A GENERIC UNIQUENESS RESULT FOR THE STOKES SYSTEM AND ITS CONTROL THEORETICAL CONSEQUENCES

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Abstract

We consider the Stokes system in a three-dimensional cylinder $\Omega = G \times (0, L)$ of $\mathbb{R}^3$, G being a bounded smooth domain of $\mathbb{R}^2$. We study the following uniqueness property: If u is a solution of the Stokes system in $\Omega \times (0, T)$ with Dirichlet boundary conditions, T being a positive time, and its third component vanishes, i.e. $u_3 \equiv 0$, then can we ensure that $u \equiv 0$? We prove that this property does hold for "almost every" cross-section G. By using the Fourier expansion of solutions the problem is reduced to show that, generically with respect to the cross-section G, there is no eigenfunction of the Stokes system with third component identically zero. We also show how this uniqueness result can be applied to obtain approximate controllability properties of the Stokes system with scalar controls oriented in the direction (0,0,1) of $\mathbb{R}^3$. Actually, it was while working on the approximate controllability problem that we were led to the problem of generic uniqueness studied in the present paper. We also prove that the results above fail when G is a ball of $\mathbb{R}^2$.

1. INTRODUCTION AND MAIN RESULTS

Let G be a smooth, bounded domain of $\mathbb{R}^2$ and $L > 0$. We consider the three-dimensional cylinder $\Omega = G \times (0, L) \subset \mathbb{R}^3$. Let $T > 0$ be a positive number and let us consider the evolution Stokes system:

$$\begin{cases} u_t - \Delta\, u = -\nabla\, p & \text{in} \quad \Omega \times \, (0,T) \\ \text{div } u = 0 & \text{in} \quad \Omega \times \, (0,T) \\ u = 0 & \text{in} \quad \partial\,\Omega \times \, (0,T) \\ u\,(x,0) = u^0\,(x) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Let $\omega \subset \Omega$ be an open and non-empty subset of Ω.

This paper is devoted to study the following uniqueness property:

$$\textit{Does} \quad u_3 = 0 \quad \textit{in} \quad \omega \times \, (0,T), \quad \textit{imply that} \quad u \equiv 0 \quad \textit{in} \quad \Omega \times \, (0,T)\textit{?} \tag{1.2}$$

The following main result shows that this holds for “almost every” smooth cross section G:

Theorem 1

Generically with respect to the smooth, bounded cross-section G, the unique-continuation property (1.2) holds for all $L > 0$ and $T > 0$.

More precisely, given any bounded domain G of $\mathbb{R}^2$ of class C^k, with $k \geq 3$, we can find another domain $\tilde{G}$ arbitrarily close to G in the C^k topology such that (1.2) holds in $\tilde{G}$ for any $T > 0$ and $L > 0$.

Remarks 1.1

a) The method of proof of Theorem 1 we shall use shows that, in fact, Theorem 1 holds if all the eigenvalues of the Laplacian in $H_0^1(G)$ are simple. This is well known to be a generic property of smooth domains (see J. H. Albert [A], A. M. Micheletti [M] and K. Uhlenbeck [U]).

b) In Section 3 we shall show that the uniqueness property (1.2) fails when G is a ball of $\mathbb{R}^2$.

c) If we impose periodic or Neumann type boundary conditions at $x_3 = 0, L$ for $x \in G$ and Dirichlet ones on $\partial G \times (0,L)$ it is easier to construct counter-examples. In fact, in those cases there is no cross-section G such that the analogue of (1.2) holds. To see this it is sufficient to take a solution (u,p) of the two-dimensional Stokes problem in $G \times (0,T)$, with $u = (u_1, u_2)$, u_j and p being functions of x_1, x_2 and t, and to observe that $(\tilde{u}, p)$ with $\tilde{u} = (u_1, u_2, 0)$ solves the Stokes problem in $\Omega \times \, (0,T)$ for those boundary conditions. This type of counter-example is developed in detail by Díaz and Fursikov [DF].

As a consequence of Theorem 1, applying Hahn-Banach Theorem we can deduce immediately an approximate controllability result for the Stokes system with controls supported in ω and oriented in the direction $e_3 = (0,0,1)$.

Indeed, let us denote by $v = v(x,t)$ the scalar control function and consider the controlled Stokes system:

$$\begin{cases} y_t - \Delta y = -\nabla p + v\chi_\omega e_3 & \text{in} \quad \Omega \times (0,T) \\ \text{div } y = 0 & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{in} \quad \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in} \quad \Omega \end{cases} \tag{1.3}$$

where χ_ω denotes the characteristic function of ω.

