffusive scale, the second the polynomial relaxation of the half-line dynamics toward the universal profile. Together they exhibit the window $1 \ll t \ll L^2$ in which v^L is close to f_M.

Remark 3.17. *Fix $R > 0$ and write $c = t/L^2$.*

1. Self-similar regime, $1 \ll t \ll L^2$. *By Theorem 3.15, $\|v^L(\cdot,t)-v^\infty(\cdot,t)\|_{C^k([0,R])} \leq C_k e^{-(1-\varepsilon)L^2/t} \to 0$, while $v^\infty(\cdot,t) \to f_M$ in C^k_{loc} by Theorem 2.9; hence $v^L(\cdot,t) \to f_M$. Corollary 3.16 quantifies the window, and Theorem 3.8 shows the rate $e^{-L^2/t}$ is sharp.*
2. Transition regime, $t \sim L^2$. *For each fixed $c > 0$, Theorem 3.3 gives $v^L(\cdot, cL^2) \to \Phi_c$ in $C^k_{\mathrm{loc}}([0, c^{-1/2}))$, with Φ_c explicit. The family tends to f_M in similarity coordinates as $c \to 0$ and to $\mathcal{U}_M$ only after the domain-scale rescaling as $c \to \infty$ (Corollary 3.9 and Proposition 3.4), and its existence identifies $t \sim L^2$ as the critical scale.*
3. Stationary regime, $t \gg L^2$. *By Theorem 2.2, $\|u^L(\cdot,t) - U^L_M\|_{L^\infty(0,L)} \leq C(L)\, e^{-\pi^2 t/L^2}$ for $t \geq t_0$, consistently with Proposition 3.4.*

The three regimes are encoded by the single family Φ_c, with the qualification that its $c \to \infty$ limit must be taken after domain-scale rescaling.

Remark 3.18. *The transient described here lasts $t \sim L^2$, the diffusive time of a domain of diameter L: it is the natural time scale of the problem rather than an anomalously long one, and the solution follows the half-line dynamics for exactly as long as the boundary is out of diffusive reach. This is the principle of not feeling the boundary [20], and the accurate description is Barenblatt's intermediate asymptotics [2]. It is distinct from the metastability of [21, 4], where the small parameter is the viscosity and the transient exceeds the natural scale, and from the exponentially slow, boundary-driven shock motion for Burgers on an interval studied in [26, 22, 24].*

Remark 3.19. *The compact support assumption* (14) *is made for simplicity. The results extend to data with exponential decay, $|u_0(x)| \leq Ce^{-\alpha x}$: the Gaussian tail estimate of Lemma 3.11 is replaced by the corresponding heat-kernel estimate against an exponentially decaying datum, the layer in Lemma 3.1 has width $O(\alpha^{-1}\log L/L)$, and the exponent $\gamma = 1$ is unchanged provided $\alpha^{-1} = o(L)$. For data with polynomial decay the exponent is dictated by the tail rather than by the image charge.*

4. Robust interval conclusions for space-dependent diffusion

The explicit analysis of the previous sections rests on the constant-coefficient heat kernel. In many models the diffusion is heterogeneous, and it is natural to ask which of the phenomena described above — the two attractors, the relaxation rate, the diffusive transition — survive when the diffusivity varies in space. We consider

$$u_t = \partial_x\big(a(x)\,(u_x + u^2)\big), \qquad (x,t) \in (0,L) \times (0,\infty), \tag{71}$$

with the conservative boundary conditions

$$a(x)\,(u_x + u^2) = 0 \quad \text{at } x = 0, L, \tag{72}$$

which, since $a > 0$, read again $u_x + u^2 = 0$ at the endpoints and make the flux vanish there, so that the mass $M = \int_0^L u\,dx$ is conserved. Throughout this section a is smooth and uniformly elliptic,

$$0 < a_0 \le a(x) \le a_1 < \infty. \tag{73}$$

This is the only assumption used for the interval results below, which therefore hold for any diffusivity satisfying (73), including one varying on the scale of the domain. The half-line discussion at the end of the section requires in addition that a settle to a constant at infinity, in the spirit of Duro and Zuazua [10],

$$|a(x) - a_\infty| \le C(1+x)^{-\delta}, \qquad |a_x(x)| \le C(1+x)^{-\delta-1}, \qquad a_\infty > 0,\ \delta > 0; \tag{74}$$

it is introduced there and not before.

