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EXPONENTIAL DECAY FOR THE SEMILINEAR WAVE EQUATION WITH LOCALIZED DAMPING IN UNBOUNDED DOMAINS [1]

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ABSTRACT. — We consider the damped semilinear wave equation

$$u_{tt}-\Delta u+\alpha u+f(u)+a(x)u_t=0 \quad \text{in } \mathbf{R}^n\times(0,\infty)$$

with $\alpha>0$, $a\in L_+^\infty(\mathbf{R}^n)$, $a(x)\geqq a_0>0$ as $|x|\to+\infty$ and $f\in C^1(\mathbf{R})$ satisfying $f(s)s\geqq 0$ for every $s\in\mathbf{R}$ and the usual growth conditions at infinity.

The exponential decay of the energy of every solution is proved by assuming either that

(i) f is globally Lipschitz and that either $f'(s)$ has some limits as s goes to $+\infty$ and $-\infty$ or the limit of $f(s)/s$ as $|s|\to+\infty$ exists

or

(ii) f is superlinear.

The method of proof is based on multiplier techniques and on unique continuation results that allow us to estimate the global energy of any solution in terms of the "energy concentrated at infinity".

Analogous results may be proved for equations in unbounded domains, when the damping term is effective both at infinity and on a neighbourhood of the boundary of the domain.

RÉSUMÉ. — On considère l'équation des ondes semilinéaire dissipative suivante

$$u_{tt}-\Delta u+\alpha u+f(u)+a(x)u_t=0 \quad \text{in } \mathbf{R}^n\times(0,\infty)$$

avec $\alpha>0$, $a\in L_+^\infty(\mathbf{R}^n)$, $a(x)\geqq a_0>0$ lorsque $|x|\to+\infty$ et $f\in C^1(\mathbf{R})$ vérifiant $f(s)s\geqq 0$ pour tout $s\in\mathbf{R}$ et les conditions de croissance à l'infini habituelles.

On démontre la décroissance exponentielle de l'énergie des solutions lorsque $t\to+\infty$ en supposant que soit

(i) f est globalement Lipschitz et soit la limite de $f'(s)$ existe lorsque $s\to+\infty$ et $s\to-\infty$ ou bien la limite de $f(s)/s$ existe lorsque $|s|\to+\infty$

ou bien

(ii) f est surlinéaire.

La méthode de démonstration est basée sur des techniques de multiplicateurs et un principe de continuation unique qui permettent d'estimer l'énergie totale des solutions en fonction de « l'énergie concentrée à l'infini ».

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On démontre des résultats analogues pour l'équation des ondes dans des domaines non bornés en supposant que la dissipation est effective à l'infini et sur un voisinage du bord du domaine.

1. Introduction

This paper is devoted to the study of the exponential decay of solutions of the following semilinear damped wave equation:

$$\begin{cases} u_{tt}-\Delta u+\alpha u+f(u)+a(x)u_t=0 & \text{in } \mathbf{R}^n\times(0,\infty) \\ u(0)=u_0\in H^1(\mathbf{R}^n), & u_t(0)=u_1\in L^2(\mathbf{R}^n). \end{cases} \tag{1.1}$$

We assume

$$a\in L^\infty_+(\mathbf{R}^n),\quad a(x)\geqq a_0>0 \qquad \text{a.e. in}\quad \Omega_R=\mathbf{R}^n\setminus B_R=\{|x|\geqq R\} \tag{1.2}$$

for some $R>0$ where $B_R=\{x\in\mathbf{R}^n:|x|<R\}$,

$$\alpha>0;\ f(s)s\geqq 0 \text{ for every } s\in\mathbf{R}, \tag{1.3}$$

$$\begin{cases} f\in C^1(\mathbf{R}) \text{ and there exist some constants } C>0,\ p>1,\ (n-2)p\leqq n \text{ such that} \\ |f(s_1)-f(s_2)|\leqq C(1+|s_1|^{p-1}+|s_2|^{p-1})|s_1-s_2| \quad \text{for every } s_1,s_2\in\mathbf{R}. \end{cases} \tag{1.4}$$

Condition (1.2) ensures that the damping term $a(x)u_t$ is effective on the set Ω_R. From (1.4) we deduce that problem (1.1) is well posed in $H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$. Hypothesis (1.3) ensures the coercivity of the energy

$$E(t)=\frac{1}{2}\int_{\mathbf{R}^n}[|\nabla u(x,t)|^2+|u_t(x,t)|^2+\alpha|u(x,t)|^2]\,dx+\int_{\mathbf{R}^n}F(u(x,t))\,dx \tag{1.5}$$

with

$$F(z)=\int_0^z f(s)\,ds \quad \text{for all } z\in\mathbf{R}. \tag{1.6}$$

Therefore, under conditions (1.2)-(1.4) problem (1.1) has a unique solution in $C([0,\infty);H^1(\mathbf{R}^n))\cap C^1([0,\infty);L^2(\mathbf{R}^n))$ and the energy $E(t)$ is a non increasing function of the time variable t. More precisely,

$$E(t_2)-E(t_1)=-\int_{t_1}^{t_2}\int_{\mathbf{R}^n}a(x)|u_t(x,t)|^2\,dx\,dt \quad \text{for all } t_2>t_1\geqq 0. \tag{1.7}$$

