Explicit decay rates for discrete velocity BGK models

Tobias Wöhrer

in collaboration with Anton Arnold, Amit Einav and Beatrice Signorello

FAU Erlangen-Nürnberg
Department of Data Science

ASC Institute

Chair
DYNAMICS, CONTROL
AND NUMERICS
FAU

TU
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Overview

Part 1: Sharp decay rates for “hypocoercive ODEs”

- Lyapunov functionals for linear ODEs

Part 2: Multi-velocity BGK equations

- Spatially homogeneous relaxation
- Extension to non-homogeneous relaxation
- Extension multi-velocities.

Part 1: Lyapunov functionals for linear ODEs

Decay for nonsymmetric ODEs: Lyapunov functionals

linear ODEs with constant coefficients

$$\dot{x}(t) = -\mathbf{C}x(t), \qquad \mathbf{C} \in \mathbb{C}^{n\times n}, t \geq 0. \tag{ODE}$$

Objective: Construct functional $E[\cdot]$ such that

$$\frac{d}{dt}E[x(t)] \leq -2\mu E[x(t)] \implies E[x(t)] \leq e^{-2\mu t}E[x(0)]$$

with best possible decay rate $\mu > 0$, i.e. sharp long time behavior.

We look at modified norms:

$$E[x(t)] = |x(t)|_{?}^{2}$$

Decay for nonsymmetric ODEs: Lyapunov functionals

Definition:

C is *coercive* if $\mathrm{Re}(x^H \mathsf{C} x) = x^H \mathsf{C}_{\mathrm{sym}} x \geq \mu |x|_2^2 \; \forall x$ (for some $\mu > 0$).

Notation $x^H := \overline{x}^T$.

Example 1: $\mathsf{C} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \;\; \lambda_{\mathsf{C}} = 1 \pm i \Rightarrow$ decay rate $= 1$.

- Choose functional $E[x] = |x|_2^2$:

$$\frac{d}{dt}|x(t)|_2^2 = -x^H(\mathsf{C}^H + \mathsf{C})x = -2x^H \mathsf{C}_{\mathrm{sym}} x = -2|x|_2^2$$

$$\Longrightarrow \;\; |x(t)|_2^2 \leq e^{-2t}|x(0)|_2^2 \, .$$

Decay for nonsymmetric ODEs: Lyapunov functionals

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Notation $x^H := \overline{x}^T$.

Example 2: $\mathsf{C} = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}$, $\lambda_{\mathsf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate $= \frac{1}{2}$.

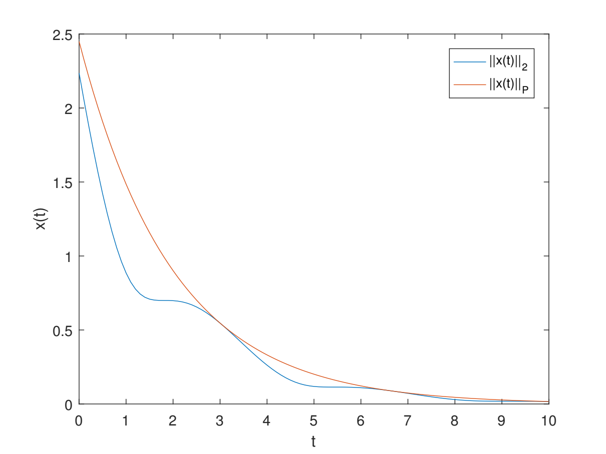
- Choose functional $E[x] = |x|_2^2$:

$$\frac{d}{dt}|x(t)|_2^2 = -x^H(\mathsf{C}^H + \mathsf{C})x = -2x^H \mathsf{C}_{\mathrm{sym}} x \leq 0$$

C is "hypocoercive" $\implies$ no decay of $|\cdot|_2^2$.

ad Example 2:

But decay of modified norm $|x(t)|_{\mathbf{P}}^2 := x^H \mathbf{P} x$; $\mathbf{P} := \begin{pmatrix} 1 & -\frac{i}{2} \\ \frac{i}{2} & 1 \end{pmatrix}$



Choice of P

Q: Can we always find an Euclidean norm modification?