The flux in (71) is the one compatible with the Hopf–Cole transformation: setting $v = \int_0^x u$ and $w = e^v$, the boundary conditions become $w(0,t) = 1$, $w(L,t) = e^M$ as in (3), and a direct computation gives

$$w_t = a(x)\, w_{xx}, \qquad (x,t) \in (0,L) \times (0,\infty), \tag{75}$$

the heat equation with variable diffusivity, in non-divergence form, with the same constant Dirichlet data as before. Since (75) is uniformly parabolic with smooth coefficient, for continuous positive w_0 it has a unique classical solution, positive by the maximum principle, and $u = w_x/w$ is the unique solution of (71)–(72) in the class of Section 2, with $\int_0^L u\,dx = M$ for all $t > 0$; the proof of Theorem 2.1 applies verbatim, the coefficient a entering only through the linear theory.

A stationary solution of (71) has constant flux, $a(x)\,(U_x + U^2) \equiv \kappa$. Evaluating (72) at $x = 0$ gives $\kappa = 0$, so $a(x)(U_x + U^2) \equiv 0$ throughout and, since $a > 0$,

$$U_x + U^2 = 0 \quad \text{on all of } (0,L),$$

independently of a. The unique stationary state of mass M is therefore again (4). This propagation from the boundary to the interior is special to the stationary problem, where the flux is constant in x; the time-dependent solution does feel a in the interior. Thus the geometry of the diffusion affects the approach to equilibrium, but not the equilibrium itself.

4.1. Relaxation rate: the principal eigenvalue

The rate of approach to U_M^L is set by the spectral gap of the operator $a(x)\partial_{xx}$ with Dirichlet conditions. It is symmetric in the weighted space $L^2\big((0,L), a^{-1}dx\big)$, the weight cancelling the coefficient so that two integrations by parts move ∂_{xx} from ϕ to ψ, the boundary terms vanishing by the Dirichlet condition. Its principal eigenvalue $\lambda_1(a)$, the least λ for which $a\phi'' + \lambda\phi = 0$, $\phi(0) = \phi(L) = 0$ admits a nontrivial solution, has the variational characterization

$$\lambda_1(a) = \inf_{\substack{\phi \in H_0^1(0,L) \\ \phi \neq 0}} \frac{\displaystyle\int_0^L (\phi_x)^2\,dx}{\displaystyle\int_0^L \phi^2\, a^{-1}\,dx}. \tag{76}$$

Since the numerator does not involve a and the denominator is monotone in a^{-1}, (73) yields

$$\frac{\pi^2}{L^2}\,a_0 \;\le\; \lambda_1(a) \;\le\; \frac{\pi^2}{L^2}\,a_1, \tag{77}$$

with equality when a is constant, recovering $\lambda_1 = \pi^2/L^2$ of Theorem 2.2.

Theorem 4.1. *Let $u_0 \in L^1(0,L)$ have mass M and let u solve (71)–(72). For every $p \in [1,\infty]$ there is $C = C(\|u_0\|_{L^1}, a, L, p) > 0$ such that*

$$\|u(\cdot,t) - U_M^L\|_{L^p(0,L)} \le C\left(1 + t^{-3/4}\right) e^{-\lambda_1(a)\,t}, \qquad t > 0. \tag{78}$$

Proof. The proof of Theorem 2.2 carries over once π^2/L^2 is replaced by $\lambda_1(a)$. With $z := w - W$, $W(x) = 1+(e^M-1)x/L$, expanding $z(\cdot,0)$ in the eigenbasis of $a\partial_{xx}$ gives decay at the rate $e^{-\lambda_1(a)t}$ in $L^2(a^{-1}dx)$, a norm equivalent to L^2 by (73). Since (75) is uniformly parabolic, the parabolic estimates give (8) with $\lambda_1(a)$ in place of λ_1 and with C depending also on $a_0, a_1, \|a\|_{C^1}$, the two time regimes being separated as there. The transfer to $u = w_x/w$ is unchanged. □

The bounds (77) show that the diffusivity enters the relaxation rate only through $\lambda_1(a)$, squeezed between the rates of the constant diffusivities a_0 and a_1; a slower diffusion anywhere in the interval lowers the gap, the effect being global, mediated by the principal eigenfunction, rather than localized where a is small.