Assume that

$$y^0 \in L^2_{\text{div}}(\Omega) = \left\{ y^0 \in \left(L^2(\Omega)\right)^3 : \ \text{div } y^0 = 0 \right\}$$

and let us define the reachable set

$$R(y^0;T) = \left\{ y(x,T) : \ v \in L^2\left(\omega \times (0,T)\right) \right\}. \tag{1.4}$$

The system is said to be approximately controllable if $R(y^0,T)$ is dense in $L^2_{\text{div}}(\Omega)$ for any $y^0 \in L^2_{\text{div}}(\Omega)$.

We have the following result:

Theorem 2

If Ω, ω and T are such that the uniqueness property (1.2) holds, the system (1.3) is approximately controllable.

Remarks 1.2

Several open problems naturally arise. We list below some of them.

a) We consider the system given by

$$y_t - \Delta y + b(x,t)\nabla y = -\nabla p + v\chi_\omega e_3 \tag{1.5}$$

where $b(x,t)$ is a given (possibly smooth) vector function such that $\text{div} b(x,t) = 0$, with all other conditions in (1.3) unchanged.

Do we still have the same result than in Theorem 2, i. e. the generic uniqueness property? Or, in other words, *do we have the generic approximate controllability*?

b) Do we have generic approximate controllability when Ω is arbitrary in $\mathbb{R}^3$ and not of the particular cylindrical structure above?

c) Let us consider now the system given by

$$\begin{cases} y_t - \Delta y = -\nabla p & \text{in} \quad \Omega \times (0,T) \\ \text{div } y = v & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{in} \quad \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in} \quad \Omega \end{cases}$$

with a *boundary control* v of the form

$$v = (0,0,v_3); \qquad \int_\Gamma v_3 n_3 d\Gamma = 0.$$

Do we still have generic approximate controllability?

d) Let us now consider the "full" Navier-Stokes system

$$\begin{cases} y_t - \Delta y + y\nabla y = -\nabla p + v\chi_\omega e_3 & \text{in} \quad \Omega \times (0,T) \\ \text{div } y = 0 & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{in} \quad \partial\Omega \times (0,T) \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases}$$

Do we still have generic approximate controllability? It has been conjectured (J. L. Lions [L3]) that the non linear terms in the Navier Stokes system actually *help* for the controllability (because, among other things, of the "mixing properties" they bring to the system). An interesting result by J. M. Coron [C1,2] for the $2D$ Euler equations goes in this direction.

The plan of the paper is as follows. We present the proof of Theorem 1 in Section 2 below. We show in Section 3 that there exist domains such that Theorem 1 is false making the "generic statement" a necessity. The proof of Theorem 2 is given in Section 4 and it is followed by further comments on (approximate) controllability questions.

2. PROOF OF THE UNIQUENESS RESULT

We proceed in several steps.

Step 1

Fist of all we observe that if (u,p) solve (1.1), then $p(x,t)$ is harmonic in Ω with respect to x for all $t \in (0,T)$. Indeed, applying the divergence operator to the first equation in (1.1) and taking into account that div $u = 0$ we deduce that

$$\Delta p = 0 \quad \text{in} \quad \Omega \times (0,T)$$

On the other hand, from the fact that

$$u_3 = 0 \quad \text{in} \quad \omega \times (0,T)$$

we deduce immediately that

$$\frac{\partial p}{\partial x_3} = 0 \quad \text{in} \quad \omega \times (0,T).$$

Therefore, by elliptic unique-continuation we deduce that

$$\frac{\partial p}{\partial x_3} \equiv 0 \quad \text{in} \quad \Omega \times (0,T) \tag{2.1}$$

i.e.

$$p = p\,(x_1, x_2, t) \quad \text{in} \quad \Omega \times (0,T) \quad \text{ is independent of } x_3.$$

But then u_3 satisfies the heat equation

$$u_{3,t} - \Delta\, u_3 = 0 \quad \text{in} \quad \Omega \times (0,T).$$

Since $u_3 = 0$ in $\omega \times (0,T)$, by the parabolic unique continuation property

$$u_3 \equiv 0 \quad \text{in} \quad \Omega \times (0,T). \tag{2.2}$$

Notice that we have applied the unique continuation theorem to equations with constant coefficients. Thus, Holmgren's theorem suffices.