The aim of this paper is to give sufficient conditions on f implying the existence of some constants $C>1$, $\gamma>0$ such that

$$(1.8)\qquad E(t)\leqq CE(0)e^{-\gamma t}\quad \text{for all } t\geqq 0 \text{ and for every solution of } (1.1).$$

This problem is by now well understood in which concerns the linear ($f=0$) wave equation on a bounded domain Ω with Dirichlet or Neumann homogeneous boundary conditions. In that case, by adapting the methods by C. Bardos, G. Lebeau and J. Rauch ([2], [3]) the exponential decay may be proved when $a(x)\geqq a_0>0$ on some open subset ω of Ω which satisfies the following *geometric control condition*: there exists some $T>0$ such that every ray of geometric optics intersects the region $\omega\times(0, T)$ (C. Bardos [1], *see* also J. Rauch and M. Taylor [8]). Always in the linear context and in the particular case where ω is a neighbourhood of the boundary of Ω, this result may be proved by using multiplier techniques (*cf.* J.-L. Lions [7], Chap. VII, and A. Haraux [5]).

In [11], by using multiplier techniques, we have proved the exponential decay for solutions of (1.1) in bounded domains with damping in a neighbourhood of the boundary for a large class of nonlinearities and various boundary conditions. In the present paper we extend these results to unbounded domains.

Hypothesis (1.2) is natural for the exponential decay of solutions in all of $\mathbf{R}^n$. Indeed, if (1.2) is not satisfied, a ray of geometric optics may escape to the damping effect and the exponential decay may fail even in the simplest case where $f=0$.

As far as we know, the only positive result for the exponential decay of solutions of (1.1) that exists concerns the simplest situation where

$$(1.9)\qquad a(x)\geqq a_0>0\quad \text{in all of } \mathbf{R}^n$$

i.e. the damping term is effective everywhere in $\mathbf{R}^n$. In this particular case, the exponential decay may be easily obtained by constructing modified energy functionals of the form

$$E_\varepsilon(t)=E(t)+\varepsilon\int_{\mathbf{R}^n} u(x, t)\,u_t(x, t)\,dx$$

with $\varepsilon>0$ small enough. Indeed, in this case it is easy to prove a differential inequality for $E_\varepsilon(t)$ leading to its exponential decay and therefore, to the exponential decay of $E(t)$ (*cf.* A. Haraux [4], A. Haraux and E. Zuazua [6] and E. Zuazua [10]).

In this paper we shall prove that hypothesis (1.9) may be relaxed to (1.2) provided f satisfies some additional properties.

We shall distinguish the cases where f is globally Lipschitz and f is superlinear since different hypotheses are needed in each of them.

Our main result is as follows.

THEOREM 1. — *Assume that hypotheses* (1.2)-(1.4) *are verified. Assume also that either*

(i) (*The globally Lipschitz case*), $f' \in L^\infty(\mathbf{R})$ *and one of the following two conditions* (1.10 *a*) *or* (1.10 *b*) *holds*

$$(1.10a) \qquad \exists \lim_{s \to +\infty} f'(s) = f'_+; \qquad \lim_{s \to -\infty} f'(s) = f'_-$$

$$(1.10b) \qquad \exists \lim_{|s| \to +\infty} f(s)/s = l$$

or

(ii) (*The superlinear case*). *There exists some* $\delta > 0$ *such that*

$$(1.11) \qquad f(s)\, s \geqq (2+\delta)\, F(s) \quad \text{for every } s \in \mathbf{R}.$$

Then, there exists some constants $C > 1$ *and* $\gamma > 0$ *such that the estimate* (1.8) *holds for every solution* $u = u(x, t)$ *of* (1.1) *with initial data in* $H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$.

Our proof is inspired by the multiplier methods of [11]. We shall estimate the global energy of a solution in terms of the energy concentrated on a set of the form $\{|x| \geqq R\} \times (0, T)$. This estimates will not hold directly since lower order additional terms will appear. In order to absorb them we shall use a *compactness-uniqueness argument* that reduces the question to a unique continuation problem that will be solved by applying recent results by A. Ruiz [9]. At this level and when f is globally Lipschitz we shall need (the technical and probably unnecessary) hypothesis (1.10 *a*) or (1.10 *b*).

The rest of the paper is divided in two parts. In section 2 we prove Theorem 1. In section 3 we discuss some possible extensions of this result. In particular, we show (Theorem 2) how the methods of this paper and [11] may be combined to obtain exponential decay results in unbounded domains when the damping term is effective both at infinity and on a neighbourhood of the boundary.

2. Proof of Theorem 1

Inspired by J. Rauch and M. Taylor [8] we observe that it suffices to prove the existence of a time $T > 0$ and a positive constant $C_0 > 0$ so that the following estimate holds for every solution of (1.1):

$$(2.1) \qquad E(T) \leqq C_0 \int_0^T \int_{\mathbf{R}^n} a(x)\, |u_t(x, t)|^2\, dx\, dt.$$

Indeed, from (1.7) and (2.1) we deduce

$$E(T) \leqq \frac{C_0}{1 + C_0} E(0).$$

This last estimate, combined with the semigroup property, gives (1.8) with

$$(2.2) \qquad C = 1 + \frac{1}{C_0}; \qquad \gamma = \frac{1}{T} \log\left(1 + \frac{1}{C_0}\right).$$

Inequality (2.1) signifies, essentially, that we may estimate the total energy of the solution in terms of the energy concentrated on the set $\{|x|\geqq R\}\times(0, T)$.