4.2. Limits of the present argument on the half-line

On the half-line the relevant regime is self-similar, and there the diffusivity must be compared with a constant one; we assume (74). For the constant coefficient a_∞, the self-similar profile of mass M is obtained from f_M by pure scaling: substituting $u = t^{-1/2}F(x/\sqrt{t})$ in $u_t = a_\infty(u_{xx} + (u^2)_x)$ gives $-\frac{1}{2}F - \frac{1}{2}\xi F' = a_\infty(F'' + (F^2)')$, solved by

$$f_M^{(a_\infty)}(\xi) = \frac{1}{\sqrt{a_\infty}}\, f_M\Big(\frac{\xi}{\sqrt{a_\infty}}\Big), \qquad \int_0^\infty f_M^{(a_\infty)} = M. \tag{79}$$

For the variable coefficient the picture is best seen through $h := w_x$, which by (75) solves the divergence-form equation

$$h_t = (a(x)\, h_x)_x \qquad \text{on } (0,\infty), \tag{80}$$

with the no-flux condition inherited at $x = 0$ and conserved mass $\int_0^\infty h(\cdot,0) = e^M - 1$, exactly as in the constant-coefficient analysis of Theorem 2.7. Under (74) the coefficient of (80) settles to a_∞, and (80) is then a divergence-form diffusion with asymptotically constant coefficient, the setting of Duro and Zuazua [10] for the analogous whole-space problem. This analogy suggests convergence toward $f_M^{(a_\infty)}$, but neither a half-line convergence theorem nor a rate is needed for the results of this paper. We therefore do not formulate an unproved rate as part of the main claims.

The boundary places (80) outside the whole-space statements of [10]: a proof would combine their scaling analysis with Gaussian bounds for the Neumann heat kernel of the divergence-form operator $\partial_x(a\,\partial_x)$ [1], the point requiring care being the uniformity of these bounds near the origin, where a departs from a_∞. For slowly decaying coefficients (δ small) the tail of a may itself limit the rate. We leave this separate problem open.

The confinement mechanism does not survive in explicit form. The change of variable $z(x) = \int_0^x ds/\sqrt{a(s)}$ turns (75) into a heat equation with a first-order drift $-\frac{1}{2}(a_x/\sqrt{a})\,W_z$ that is integrable in z under (74), so that in these coordinates the interval $(0,L)$ has effective length $d(0,L) = \int_0^L ds/\sqrt{a(s)}$ and the Gaussian cost of reaching the boundary and returning is measured in the metric $ds/\sqrt{a}$. But the family Φ_c, the sharp exponent and the matching lower bound of Theorem 3.8 all rest on the closed-form image representation of Lemma 3.6, which is unavailable for variable a. The determination of the sharp confinement constant, presumably a functional of the metric $ds/\sqrt{a}$, is open.

Finally, the reduction above tames the coefficient only because (74) forces a to settle at infinity. The genuinely different case is a diffusivity varying on the scale of the domain, $a(x) = \alpha(x/L)$ for a fixed profile α: the rescaled coefficient does not converge as $L \to \infty$, the similarity structure is lost, and no self-similar profile is available. The interval results of this section (well-posedness, mass conservation, the stationary state (4), and the relaxation rate (78)) already cover this regime, since they use only (73). What is open is the transition: whether a family analogous to Φ_c exists, and whether the confinement exponent is then a genuine functional of α.