Lemma 1 (Arnold, Erb '14)

Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ *be positive stable, i.e.*
$\mu_{\mathbf{C}} := \min\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(\mathbf{C})\} > 0\,.$

1. *If all* $\lambda_\mu \in \{\lambda \in \sigma(\mathbf{C}) \mid \mathrm{Re}\,\lambda = \mu_{\mathbf{C}}\}$ *are not defective (i.e. geometric = algebraic multiplicity)*
   $$\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{C}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{PC} + \mathbf{C}^H\mathbf{P} \geq 2\mu_{\mathbf{C}}\mathbf{P}\,.$$

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2. *If (at least) one* λ_μ *is defective* $\Rightarrow$
   $\forall \varepsilon > 0 \quad \exists\, \mathsf{P} = \mathsf{P}(\varepsilon) > 0\ : \quad \mathsf{PC} + \mathsf{C}^H\mathsf{P} \geq 2(\mu_{\mathsf{C}} - \varepsilon)\mathsf{P}\,.$

In [AE '14]: Used to prove decay rates for hypocoercive Fokker–Planck equations with linear drift.

Choice of P / Lyapunov's direct method

Proof: P can be constructed explicitly.
E.g. for C non-defective / diagonalizable:

$$\mathsf{P} := \sum_{j=1}^{n} \beta_j v_j \otimes v_j \ ;$$

- v_j are eigenvectors of C^H
- $\beta_j > 0$ are arbitrary weights

□

P not unique; but the decay rates μ_{C} (or $\mu_{\mathsf{C}} - \varepsilon$) are independent of P.

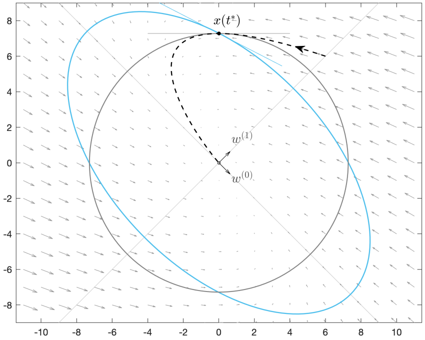
Decay of P–norm, non-defective case

Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case):

Let $|x|_{\mathbf{P}}^2 := x^H \mathbf{P}\, x$, with $\mathbf{P}$ from Lemma 1

$$\frac{d}{dt}\,|x|_{\mathbf{P}}^2 \quad = \quad -x^H \big(\underbrace{\mathbf{P}\mathbf{C} + \mathbf{C}^H\mathbf{P}}_{\geq 2\mu_{\mathbf{C}}\mathbf{P}}\big) x \leq -2\mu_{\mathbf{C}} |x|_{\mathbf{P}}^2$$

$$\Rightarrow \quad |x(t)|_{\mathbf{P}} \quad \leq \quad |x(0)|_{\mathbf{P}}\, e^{-\mu_{\mathbf{C}} t}, \qquad t \geq 0.$$



$\rightarrow$ uniform decay with sharp rate $\frac{1}{2}$

P as coordinate transformation:

$$\tilde{\mathbf{C}} := \sqrt{\mathbf{P}}\mathbf{C}\sqrt{\mathbf{P}}^{-1}, \quad \tilde{\mathbf{C}}_{\mathrm{sym}} \geq \mu_{\mathbf{C}}\mathbf{I}.$$

Decay of P–norm, defective case

Example 3:

$\dot{x} = -\mathbf{C}x$ with defective $\mathbf{C} = \begin{pmatrix} 0 & i \\ i & 2 \end{pmatrix} = \mathbf{V} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{V}^{-1}$

- $\lambda_{1,2} = 1$, algebraic multiplicity $>$ geometric multiplicity
- Lemma 1 $\implies |x(t)|^2_{\mathbf{P}_\varepsilon} \leq c_\varepsilon e^{-2(1-\varepsilon)t}$
- True behavior: $|x(t)|_2^2 \leq c(1+t^2)e^{-2t}$
- $\rightarrow P_\varepsilon$-norm rate not sharp!