This estimate will be a consequence of several lemmas that we prove below.

In which follows, for notational simplicity, we shall drop the dependence on x and t of the functions under integral sign and we shall use the convention of summation of repeated indexes as well as the following notation:

(i) $\int=\int_{\mathbf{R}^n} dx$; $\int_\Theta=\int_\Theta dx$ for every open subset Θ of $\mathbf{R}^n$.

(ii) $\iint=\int_0^T\int_{\mathbf{R}^n} dx\, dt$; $\iint_\Theta=\int_0^T\int_\Theta dx\, dt$ for every open subset Θ of $\mathbf{R}^n$.

(iii) $\iint_{S_r}=\int_0^T\int_{S_r} d\Gamma\, dt$ where $S_r=\{|x|=r\}$ and $d\Gamma$ its surface measure.

(iv) $(.)\,|_0^T=.(T)-.(0)$.

(v) $\partial./\partial r=$ derivative on the radial direction.

(vi) div $q=$ divergence of $q=\partial q_k/\partial x_k$.

We shall establish the estimate (2.1) for smooth solutions of (1.1) with initial data in $H^2(\mathbf{R}^n)\times H^1(\mathbf{R}^n)$, for which the integrations by parts below are justified. The estimate will extend to general weak solutions with initial data in $H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$ by standard density arguments. Therefore, in which follows, we shall only consider smooth solutions.

Estimate (2.1) will be a consequence of the following lemmata.

LEMMA 1. — *There exists a positive constant* $C>0$ *such that*

$$\text{(2.3)}\quad \frac{1}{2}\iint_{\Omega_{2R}}[|\nabla u|^2+\alpha|u|^2+|u_t|^2]+\iint_{\Omega_{2R}} F(u) \leqq C\left\{\iint a|u_t|^2+\|u\|^2_{L^2(B_{2R}\times(0,T))}+E(T)\right\}$$

for every $T>0$ *and smooth solution of* (1.1).

Proof of Lemma 1. — Multiplying equation (1.1) by $\varphi(x)u$ with $\varphi\in W^{2,\infty}(\mathbf{R}^n)$ and integrating by parts over $\mathbf{R}^n\times(0, T)$ we obtain:

$$\text{(2.4)}\quad \iint\varphi[|\nabla u|^2+(f(u)+\alpha u)u]=\iint\left[\frac{\Delta\varphi}{2}|u|^2+\varphi|u_t|^2\right]-\left(\int\left[\left(u_t+a\frac{u}{2}\right)\varphi u\right]\right)\Big|_0^T.$$

We apply (2.4) with $\varphi\in W^{2,\infty}(\mathbf{R}^n)$ verifying

$$\text{(2.5)}\qquad 0\leqq\varphi\leqq 1 \text{ in } \mathbf{R}^n;\qquad \varphi=0 \text{ in } B_R;\qquad \varphi=1 \text{ in } \mathbf{R}^n\setminus B_{2R}.$$

It follows:

$$(2.6)\quad \iint_{\Omega_{2R}} [|\nabla u|^2+(f(u)+\alpha u)u] \leqq C\left\{\iint_{\Omega_R} |u_t|^2+\|u\|^2_{L^2(B_{2R}\times(0,T))} + \left|\left(\int\left[\left(u_t+a\frac{u}{2}\right)\varphi u\right]\right)\right|_0^T\right\}.$$

Observing that

$$\left|\left(\int\left[\left(u_t+a\frac{u}{2}\right)\varphi u\right]\right)\right|_0^T \leqq C\{E(0)+E(T)\}=2CE(T)+C\iint a|u_t|^2$$

and the existence of some constant $\mu>0$ so that

$$(2.7)\quad (f(s)+\alpha s)s \geqq \mu\left(\frac{\alpha}{2}s^2+F(s)\right) \quad \text{for every} \quad s\in\mathbf{R}$$

the estimate (2.3) follows easily from (1.2) and (2.6). ■

Remark 1. — We note that the constant μ of (2.7) may be chosen as $\mu=2\alpha/(\alpha+\|f'\|_\infty)$ in the situation (*a*) [resp. $\mu=2$ in the situation (*b*)] of Theorem 1. ■

Remark 2. — By density, the estimate (2.3) extends to weak solutions of (1.1). ■

LEMMA 2. — *Let be* $q=(q_1,\dots,q_n)\in(W^{1,\infty}(\mathbf{R}^n))^n$. *For every* T, $r>0$ *and every smooth solution of* (1.1) *the following identity holds in* $B_r\times(0,T)$:

$$(2.8)\quad \frac{1}{2}\iint_{B_r}\operatorname{div}(q)[|u_t|^2-|\nabla u|^2-\alpha|u|^2]-\iint_{B_r}\operatorname{div}(q)F(u) + \iint_{B_r}\frac{\partial q_k}{\partial x_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}+\iint_{B_r} au_t\, q.\nabla u = -\left(\int_{B_r}u_t\, q.\nabla u\right)\Big|_0^T + \frac{1}{2r}\iint_{S_r}(q.x)[|u_t|^2-|\nabla u|^2-\alpha|u|^2]-\frac{1}{r}\iint_{S_r}(q.x)F(u)+\iint_{S_r}\frac{\partial u}{\partial r}(q.\nabla u).$$