The semigroup generated by the Stokes system in $L^2_{\text{div}}(\Omega)$ is analytic. Therefore, the function

$$t \;\in\; (0,\infty) \;\longrightarrow\; u_3\,(t) \;\in\; L^2\,(\Omega)$$

is analytic. This fact combined with (2.2) implies that

$$u_3 \;\equiv\; 0 \quad \text{in} \quad \Omega \times (0,\infty). \tag{2.3}$$

We have now to show that (1.2) holds true when $u_3 \equiv 0$ in $\Omega \times (0,T)$ (and not only in $\omega \times (0,T)$).

Step 2

We now use a representation of the solution of (1.1) using the eigenvalues and eigenfunctions of the Stokes system (so that this method of proof does not apply to the situation described in Remarks 1.2, a).)

Let $\{\lambda_j\}$ be the distinct eigenvalues of the Stokes system (1.4). Let $m_j \geq 1$ be the multiplicity of λ_j. We denote by $\{\omega_j^k\,,\,\pi_j^k\}_{k=1}^{m_j}$ the eigenfunctions and eigenpressures associated to each eigenvalue λ_j. Le us recall that $\{\omega_j^k\}, j \in \mathbb{N}, k = 1,\ldots,m_j$ can be chosen to constitute an orthonormal basis of $L^2_{\text{div}}\,(\Omega)$.

Expanding the solution of (1.1) in Fourier series we obtain that

$$u = \sum_{j=1}^{\infty} \sum_{k=1}^{m_j} a_j^k \, e^{-\lambda_j \, t} \, \omega_j^k \, (x)$$

with

$$a_j^k = \int_\Omega u_0 \, (x) \cdot \omega_j^k \, (x) \, dx$$

(by $\cdot$ we denote the scalar product in $\mathbb{R}^3$).

In particular, if $\omega_j^k = (\omega_{j,1}^k \, , \, \omega_{j,2}^k \, , \, \omega_{j,3}^k)$, by (2.3) we have:

$$u_3 \, (x,t) = \sum_{j=1}^{\infty} \sum_{k=1}^{m_j} a_j^k \, e^{-\lambda_j t} \, \omega_{j,3}^k \, (x) = 0 \quad \text{in} \quad \Omega \times (0, \infty). \tag{2.4}$$

Multiplying in (2.4) by $e^{\lambda_1 t}$ and letting $t \to \infty$ we obtain that

$$\sum_{k=1}^{m_1} a_1^k \, \omega_{1,3}^k \, (x) = 0 \quad \text{in} \quad \Omega.$$

By induction over j, repeating this argument we deduce that

$$\sum_{k=1}^{m_j} a_j^k \, \omega_{j,3}^k \, (x) = 0 \quad \text{in} \quad \Omega \, , \quad \forall \, j \geq 1. \tag{2.5}$$

We define

$$z = \sum_{k=1}^{m_j} a_j^k \, \omega_j^k; \quad \sigma = \sum_{k=1}^{m_j} a_j^k \, \pi_j^k. \tag{2.6}$$

Then

$$\begin{cases} -\Delta \, z = \lambda_j \, z - \nabla \, \sigma & \text{in} \quad \Omega \\ z = 0 & \text{on} \quad \Gamma \\ \text{div} \; z = 0 & \text{in} \quad \Omega \end{cases} \tag{2.7}$$

and by virtue of (2.5)

$$z_3 = 0 \quad \text{in} \; \Omega. \tag{2.8}$$

We shall have proven Theorem 1 if we verify the following Lemma:

Lemma 1

Generically with respect to the bounded, smooth cross-section G, for any $L > 0$, there is no solution of (2.7)–(2.8) except $z = 0$, $\nabla z = 0$.

Step 3

We write λ instead of λ_j in (2.7). Because of (2.8)

$$\frac{\partial \sigma}{\partial x_3} = 0$$

so that

$$\left\{\begin{array}{l} -\Delta\, z_1 = \lambda\, z_1 - \partial_1\, \sigma \quad \text{in} \quad \Omega \quad ; \quad z_1 = 0 \quad \text{on} \quad \partial\,\Omega \\ -\Delta\, z_2 = \lambda\, z_2 - \partial_2\, \sigma \quad \text{in} \quad \Omega \quad ; \quad z_2 = 0 \quad \text{on} \quad \partial\,\Omega \\ \quad \sigma = \sigma\,(x_1, x_2)\ , \quad \partial_1\, z_1 + \partial_2\, z_2 = 0 \quad \text{in} \quad \Omega. \end{array}\right. \tag{2.9}$$