5. Numerical illustration

We illustrate the three regimes numerically, always with mass $M = 1$ and initial datum $u_0(x) = 2(1-x)\mathbf{1}_{[0,1]}(x)$, from which $w_0(x) = \exp(\int_0^x u_0)$. The observation window is stated in each caption.

The simulations use the Hopf–Cole transformation to avoid discretizing the nonlinear boundary conditions: we solve the linear heat equation for w by an explicit finite difference scheme and recover $u = w_x/w$. Since the data $w(0,t) = 1$, $w(L,t) = e^M$ are imposed exactly, the mass $M = \ln w(L,t) - \ln w(0,t)$ is exact at the continuous level, and at the discrete level it is conserved up to the $O(\Delta x^2 + \Delta t)$ error of the reconstruction of u. In higher dimensions, the absence of such a transformation would require treating the nonlinear boundary condition directly, which typically introduces mass loss.

For reproducibility: the same mesh $x_j = j\Delta x$ is used for every L, with $\Delta x = 0.02$ and $\mu = \Delta t/\Delta x^2 = 0.4$, within the stability constraint $\mu \leq 1/2$; interior values are updated by $w_j^{n+1} = w_j^n + \mu(w_{j+1}^n - 2w_j^n + w_{j-1}^n)$ with $w_0^n = 1$, $w_N^n = e^M$; w_x is approximated by second-order centered differences at interior nodes and second-order one-sided differences at the endpoints; discrete supremum norms are maxima over nodes in the stated window and discrete L^p norms use composite trapezoidal quadrature. Both U_M^L and f_M are known explicitly and serve as references. Figure 2 uses the closed formulas and no half-line truncation.

As a grid-refinement check, Table 1 compares the finite-difference solution at $L = 20$, $t = 0.5$ with the sine-series representation of $w - W$ (500 modes), the supremum error being computed for $u = w_x/w$ on the full interval and Δt being in each run the largest subdivision of t not exceeding $0.4\Delta x^2$. The observed orders are close to two, as expected. Raising the reference truncation to 750 and 1000 modes changes the reference values by less than 10^{-15} at all nodes used.

Δx	Δt	$\|u_{\Delta x} - u_{\mathrm{ref}}\|_\infty$	observed order
0.04	6.3939×10^{-4}	4.4952×10^{-4}	–
0.02	1.6000×10^{-4}	1.1666×10^{-4}	1.95
0.01	4.0000×10^{-5}	2.9697×10^{-5}	1.97

Table 1: Grid refinement for the finite-difference reconstruction of u at $L = 20$ and $t = 0.5$.

We first illustrate the sharp rate of Subsection 3.2.