Proof of Lemma 2. — It suffices to multiply equation (1.1) by $q.\nabla u$ and to integrate by parts on $B_r\times(0,T)$. ■

LEMMA 3. — *Let be* T, $r>0$ *and* $\varphi\in W^{1,\infty}(\mathbf{R}^n)$. *The following identity holds for every smooth solution of* (1.1):

$$(2.9)\quad \iint_{B_r}\varphi[|\nabla u|^2+(f(u)+\alpha u)u]=\iint_{B_r}[\varphi|u_t|^2-u\nabla\varphi.\nabla u] + \iint_{S_r}\varphi\frac{\partial u}{\partial r}u-\left(\int_{B_r}\left[\left(u_t+a\frac{u}{2}\right)\varphi u\right]\right)\Big|_0^T.$$

Proof of Lemma 3. — It suffices to multiply the equation (1.1) by $\varphi(x)u$ and to integrate by parts over $B_r \times (0, T)$. ■

LEMMA 4. — *For every $r > R$ there exists a positive constant $C_r > 0$ so that the following estimate holds for every $T > 0$ and every smooth solution of* (1.1):

$$(2.10)\quad \frac{1}{2}\iint_{B_r}[|\nabla u|^2+\alpha|u|^2+|u_t|^2]+\iint_{B_r}F(u) \leqq C_r\left\{\iint a|u_t|^2+\|u\|^2_{L^2(B_{2r}\times(0,T))}+E(T)\right\}.$$

Proof of Lemma 4. — Applying identity (2.8) with $q = x$ we deduce:

$$(2.11)\quad \frac{n}{2}\iint_{B_r}[|u_t|^2-|\nabla u|^2-\alpha|u|^2]-n\iint_{B_r}F(u)+\iint_{B_r}|\nabla u|^2=-\iint_{B_r}au_t x.\nabla u+A+B$$

with

$$(2.12)\quad A=-\left(\int_{B_r}u_t x.\nabla u\right)\Big|_0^T$$

and

$$(2.13)\quad B=r\iint_{S_r}\left[\frac{1}{2}|u_t|^2-\frac{1}{2}|\nabla u|^2+\left|\frac{\partial u}{\partial r}\right|^2-\frac{\alpha}{2}|u|^2-F(u)\right].$$

On the other hand, applying (2.9) with $\varphi = 1$ we get

$$(2.14)\quad \iint_{B_r}[|\nabla u|^2+(f(u)+\alpha u)u]=\iint_{B_r}|u_t|^2+\iint_{S_r}\frac{\partial u}{\partial r}u-\left(\int_{B_r}\left[\left(u_t+a\frac{u}{2}\right)u\right]\right)\Big|_0^T.$$

We choose a positive constant

$$(2.15)\quad \beta\in\left(\frac{n-2}{2},\frac{n}{2}\right).$$

In the situation (ii) of Theorem 1, the constant β is chosen so that there exists some $\eta > 0$ for which

$$(2.16)\quad \beta f(s)s\geqq(n+\eta)F(s)\quad \text{for every } s\in\mathbf{R}.$$

Note that, because of (1.11), conditions (2.15) and (2.16) are compatible.

Combining (2.10) and (2.14) we deduce

$$(2.17)\quad \left(\frac{n}{2}-\beta\right)\iint_{B_r}|u_t|^2+\left(1+\beta-\frac{n}{2}\right)\iint_{B_r}|\nabla u|^2+\alpha\left(\beta-\frac{n}{2}\right)\iint_{B_r}|u|^2$$
$$+\iint_{B_r}(\beta f(u)u-n\,F(u))\leqq-\iint_{B_r}au_t x.\nabla u+\beta\iint_{S_r}\frac{\partial u}{\partial r}u+B+D$$

with

$$(2.18)\qquad D=A-\beta\left(\int_{B_r}\left[\left(u_t+a\frac{u}{2}\right)u\right]\right)\Bigg|_0^T.$$

We now remark that

$$(2.19)\qquad \left|\iint_{B_r}au_t x.\nabla u\right|\leqq\varepsilon\iint_{B_r}|\nabla u|^2+\frac{r^2\|a\|_{L^\infty(B_r)}}{4\varepsilon}\iint_{B_r}a|u_t|^2$$

for every $\varepsilon>0$.

Combining (2.17) and (2.19) with the existence of some $c>0$ so that

$$(2.20)\qquad \alpha\left(\beta-\frac{n}{2}\right)s^2+\beta f(s)s-n\,F(s)\geqq\eta\,F(s)-cs^2\quad\text{for every } s\in\mathbb{R}$$

we deduce easily the following estimate

$$(2.21)\quad \frac{1}{2}\iint_{B_r}[|\nabla u|^2+\alpha|u|^2+|u_t|^2]+\iint_{B_r}F(u)$$
$$\leqq C\left\{\iint a|u_t|^2+\|u\|^2_{L^2(B_r\times(0,T))}+\left|\iint_{S_r}\frac{\partial u}{\partial r}u\right|+|B|+|D|\right\}$$

for some constant $C>0$ which does not depend on T.