Lemma 2 (Arnold, Jin, W. '19)

Let $\mathsf{C} \in \mathbb{C}^{n\times n}$ *be positive stable, i.e.* $\mu_{\mathsf{C}} := \min\{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(\mathsf{C})\} > 0$. *If (at least) one* λ_μ *is defective there exists a time-dependent Hermitian positive-definite matrix* $\mathsf{P}(t) \in \mathbb{C}^{d\times d}$ *such that solutions to* $\dot{x} = -\mathsf{C}x$ *satisfies*

$$|x(t)|^2_{\mathsf{P}(t)} \leq e^{-2\mu_{\mathsf{C}} t} |x(0)|^2_{\mathsf{P}(0)}.$$

For Fokker–Planck setting in $\mathbb{R}^d$: [Monmarché, '15]

Multi-velocity BGK equations

linear BGK models[1]

$$\partial_t f(v,x,t) + v\partial_x f(v,x,t) = \mathrm{Q}_{\mathrm{relax}} f(v,x,t)$$
$$:= \sigma \left[M_T(v) \int_{\mathbb{R}} f(x,v,t)dv - f(x,v,t) \right]$$

Idea: Replace nonlinear Boltzmann collision by linear relaxation term.

- $f(v,x,t)$ particle density at velocity $v \in \mathbb{R}$, position $x \in \Omega$, time $t \geq 0$.
- $M_T(v)$ is the normalized Maxwellian at temperature T.
- Prescribed relaxation rate $\sigma > 0$.
- Conservation of mass and momentum.

[1] named after the three physicists Bhatnagar, Gross, and Krook

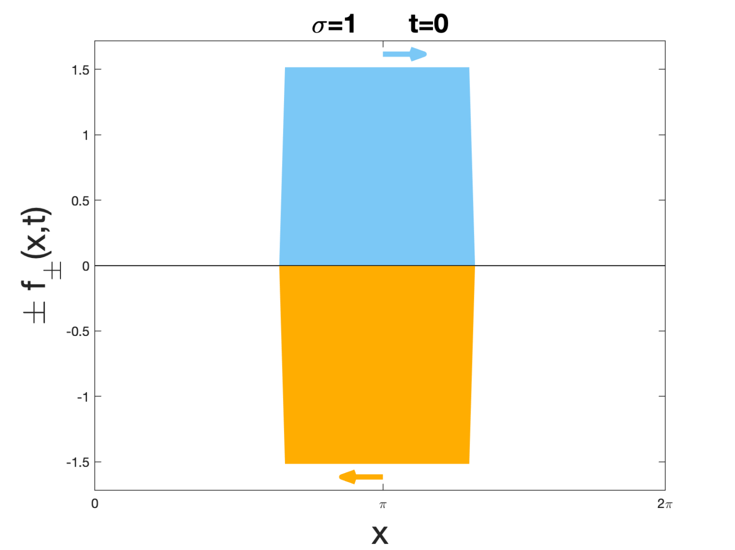
Further simplification: From $v \in \mathbb{R}$ to $v \in \{-1, +1\}$:

Goldstein–Taylor Model for $x \in \mathbb{T}^1$:

$$\begin{aligned} \partial_t f_+(x,t) + \partial_x f_+(x,t) &= \frac{\sigma(x)}{2}\big(f_-(x,t) - f_+(x,t)\big), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) &= -\frac{\sigma(x)}{2}\big(f_-(x,t) - f_+(x,t)\big), \\ f_\pm(x,0) &= f_{\pm,0} \in L^2(\mathbb{T}) \end{aligned} \tag{GT}$$

- $f_\pm(\cdot, t)$ probability density of particles with $v = \pm 1$.
- Relaxation coefficient $\sigma(x) > 0$.

Equation exhibits **hypocoercive dynamics**.



Global (normalized) equilibrium: $(f_+^\infty, f_-^\infty)^T = (\frac{1}{2}, \frac{1}{2})^T$

We observed: Interplay of conservative force (transport) and partially dissipative force (relaxation) leads to (exponential) convergence to equilibrium.

Hypocoercivity:

- [Villani '09]: Abstract concept for operators on Hilbert spaces
$$L = T + Q,$$
where T is antisymmetric and Q coercive on a subspace.
- L is **hypocoercive** if T and Q fulfil "mixing condition" expressed through commutator relations.

$$\Longrightarrow \|e^{-Lt} f_0\|_{\mathcal{H}} \leq \underbrace{\mathcal{C}}_{\geq 1} e^{-\lambda t} \|f_0\|_{\mathcal{H}}, \quad t \geq 0.$$

- E.g. BGK, Fokker–Planck operators with deg. diffusion.