We define

$$z^n\,(x_1, x_2) = \int_0^L \sin\Big(\frac{\pi\, n\, x_3}{L}\Big)\, z\,(x_1, x_2, x_3)\, d\, x_3 \tag{2.10}$$

and

$$\sigma^n\,(x_1, x_2) = \int_0^L \sin\Big(\frac{\pi\, n\, x_3}{L}\Big)\, \sigma\,(x_1, x_2)\, d\, x_3 = \frac{L}{n\,\pi}\,(1 - (-1)^n)\, \sigma\,(x_1, x_2). \tag{2.11}$$

Multiplying in (2.9) by $\sin\Big(\frac{\pi\, n\, x_3}{L}\Big)$ and integrating with respect to x_3 we deduce that

$$\left\{\begin{array}{lll} -\Delta'\, z^n = \Big[\lambda - \Big(\frac{\pi\, n}{L}\Big)^2\Big]\, z^n - \nabla'\, \sigma^n & \text{in} & G \\ \qquad z^n = 0 & \text{on} & \partial\, G \\ \qquad \operatorname{div}'\, z^n = 0 & \text{in} & G \end{array}\right. \tag{2.12}$$

where we have denoted by Δ', ∇' and div' the Laplacian, gradient and divergence respectively in the variables (x_1, x_2).

Thus, (z^n, σ^n) are eigenfunctions of the two-dimensional Stokes system in G with eigenvalue $\mu_n = \lambda - \Big(\frac{n\,\pi}{L}\Big)^2$.

However, it is well-known that all the eigenvalues of the Stokes system are strictly positive. This implies

$$z^n = 0; \quad \nabla' \sigma^n = 0$$

except possibly if $\mu_n = \lambda - \Big(\frac{n\,\pi}{L}\Big)^2 > 0$ i.e.

$$n < \frac{L}{\pi}\sqrt{\lambda}$$

Then

$$z\,(x_1,x_2,x_3) = \frac{2}{L} \sum_{n<L/\pi\sqrt{\lambda}} \sin\Big(\frac{\pi\,n\,x_3}{L}\Big)\, z^n\,(x_1,x_2)$$

and

$$\sigma\,(x_1,x_2,x_3) = \frac{2}{L} \sum_{n<L/\pi\sqrt{\lambda}} \sin\Big(\frac{\pi\,n\,x_3}{L}\Big)\, \sigma^n\,(x_1,x_2).$$

Using (2.11) this last identity writes

$$\sigma = \frac{2}{\pi} \sum_{n<L/\pi\sqrt{\lambda}} (1-(-1)^n) \sin\Big(\frac{n\pi x_3}{L}\Big)\sigma$$

so that

$$\sigma = 0. \tag{2.13}$$

Then $\sigma^n = 0$. Therefore (2.12) reduces to

$$\begin{cases} -\Delta'\,z^n = \Big[\lambda - \Big(\dfrac{\pi\,n}{L}\Big)^2\Big]\, z^n & \text{in} \quad G \\ z^n = 0 & \text{on} \quad \partial\,G \\ \operatorname{div}'\; z^n = 0 & \text{in} \quad G. \end{cases} \tag{2.14}$$

In order to show Lemma 1, we cover the "genericity" part of that Lemma if we prove it under the hypothesis

$$\text{the spectrum of } -\Delta \text{ for Dirichlet in } G \text{ is simple} \tag{2.15}$$

(since we know, as recalled in Remarks 1.1, a), that (2.15) is generically true).

We then argue as in [LZ]. In view of (2.14), for every $n\in\mathbb{N}$, $n < L/\pi\sqrt{\lambda}$, there are two possibilities: either $\lambda - \Big(\dfrac{\pi\,n}{L}\Big)^2$ is not an eigenvalue of $-\Delta'$ in $H_0^1(G)$, in which case $z^n \equiv 0$, or it is an eigenvalue. If $\lambda - \Big(\dfrac{\pi\,n}{L}\Big)^2$ is an eigenvalue of $-\Delta'$ in $H_0^1(G)$ since we have assumed G to be such that the spectrum of $-\Delta'$ in $H_0^1(G)$ is simple, we conclude the existence of some constant $\alpha^n \in \mathbb{R}$ such that

$$z_1^n = \alpha^n\, z_2^n \quad \text{in} \quad G$$

But then, the condition $\operatorname{div}'\; z^n = 0$ implies that

$$\alpha^n\,\partial_1\, z_2^n + \partial_2\, z_2^n = 0 \quad \text{in} \quad G$$

Taking into account that $z_2^n = 0$ on $\partial\,G$ this implies that $z_2^n \equiv 0$ in G.

We have proved that $z^n \equiv 0$ for all $n\in\mathbb{N}$. Thus $z\equiv 0$. This concludes the proof of Lemma 1 and Theorem 1.