Note that in the case (i) (resp. case (ii)) of Theorem 1, (2.20) is valid for

$$c=\left(\frac{n+\eta}{2}\right)\|f'\|_\infty+\left(\frac{n}{2}-\beta\right)\alpha\qquad\left[\text{resp. } c=\left(\frac{n}{2}-\beta\right)\alpha\right].$$

We now estimate the integrals over $S_r\times(0,T)$ on the right hand side of (2.21). Applying the identities (2.8) and (2.9) with $\varphi\in W^{1,\infty}(B_r)$ and φx respectively with

$$(2.22)\qquad 0\leqq\varphi\leqq 1\ \text{ in } B_r;\qquad \varphi=0\ \text{ in } B_{r'}\ \text{ with } R<r'<r;\ \varphi=1\ \text{ on } S_r$$

we deduce for some constant $C>0$ which does not depend on T,

$$\left|\iint_{S_r} \frac{\partial u}{\partial r} u\right| + |B| \leqq C\left\{\iint_{B_r \setminus B_{r'}} [|\nabla u|^2 + \alpha |u|^2 + |u_t|^2 + f(u)u] + |G|\right\} \tag{2.23}$$

with

$$G = -\left\{\int_{B_r}\left[u_t \varphi . \nabla u + \left(u_t + \frac{au}{2}\right)\varphi u\right]\right\}\Big|_0^T. \tag{2.24}$$

Finally we apply (2.9) in $B_{2r} \times (0, T)$ with $\varphi \in W^{1,\infty}(B_{2r})$ such that

$$\begin{cases} 0 \leqq \varphi \leqq 1 \text{ in } B_r; \quad \varphi = 0 \text{ in } B_R; \quad \varphi = 1 \text{ in } B_r \setminus B_{r'} \text{ with } R < r' < r; \ \varphi = 0 \text{ on } S_{2r}; \\ \dfrac{|\nabla \varphi|^2}{\varphi} \in L^\infty(B_{2r}). \end{cases} \tag{2.25}$$

We obtain

$$\iint_{B_{2r}} \varphi [|\nabla u|^2 + \alpha |u|^2 + f(u)u] \leqq C\left\{\iint a|u_t|^2 + \iint_{B_{2r}} |\nabla \varphi . \nabla uu| + |H|\right\} \tag{2.26}$$

with

$$H = -\left\{\iint_{B_{2r}}\left(u_t + \frac{au}{2}\right)\varphi u\right]\right\}\Big|_0^T. \tag{2.27}$$

We now remark that, by (2.25),

$$\begin{aligned} \iint_{B_{2r}} |\nabla\varphi . \nabla uu| &\leqq \varepsilon \iint_{B_{2r}} \varphi |\nabla u|^2 + \frac{1}{4\varepsilon}\iint_{B_{2r}} \frac{|\nabla\varphi|^2}{\varphi}|u|^2 \\ &\leqq \varepsilon \iint_{B_{2r}} \varphi |\nabla u|^2 + \frac{C}{\varepsilon}\iint_{B_{2r}} |u|^2 \end{aligned} \tag{2.28}$$

for every $\varepsilon > 0$.

Combining (2.26) and (2.28) with $\varepsilon > 0$ small enough we obtain

$$\iint_{B_r \setminus B_{r'}} [|\nabla u|^2 + \alpha |u|^2 + f(u)u] \leqq C\left\{\iint a|u_t|^2 + \|u\|^2_{L^2(B_{4r} \times (0,T))} + |H|\right\}. \tag{2.29}$$

From (2.21), (2.23) and (2.29) we deduce

$$\begin{aligned} \frac{1}{2}\iint_{B_{2r}} [|\nabla u|^2 + \alpha |u|^2 + |u_t|^2] + \iint_{B_{2r}} F(u) \\ \leqq C\left\{\iint a|u_t|^2 + \|u\|^2_{L^2(B_{4r} \times (0,T))} + |D| + |G| + |H|\right\}. \end{aligned} \tag{2.30}$$

We now observe that

$$|D|+|G|+|H|\leqq C\{E(0)+E(T)\}=C\left\{2E(0)+\iint a|u_t|^2\right\} \tag{2.31}$$

and the estimate (2.10) holds. ■

Remark 3. — By density, the estimate (2.10) extends for weak solutions of (1.1). ■

LEMMA 5. — *There exists some* $T_1>0$ *such that for every* $T>T_1$ *there exists a constant* $C(T)>0$ *so that the following estimate holds*

$$E(T)\leqq C\left\{\iint a|u_t|^2+\|u\|^2_{L^2(B_{4R}\times(0,T))}\right\}. \tag{2.32}$$

Proof of Lemma 5. — Combining Lemma 1 and Lemma 4 for $r=2R$ we deduce

$$\int_0^T E(t)\,dt\leqq C_1\left\{\iint a|u_t|^2+\|u\|^2_{L^2(B_{4R}\times(0,T))}+E(T)\right\} \tag{2.33}$$

with $C_1>0$ which does not depend on T.