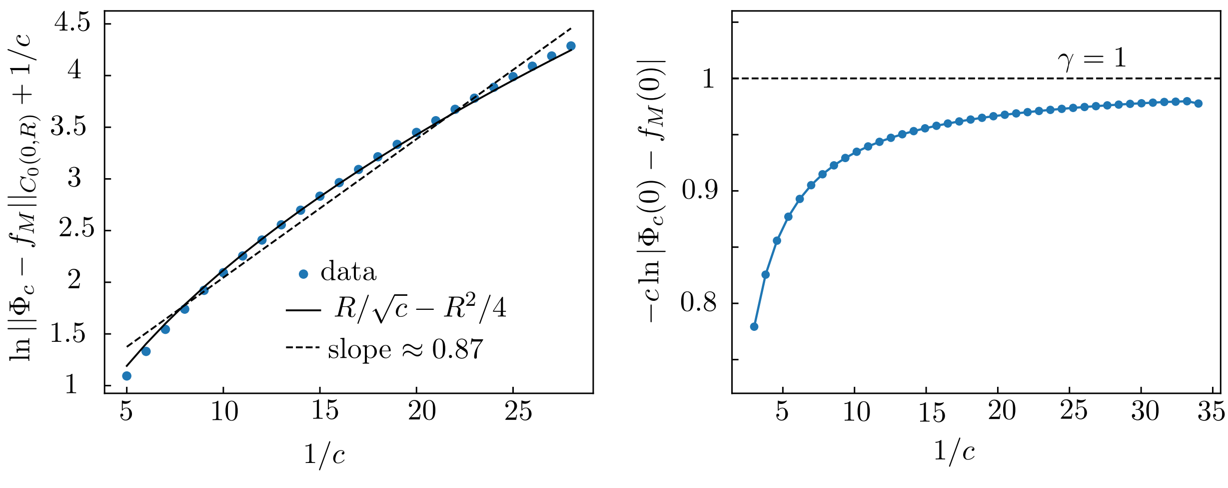

Figure 2: The sharp rate $\gamma = 1$ at $M = 1$, $R = 1$, computed from the closed forms of Φ_c and f_M.

Since Φ_c and f_M are known in closed form, Figure 2 is computed to machine precision, with no discretization; as Φ_c is the $L \to \infty$ limit of $v^L(\cdot, cL^2)$ (Theorem 3.3), it displays the transition in the limit that isolates the exponent. The right panel is decisive: $-c\ln|\Phi_c(0) - f_M(0)|$, which Theorem 3.8(ii) asserts converges to 1, reaches 0.98 by $1/c = 34$. The left panel shows why a straight-line fit of $\ln E$ against $1/c$ misleads: it returns a slope near 0.87 that drifts with the window, because the leading image contribution and the upper bound carry the factor $e^{-1/c+R/\sqrt{c}-R^2/4}$, whose subexponential part $e^{R/\sqrt{c}}$ flattens the apparent slope. Removing it leaves 0.995, in agreement with $\gamma = 1$ (Remark 3.10).

The next simulation illustrates Corollary 3.16, where $c = t/L^2$ in (70). Letting $c \to 0$ is not by itself enough: the estimate is informative only inside the window $1 \ll t \ll L^2$, so L and t must grow together, with $t = cL^2 \to \infty$ and $c \to 0$. Figure 3 compares finite values of L and t and is a qualitative illustration of that separation of scales rather than a test of the limit itself, since the smallest displayed times do not satisfy $1 \ll t$. It nevertheless shows the predicted ordering: profiles with smaller t/L^2 stay closer to the half-line self-similar profile, and the shorter intervals feel the remote boundary first.