Overview

Goals for Goldstein-Taylor model:

- Analyze the long-time behavior for **non-homogeneous relaxation** $\sigma(x)$.
- Obtain **explicit decay rates**.
- Extend results to **multi-velocity** BGK setting.

Strategy:

- First **homogeneous relaxation:**
  Mode-by-Mode Lyapunov functional for lin. ODEs.
- **Non-homogeneous relaxation:**
  Functional via pseudo-differential operator.

Constant Relaxation $\sigma > 0$

Reformulate equation for mass density and flux density

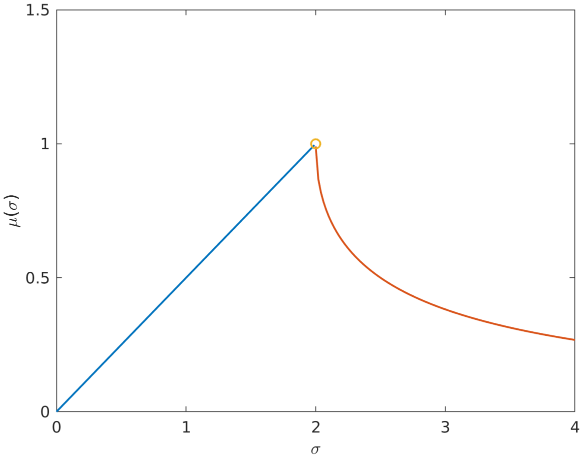
$$u := f_+ + f_-, \qquad w := f_+ - f_-.$$

Goldstein–Taylor Model in Fourier Space:

$$\partial_t \begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix} = -\underbrace{\begin{pmatrix} 0 & ik \\ ik & \sigma \end{pmatrix}}_{C_k(\sigma)} \underbrace{\begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix}}_{\hat{y}_k(t)}, \qquad k \in \mathbb{Z}.$$

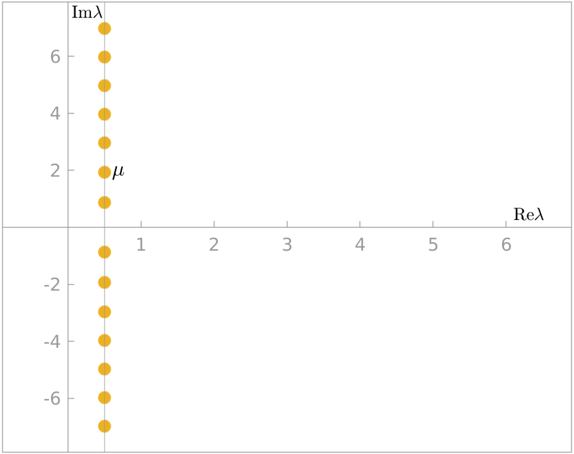
Connection to Part 1:

- Eigenvalues of $C_k(\sigma)$ determine decay behavior of $\hat{y}_k(t)$.
- $C_k(\sigma)$ is “hypocoercive”
- Slowest decaying Fourier mode determines decay for $y(t)$.

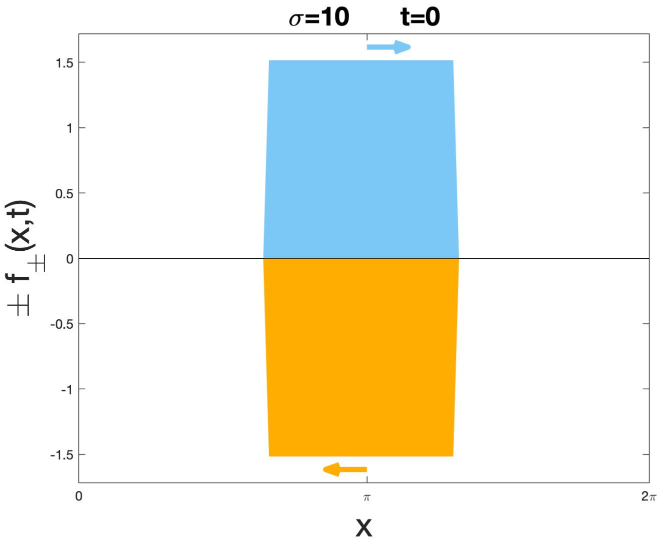


uniform-in-k spectral gap

$$\mu(\sigma) := \min_{k\in\mathbb{Z}} \mu_k(\sigma) = \mu_1(\sigma) = \mathrm{Re}\left(\frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}\right) \in (0,1].$$



Eigenvalues of $C_k(\sigma)$ for Fourier modes $k \in \mathbb{Z} \setminus \{0\}$.