Taking into account that, from the nonincreasing character of the energy,

$$\int_0^T E(t)\,dt\geqq TE(T)$$

inequality (2.32) follows from (2.33) for every $T>T_1=C_1$ with $C(T)=(T-T_1)^{-1}$. ■

Remark 4. — Let us now introduce the rescaled nonlinearities

$$f_\lambda(s)=\frac{1}{\lambda}f(\lambda s),\qquad \forall s\in\mathbf{R},\lambda>0 \tag{2.34}$$

and consider the family of problems

$$u_{tt}-\Delta u+\alpha u+f_\lambda(u)+a(x)u_t=0\quad\text{in }\mathbf{R}^n\times(0,\infty). \tag{2.35}$$

The estimate (2.32) applies to (2.35). We claim that the constant $C_\lambda(T)$ of the corresponding estimate is uniformly bounded with respect to $\lambda>0$. Indeed, as our proof of (2.32) shows, the constant $C=C(T)$ of (2.32) only depends on the following properties of the nonlinearity f:

(i) The Lipschitz constant $\|f'\|_{L^\infty(\mathbf{R})}$ in the situation (i)

and

(ii) the δ constant in (1.11) in the situation (ii).

We note that both are uniform with respect to the family (2.34) for $\lambda>0$. ■

We may now prove the final estimate

LEMMA 6. — *Let* T_0 *be given by*

$$T_0=\max\{T_1, 2R\}. \tag{2.36}$$

Then, for every $T>T_0$ *there exists a constant* $C(T)>0$ *so that*

$$\|u\|^2_{L^2(B_{4R}\times(0,T))} \leqq \iint a|u_t|^2. \tag{2.37}$$

Proof of Lemma 6. — We argue by contradiction. Let be $T>T_0$ and assume that (2.37) does not hold. Then, there exists a sequence of initial data $\{u_{0,n}, u_{1,n}\}\in H^1_0(\Omega)\times L^2(\Omega)$ such that the corresponding sequence $\{u_n\}$ of solutions of (1.1) satisfies

$$\frac{\|u_n\|^2_{L^2(B_{4R}\times(0,T))}}{\iint a|(u_n)_t|^2} \to +\infty. \tag{2.38}$$

We define

$$v_n=\frac{u_n}{\lambda_n} \tag{2.39}$$

where

$$\lambda_n=\|u_n\|_{L^2(B_{4R}\times(0,T))}. \tag{2.40}$$

Clearly

$$\|v_n\|_{L^2(B_{4R}\times(0,T))}=1 \quad \text{for every } n\in\mathbf{N} \tag{2.41}$$

and v_n solves (2.35) with $\lambda=\lambda_n$, *i.e.*

$$(v_n)_{tt}-\Delta v_n+\alpha v_n+f_n(v_n)+a(x)(v_n)_t=0 \quad \text{in } \mathbf{R}^n\times(0,\infty) \tag{2.42}$$

with $f_n=f_{\lambda_n}$.

On the other hand, (2.38) ensures that

$$\iint a|(v_n)_t|^2\to 0 \quad \text{as } n\to\infty. \tag{2.43}$$

Combining (2.41)-(2.43) and the fact that, as we have mentioned in Remark 4, the estimate (2.32) is uniform with respect to the family of problems (2.42) we conclude

$$\frac{1}{2}\iint[|\nabla v_n|^2+\alpha|v_n|^2+|(v_n)_t|^2]+\iint F_n(v_n)\leqq C. \tag{2.44}$$

In particular $\{v_n\}$ is uniformly bounded in $H^1(\mathbf{R}^n\times(0,T))$.

Let us extract a subsequence (for simplicity still denoted by the subscript n) such that:

$$(2.45)\qquad \begin{cases} v_n \to v & \text{weakly in } H^1(\mathbf{R}^n \times (0, T)) \\ v_n \to v & \text{strongly in } L^2(B_{4R} \times (0, T)) \\ v_n \to v & \text{a.e. in } B_{4R} \times (0, T). \end{cases}$$

From (2.41) and (2.45_2) we deduce that

$$(2.46)\qquad \| v \|^2_{L^2(B_{4R} \times (0, T))} = 1$$

and from (2.43)-(2.45_1) we obtain

$$\iint a \,| v_t |^2 = 0$$

and therefore

$$(2.47)\qquad v_t = 0 \quad \text{a.e. in } \{ a > 0 \} \times (0, T).$$

In order to pass to the limit in (2.42) we distinguish the situations (i) and (ii) of Theorem 1.

(i) The limit equation depends on the behaviour of the sequence $\{ \lambda_n \}$. Therefore we distinguish the following three possibilities.