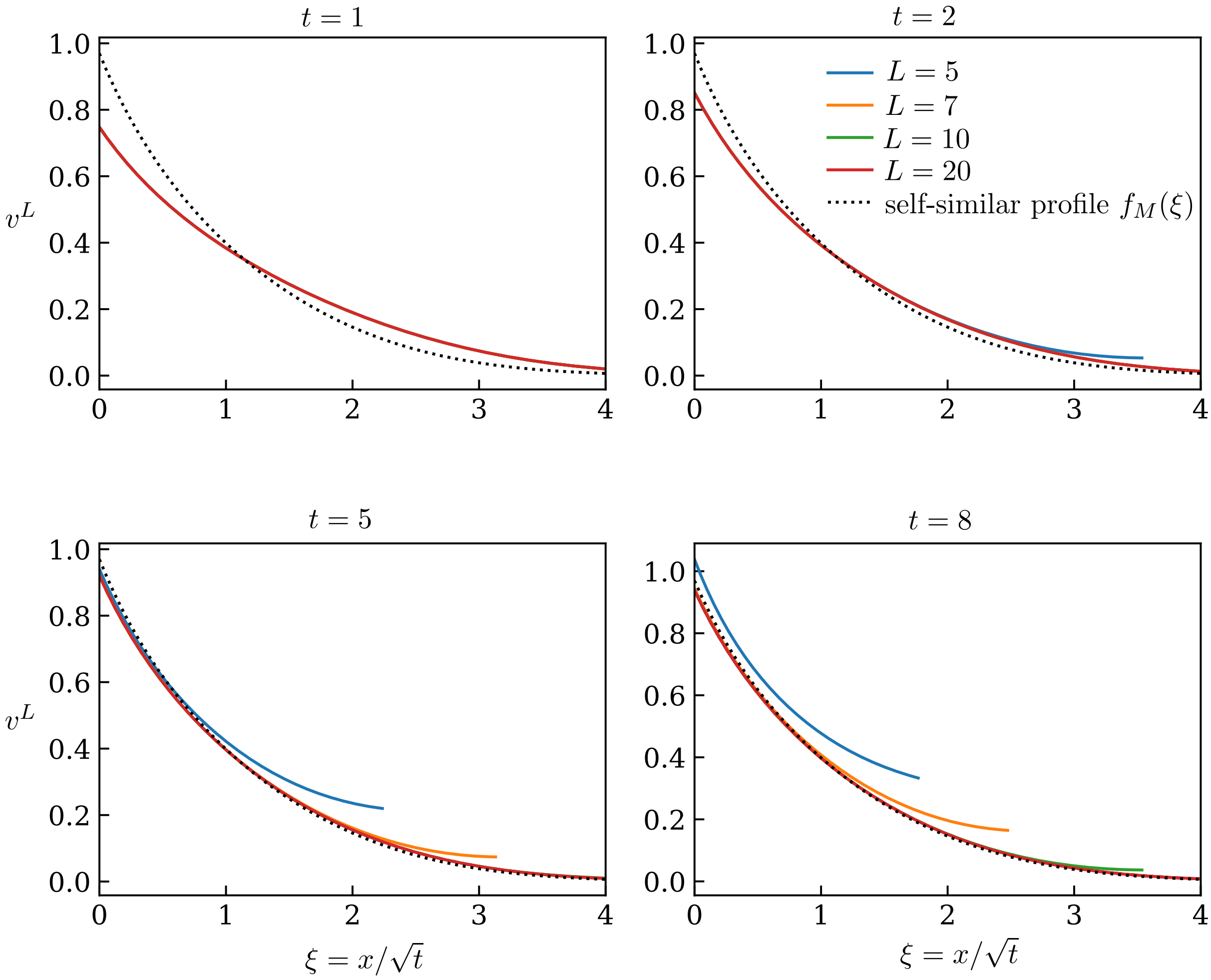

Figure 3: Profiles of $v^L(\xi, t) = \sqrt{t}\, u^L(\xi\sqrt{t}, t)$ in similarity variables $\xi = x/\sqrt{t}$ for $L \in \{5, 7, 10, 20\}$ at times $t \in \{1, 2, 5, 8\}$.

At $t = 1$ the four profiles are clustered, still under the influence of the initial datum; at $t = 2$ they have moved closer to f_M, and v^5 already detaches, being the first to feel the boundary (its similarity interval is then $0 < \xi < 5/\sqrt{2} \approx 3.5$). At $t = 5$ the deviation of v^5 is pronounced and v^7 starts to follow, while v^{10} and v^{20} remain very close to f_M; at $t = 8$ so does v^{10}, whereas v^{20} is still clustered with f_M.

Fixing $L = 20$, we now let t increase, expecting u^L to approach the self-similar profile first and the interval equilibrium afterwards. To make the second convergence visible we return to the physical variable x, in which the self-similar profile reads $\tilde{f}_M(x, t) = t^{-1/2} f_M(x/\sqrt{t})$.

At $t = 0.5$ the profile is still dominated by the initial datum; by $t = 5$ it is very close to $\tilde{f}_M$, and it departs visibly from that diffusive regime around $t = 50$. The panel $t = 100$ shows the transition