σ=10 t=0
1.5
1
0.5
0
-0.5
-1
-1.5
0
π
2π
x
± f±(x,t)

Case $\sigma \in (0,2)$:

Plan: For each Fourier mode $\hat{y}_k = (\hat{u}_k, \hat{w}_k)^T$, $k \in \mathbb{Z} \setminus \{0\}$, define the **modified norm** (constructed according to Lemma 1)

$$|\hat{y}_k|^2_{P_k} := \hat{y}_k^H P_k \hat{y}_k, \qquad \text{with} \qquad P_k := \begin{pmatrix} 1 & \frac{\sigma}{2ik} \\ -\frac{\sigma}{2ik} & 1 \end{pmatrix} > 0.$$

$$\frac{d}{dt}|\hat{y}_k|^2_{\mathsf{P}_k} = -\hat{y}_k(\mathsf{C}_k^H \mathsf{P}_k + \mathsf{P}_k \mathsf{C}_k)\hat{y}_k \leq -\sigma\, \hat{y}_k^H \mathsf{P}_k \hat{y}_k$$
$$\implies |\hat{y}_k(t)|^2_{\mathsf{P}_k} \leq e^{-\sigma t}|\hat{y}_k(0)|^2_{\mathsf{P}_k}.$$

$|\cdot|^2_{P_k}$ is Lyapunov functional with sharp decay rate for each mode $k \in \mathbb{Z} \setminus \{0\}$.

Case $\sigma \in (0,2)$: Spatial Lyapunov functional

Fourier transform:

- $\mathcal{F}[\partial_x f](k) = ik\mathcal{F}[f](k)$, $k \in \mathbb{Z}$.
- $\implies \frac{1}{ik}\mathcal{F}[f](k) = \mathcal{F}[\partial_x^{-1} f](k)$ $k \neq 0$.

Idea: Recast functional in position space

$$\sum_{k\in\mathbb{Z}} \|(\hat{u}_k - \hat{u}_k^\infty, \hat{w}_k)^T\|_{P_k}^2 = \sum_{k\in\mathbb{Z}} |\hat{u}_k - \hat{u}_k^\infty|^2 + |\hat{w}_k|^2 - \sigma \operatorname{Re}(\overline{\hat{w}_k} \frac{\hat{u}_k - \hat{u}_k^\infty}{ik})$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \Big((u-u^\infty)^2 + w^2 - \sigma w \partial_x^{-1}(u-u^\infty)\Big) dx$$

$$=: E_\sigma[u-u^\infty, w],$$

where $u^\infty \equiv 1$.

Spatial Entropy Functional

For parameter $\theta \in (0, 2)$:

$$E_{\theta}[u, w] := \|u\|_{L^2}^2 + \|w\|_{L^2}^2 - \frac{\theta}{2\pi} \int_0^{2\pi} w \partial_x^{-1} u dx,$$

where

$$(\partial_x^{-1} u)(x) := \int_0^x u dx + c(u) \quad \text{with} \quad c(u) := -\frac{1}{2\pi} \int_0^{2\pi} \int_0^x u dy dx.$$

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Norm bounds:

$$(1-\tfrac{\theta}{2})\|(u-u^{\infty},w)\|_{L^2}^2 \leq E_{\theta}[u-u^{\infty},w] \leq (1+\tfrac{\theta}{2})\|(u-u^{\infty},w)\|_{L^2}^2.$$

Decay estimates for constant $\sigma > 0$

Lemma 3 (Arnold, Einav, Signorello, W.)

Let $(u,w)^T$ be a solution to (GT) with constant $\sigma > 0$.