Case (i1). — There exists a subsequence (still denoted by $\{ \lambda_n \}$) such that

$$(2.48)\qquad \lambda_n \to \lambda \in (0, \infty).$$

It is easy to see that the limit state v satisfies (2.35) in $B_{4R} \times (0, T)$ and therefore

$$(2.49)\qquad w = v_t$$

verifies

$$(2.50)\qquad w_{tt} - \Delta w + \alpha w + f'(\lambda v) w = 0 \quad \text{in } B_{4R} \times (0, T)$$

and by (2.47):

$$(2.51)\qquad w = 0 \quad \text{a.e. in } (B_{4R} \setminus B_R) \times (0, T).$$

Case (i2). — We are not in the situation above and there exists a subsequence satisfying

$$(2.52)\qquad \lambda_n \to 0.$$

In this case the limit state v satisfies

$$(2.53)\qquad v_{tt} - \Delta v + \alpha v + f'(0) v = 0 \quad \text{in } B_{4R} \times (0, T)$$

and therefore, in addition to (2.51), w verifies

$$w_{tt}-\Delta w+\alpha w+f'(0)\,w=0 \quad \text{in } B_{4R}\times(0,T). \tag{2.54}$$

Case (i3). — The whole sequence $\{\lambda_n\}$ goes to infinity. In this case we proceed as in [11]. Let us assume first that (1.10 *a*) holds. We note that $w_n=(v_n)_t$ verifies

$$(w_n)_{tt}-\Delta w_n+\alpha w_n+f'(\lambda_n v_n)\,w_n+a(x)\,(w_n)_t=0 \quad \text{in } B_{4R}\times(0,T). \tag{2.55}$$

Hypotheses (1.10), $(2.45_{1,2})$ and (2.47) allow to pass to the limit in (2.55) to obtain, in addition to (2.51), the limit equation

$$w_{tt}-\Delta w+\alpha w+q(x,t)\,w=0 \quad \text{in } B_{4R}\times(0,T) \tag{2.56}$$

with

$$q(x,t)=f'_{+}\,\chi\{v>0\}+f'_{-}\,\chi\{v<0\}. \tag{2.57 a}$$

When (1.10 *b*) holds, passing to the limit in (2.42) we obtain that the limit v satisfies

$$v_{tt}-\Delta v+(\alpha+l)\,v=0 \quad \text{in } B_{4R}\times(0,T)$$

and therefore, $w=v_t$ satisfies (2.56) with

$$q(x,t)=l \tag{2.57 b}$$

(ii) Let us now consider the situation (ii) of Theorem 1. In this case the third possibility where $\lambda_n\to\infty$ may be easily excluded.

From (2.44) we know that

$$\{F_n(v_n)\}\ \text{is uniformly bounded in } L^1(B_{4R}\times(0,T)) \tag{2.58}$$

with

$$F_n(z)=\int_0^z f_n(s)\,ds=\frac{1}{\lambda_n^2}\,F(\lambda_n z). \tag{2.59}$$

We note that (1.11) implies

$$F(s)\geqq c\,|s|^{2+\delta}, \qquad \forall\,|s|\geqq 1 \tag{2.60}$$

with $c=\min\{F(1),F(-1)\}$.

Combining (2.58)-(2.60) we deduce

$$\lambda_n^{\delta}\iint_{\{|v_n|\geqq\lambda_n^{-1}\}\cap\{B_{4R}\times(0,T)\}}|v_n|^{2+\delta}\leqq C \tag{2.61}$$

that implies

$$(2.62) \qquad \iint_{B_{4R} \times (0, T)} |v_n|^{2+\delta} \to 0$$

which contradicts (2.46).

The situations (ii1) where $\lambda_n \to \lambda \in (0, \infty)$ and (ii2) where $\lambda_n \to 0$ may be treated as in the situation (a) above. The same conclusions hold.

Recapitulating, we see that, in all the possible situations, the state

$$w = v_t \in L^2(B_{4R} \times (0, T))$$

satisfies

$$(2.64) \qquad \begin{cases} w_{tt} - \Delta w + \alpha w + b(x, t) w = 0 & \text{in } B_{4R} \times (0, T) \\ w = 0 & \text{a.e. in } \{B_{4R} \setminus B_R\} \times (0, T) \end{cases}$$

for some non negative potential $b(x, t) \geqq 0$. In the situation (i) the potential b is bounded, *i.e.* $b \in L_+^\infty(B_{4R} \times (0, T))$. In the situation (ii), since $(n-2)p \leqq n$, $b \in L_+^\infty(0, T; L^n(B_{4R}))$.

Therefore, since $T > 2R$, we may apply the unique continuation result by A. Ruiz [9] showing that

$$(2.65) \qquad w = v_t = 0 \quad \text{a.e. in } B_{4R} \times (0, T).$$

Combining (2.47), (2.64) we deduce that

$$(2.66) \qquad v = v(x) \in H^1(\mathbf{R}^n).$$

On the other hand passing to the limit in (2.42) (for this we shall distinguish the different possibilities above) we obtain that v satisfies

$$(2.67) \qquad v_{tt} - \Delta v + \alpha v + p(x, t) v + a(x) v_t = 0 \quad \text{in } \mathbf{R}^n \times (0, T)$$

with $p \geqq 0$ such that

(i) In the situation (i), $p \in L^\infty(\mathbf{R}^n \times (0, T))$

(ii) In the situation (ii), $p(x, t) \leqq C(1 + |v(x, t)|^{p-1})$ (note that, from (1.4), $|v(t)|^{p-1} \in L^n(\mathbf{R}^n)$).

Combining (2.66) and (2.67) we deduce that $v = v(x) \in H^1(\mathbf{R}^n)$ solves

$$-\Delta v + \alpha v + p(x, t) v = 0 \quad \text{in } \mathbf{R}^n$$

and since $\alpha > 0$ and $p \geqq 0$ we deduce that $v \equiv 0$ which contradicts (2.46).