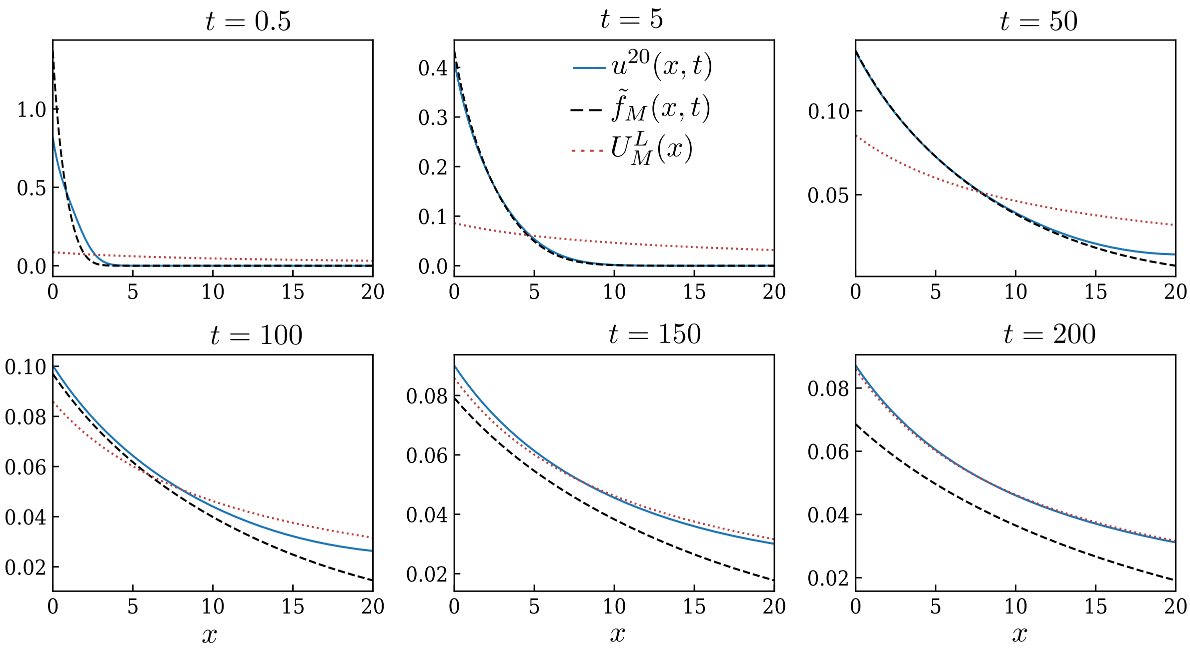

Figure 4: Evolution of u^{20} (u^L, for $L = 20$) from self-similar dynamics ($\tilde{f}_M$) to the stationary regime (U_M^L).

phase between the two regimes; by $t = 150$ the solution is already near U_M^L, and at $t = 200$ the two curves are indistinguishable.

6. Conclusions and outlook

We have resolved the transition between half-line and bounded-domain asymptotics for the viscous Burgers equation with conservative boundary conditions. At the critical scale $t = cL^2$, the rescaled solution converges to an explicit family Φ_c connecting the nonlinear self-similar profile to the nonconstant interval equilibrium. Before confinement dominates, the remote-boundary correction is exponentially small, with sharp order $e^{-L^2/t}$ and optimal constant 1. The result therefore provides the four elements of a complete crossover description: the relevant scale, the profile governing the transition, its two limiting states, and the sharp rate at which the boundary becomes visible.

The interval/half-line pair is the one-dimensional instance of a broader truncation question: whether dynamics on bounded domains Ω_L shadows that on an unbounded limit domain up to time $t \sim L^2$. Cones are a natural next setting because they preserve diffusive scaling. The same one-dimensional argument already extends to $(-L, L)$, with the whole line as limiting domain [11, 13].

For a multidimensional viscous conservation law

$$u_t = \nabla\cdot(\nabla u + F(u)), \qquad (\nabla u + F(u)) \cdot n = 0,$$

mass is still conserved, but the two tools responsible for the exact result above disappear: there is generally no Hopf–Cole transformation, and heat propagation in a cone is governed by the angular spectrum rather than a one-dimensional image expansion. Establishing a nonlinear crossover, and identifying the geometric constant replacing 1, therefore require methods beyond those of this paper.

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Data and code availability

No external data were used in this study. The numerical results are generated from the initial datum, closed formulas, and finite-difference scheme specified in Section 5. The script `numerical_refinement_v12.py`, supplied as a supplementary file, reproduces Table 1. The figure-generation code and the resulting numerical arrays are available from the corresponding author upon reasonable request and will be deposited in a public repository upon acceptance.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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