(i) *For $\sigma \in (0,2)$*

$$E_{\sigma}[u(t) - u^{\infty}, w(t)] \leq E_{\sigma}[u(0) - u^{\infty}, w(0)]e^{-\sigma t}.$$

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(ii) $\sigma = 2$, *with* $\theta_\varepsilon := \frac{2(2-\varepsilon^2)}{2+\varepsilon^2}$,

$$E_{\theta_\varepsilon}[u(t) - u^\infty, w(t)] \leq E_{\theta_\varepsilon}[u(0) - u^\infty, w(0)]e^{-2(1-\varepsilon)t}.$$

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$$E_{\theta_{\varepsilon}}[u(t) - u^{\infty}, w(t)] \leq E_{\theta_{\varepsilon}}[u(0) - u^{\infty}, w(0)]e^{-2(1-\varepsilon)t}.$$

(iii) *For* $\sigma > 2$ *with* $\mu = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}$,

$$E_{\frac{4}{\sigma}}[u(t) - u^{\infty}, w(t)] \leq E_{\frac{4}{\sigma}}[u(0) - u^{\infty}, w(0)]e^{-2\mu t}.$$

Space-dependent relaxation $\sigma(x)$

Theorem 4 (Arnold, Einav, Signorello, W.)

Let $(u,w)^T$ be a solution to (GT) with $u_0, w_0 \in L^2(\mathbb{T})$ and

$$0 < \sigma_{\min} := \inf_{x\in\mathbb{T}} \sigma(x) \le \sup_{x\in\mathbb{T}} \sigma(x) =: \sigma_{\max} < \infty.$$

Then, for $\theta^* = \min\{\sigma_{\min}, \frac{4}{\sigma_{\max}}\}$ exists an explicit decay rate $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$ such that

$$E_{\theta^*}[u(t) - u^\infty, w(t)] \le e^{-\alpha^* t} E_{\theta^*}[u_0 - u^\infty, w_0].$$

- **Proof:** perturbative approach
- Decay rate $\alpha^*(\sigma_{\min}, \sigma_{\max})$ is not sharp.

Discussion on GT result

[Bernard, Salvarani '13]:

- ⊕ For $\sigma \in L^1(\mathbb{T})$, $\sigma(x) \geq 0$ sharp decay rate

$$\alpha = \min\{\sigma_{\text{avg}}, D(0)\},$$

  where $D(0)$ is the spectral gap of Telegrapher's equation.
- ⊖ Method restricted to two velocities.
- ⊖ Rate in general not explicit.

Our approach:

- Extends to multi-velocity models with $\sigma(x)$.
- Strategy extends to $x \in \mathbb{R}$.

Three-velocity BGK model

$$\partial_t u_1(x,t) + \sqrt{\frac{2}{3}}\partial_x u_2(x,t) = 0,$$
$$\partial_t u_2(x,t) + \sqrt{\frac{2}{3}}\partial_x u_1(x,t) + \frac{1}{\sqrt{3}}\partial_x u_3(x,t) = -\sigma(x)u_2(x,t),$$
$$\partial_t u_3(x,t) + \frac{1}{\sqrt{3}}\partial_x u_2(x,t) = -\sigma(x)u_3(x).$$

(3BGK)

System matrix for $\sigma(x) \equiv \sigma$ in Fourier space and approximate modified norms:

$$\mathbf{C}_k = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}}ik & 0 \\ \sqrt{\frac{2}{3}}ik & \sigma & \frac{1}{\sqrt{3}}ik \\ 0 & \frac{1}{\sqrt{3}}ik & 0 \end{pmatrix} \qquad \mathbf{P}_k = \begin{pmatrix} 1 & \frac{\lambda}{ik} & 0 \\ -\frac{\lambda}{ik} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad k \neq 0,$$

(1)

Theorem 5

Let $u_1, u_2, u_3 \in C([0,\infty); L^2(\mathbb{T}))$ *be mild solutions to* (3BGK) *with initial datum* $u_{1,0}, u_{2,0}, u_{3,0} \in L^2(\mathbb{T})$. *For*

$$\mathfrak{E}_\theta(f,g,h) := \|f\|^2 + \|g\|^2 + \|h\|^2 - \frac{\theta}{2\pi}\int_0^{2\pi} \partial_x^{-1} f(x)\, g(x) dx,$$

we have that

$$\mathfrak{E}_\theta(u_1(t) - u_\infty, u_2(t), u_3(t)) \le \mathfrak{E}_\theta(u_{1,0} - u_\infty, u_{2,0}, u_{3,0}) e^{-\alpha t}\,, \quad t \ge 0,$$

with θ *appropriately chosen and explicit rate* $\alpha > 0$.