The proof of Lemma 6 is now complete. ■

End of proof of Theorem 1. — Combining the estimates (2.32) and (2.37) we obtain (2.1) for $T > T_0$ and some positive constant $C_0 = C(T)$. The proof of Theorem 1 is complete. ■

3. Some comments on the extensions of the main results

In this section we discuss some possible extensions of Theorem 1.

3.1. *Semilinear wave equation in unbounded domains.* — Let Ω be an unbounded domain of $\mathbf{R}^n$ with boundary of class C^2 and let us consider the semilinear wave equation

$$\text{(3.1)} \qquad \left\{ \begin{array}{l} u_{tt} - \Delta u + \alpha u + f(u) + a(x) u_t = 0 \quad \text{in } \Omega \times (0, \infty) \\ u = 0 \quad \text{on } \partial\Omega \times (0, \infty) \\ u(0) = u_0 \in H_0^1(\Omega), \qquad u_t(0) = u_1 \in L^2(\Omega) \end{array} \right.$$

with $\alpha > 0$, $a \in L_+^\infty(\Omega)$ and f as in section 1.

The energy related to the system is now

$$\text{(3.2)} \qquad E(t) = \frac{1}{2} \int_\Omega [|\nabla u(x, t)|^2 + |u_t(x, t)|^2 + \alpha |u(x, t)|^2] \, dx + \int_\Omega F(u(x, t)) \, dx.$$

In [11] we proved that when Ω is bounded and $a \geqq a_0 > 0$ on a neighbourhood of the boundary $\partial\Omega$, then the energy decays exponentially as t goes to infinity (here and in what follows, by a neighbourhood of the boundary we mean the intersection of Ω with a neighbourhood of the boundary in $\mathbf{R}^n$).

Combining the methods of [11] and of the present paper the following result may be easily proved.

THEOREM 2. — *Let Ω be an unbounded domain of $\mathbf{R}^n$ with boundary of class C^2 and assume that the nonlinearity f satisfies either the hypothesis* (i) *or* (ii) *of* Theorem 1. *Assume that $a \in L_+^\infty(\Omega)$ is such that there exists some $a_0 > 0$ so that*

$$\text{(3.3)} \qquad a(x) \geqq a_0 > 0 \quad \text{a.e. in } \omega$$

with

(3.4) ω = the union of a neighbourhood of the boundary $\partial\Omega$ and the set $\{x \in \Omega : |x| \geqq R\}$ *for some* $R > 0$.

Then, there exists some constants $C > 1$ *and* $\gamma > 0$ *such that the estimate* (1.8) *holds for every solution* $u = u(x, t)$ *of* (3.1), *the energy being given by* (3.2).

Remark 3.1. — The same result holds if we consider in (3.1) Neumann boundary conditions instead of Dirichlet boundary conditions. The method can also be adapted to treat Dirichlet-Neumann mixed boundary conditions. ■

Remark 3.2. — The same multiplier methods allow us to prove the estimate (2.32) for (3.1) if the set ω in (3.4) is the union of a set of type $\{x\in\Omega:|x|\geqq R\}$ and a neighbourhood of part of the boundary

$$\Gamma(x^0)=\{x\in\partial\Omega:(x-x^0).\nu(x)>0\}$$

for some $x^0\in\mathbf{R}^n$ [we denote by $\nu(x)$ the unit outward vector to $\partial\Omega$].

If we had a unique continuation result for solutions of (2.64_1) vanishing in $\omega\times(0, T)$ (for $T>0$ large enough) the arguments of the proof of Lemma 6 would apply and the exponential decay would hold. However, this unique continuation problem seems to be open. We note that the results of [9] only apply if the solution vanishes on the exterior of a bounded set of $\mathbf{R}^n$ during a large enough time interval and therefore, in this context, they are only valid if ω is a neighbourhood of the whole boundary. ■

3.2. *Plate models.* — Let us consider the simplified semilinear plate model

$$(3.5)\qquad \begin{cases} u_{tt}+\Delta^2 u+\alpha u+f(u)+a(x)u_t=0 & \text{in } \Omega\times(0,\infty)\\ u=\partial u/\partial\nu=0 \quad \text{on } \partial\Omega\times(0,\infty) & \\ u(0)=u_0\in H_0^2(\Omega), & u_t(0)=u_1\in L^2(\Omega)\end{cases}$$

where Ω is an unbounded domain of $\mathbf{R}^n$ with smooth boundary (we denote by $\partial./\partial\nu$ the derivative on the outward normal direction).

The multiplier methods of [11] and the present paper allow us to prove an estimate of type (2.32) under the hypotheses of Theorem 2 [the growth condition (1.4) may be relaxed to $(n-4)\,p\leqq n$].

Once again, in order to conclude the exponential decay a unique continuation result is needed. More precisely, we need a result asserting that solutions $w\in H^2(\Omega\times(0, T))$ of a linear plate equation of type

$$w_{tt}+\Delta^2 w+q(x,t)\,w=0 \quad \text{in } \Omega\times(0,T)$$

with $q\in L^\infty(0,T;L_{\mathrm{loc}}^{n/2}(\Omega))$ and satisfying

$$w=0 \quad \text{in} \quad \omega\times(0,T)$$

must necessarily be identically zero.

This problem seems to be open. ■

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