Summary

- Goldstein–Taylor equation exhibits **hypocoercive dynamics**.
- Modification of L^2 norm necessary.
- Modal analysis for homogeneous relaxation:
  modified norms for lin. ODEs to obtain functional
- **Non-homogeneous relaxation** $\sigma(x) > 0$ and multi-velocities:
  Expression of Lyapunov functional in spatial domain via **anti-derivatives**.

Outlook

- Discrete velocity $\rightarrow$ continuous velocity?
- Allow $\sigma_{\mathrm{min}} = 0$.
- Nonlinear versions of discrete velocity BGK.

Thank you for your attention.

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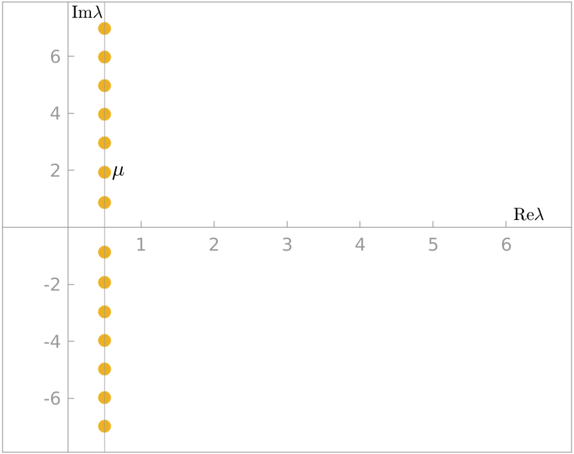
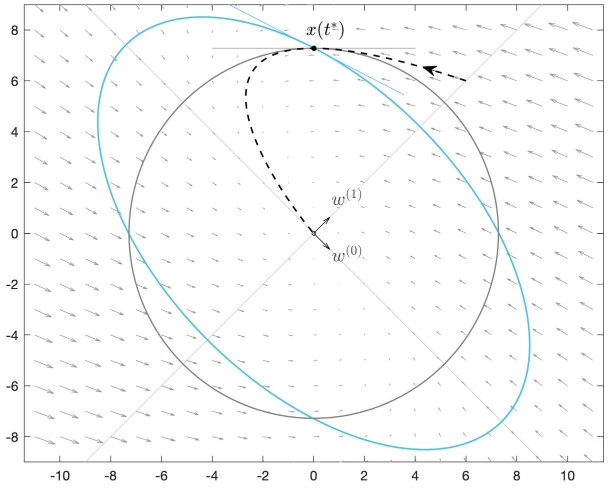
Bernard, É., Salvarani, F.: Optimal estimate of the spectral gap for the degenerate Goldstein–Taylor model. (2013).

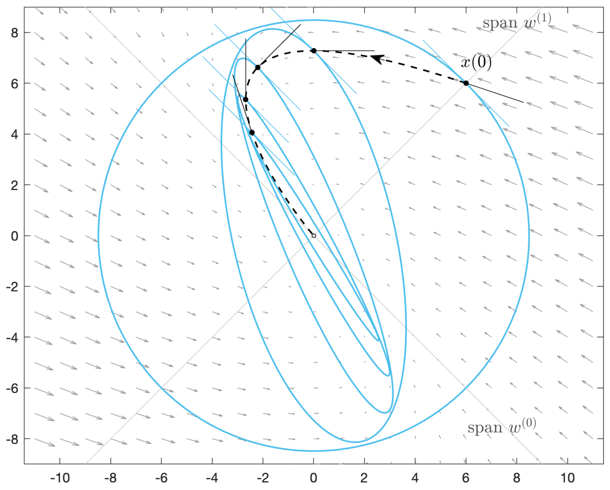
Bouin, E., Dolbeault, J., Mischler, S., Mouhot, C., Schmeiser, C.: Hypocoercivity without confinement. (2020).



Eigenvalues of $A_k(\sigma = 1)$ for Fourier modes $k \in \mathbb{Z} \setminus \{0\}$.



$x(t^*)$
$w^{(1)}$
$w^{(0)}$
8
6
4
2
0
-2
-4
-6
-8
-10
-8
-6
-4
-2
0
2
4
6
8
10



span $w^{(1)}$
$x(0)$
span $w^{(0)}$
8
6
4
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10