Remark 2.1 We thus have proven Theorem 1 assuming (2.15). This hypothesis is by no means necessary as it is shown at the end of Section 4.

3. A COUNTEREXAMPLE

In this section we are going to show that the uniqueness property fails when G is a ball of $\mathbb{R}^2$ for any $L > 0$ and $T > 0$ and even if $\omega = \Omega$.

We claim that if G is a ball of $\mathbb{R}^2$ there exists an (actually there exist infinitely many) eigenfunction $z = (z_1, z_2)$, $z_j = z_j\,(x_1, x_2)$ of the Stokes system in G such that the corresponding pressure vanishes, i.e.

$$\begin{cases} -\Delta'\, z = \lambda\, z & \text{in} \quad G \\ z = 0 & \text{on} \quad \partial\, G \\ \operatorname{div}'\, z = 0 & \text{in} \quad G. \end{cases} \tag{3.1}$$

This is a known example (see, for instance, H. Lamb [La]), a simple proof of it being presented below.

Let us assume for a moment that this is true. Then, for any $n \in \mathbb{N}$,

$$\omega^n\,(x_1, x_2, x_3) = \Big(\sin\Big(\frac{n\,\pi}{L}\,x_3\Big)\, z_1\,(x_1, x_2)\ ,\ \sin\Big(\frac{n\,\pi}{L}\,x_3\Big)\, z_2\,(x_1, x_2)\ ,\ 0\Big)$$

is an eigenfunction of the Stokes system in $\Omega = G \times (0, L)$ with zero pressure and $\omega_3^n \equiv 0$. More precisely

$$\begin{cases} -\Delta\,\omega^n = \Big(\lambda + \dfrac{n^2\pi^2}{L^2}\Big)\,\omega^n & \text{in} \quad \Omega \\ \omega^n = 0 & \text{on} \quad \partial\,\Omega \\ \operatorname{div}\,\omega^n = 0 & \text{in} \quad \Omega \end{cases} \tag{3.2}$$

and

$$\omega_3^n \equiv 0 \quad \text{in} \quad G. \tag{3.3}$$

Then,

$$u\,(x, t) = e^{(\lambda + n^2\pi^2/L^2)t}\,\omega^n\,(x)$$

is a solution of the Stokes system (1.1) such that $p \equiv 0$ and $u_3 \equiv 0$.

Let us now go back to the claim and show that there exists z such that (3.1) holds. Proceeding as in [LZ] we consider a radially symmetric eigenfunction $\varphi = \varphi\,(r)$ of the following fourth order problem:

$$\begin{cases} (\Delta')^2\,\varphi = -\lambda\,\Delta'\,\varphi & \text{in} \quad G \\ \varphi = \dfrac{\partial\,\varphi}{\partial\,\nu} = 0 & \text{on} \quad \partial\, G. \end{cases} \tag{3.4}$$

Of course, there are infinitely many radially symmetric eigenfunctions of (3.4).

We then define the vector field

$$z = \left(\frac{\partial \varphi}{\partial x_2} \, , \, -\frac{\partial \varphi}{\partial x_1} \right). \tag{3.5}$$

Then, it is obvious that

$$z = 0 \quad \text{on} \quad \partial G$$

$$\text{div}' \, z = 0 \quad \text{in} \quad G.$$

We have to show that

$$-\Delta' \, z = \lambda \, z \quad \text{in} \quad G. \tag{3.6}$$

In view of (3.4)-(3.5) we have

$$\frac{\partial}{\partial x_2} (\Delta' z_1 + \lambda z_1) - \frac{\partial}{\partial x_1} (\Delta' z_2 + \lambda z_2) = (\Delta')^2 \varphi + \lambda \Delta' \varphi = 0 \quad \text{in} \quad G.$$

This implies the existence of $\pi = \pi \, (x_1, x_2)$ such hat

$$-\Delta' \, z = \lambda \, z + \nabla' \, \pi \quad \text{in} \quad G.$$

If is then sufficient to show that π can be chosen such that $\pi \equiv 0$.

Obviously, π satisfies

$$\begin{cases} \Delta' \, \pi = 0 & \text{in} \quad G \\ \dfrac{\partial \pi}{\partial \nu} = -\Delta' \, z \cdot \nu = \dfrac{\partial \, \Delta' \, \varphi}{\partial \, \tau} & \text{on} \quad \partial G \end{cases}$$

where τ is the unit tangent vector to ∂G. But since φ is radially symmetric, $\dfrac{\partial \Delta' \varphi}{\partial \, \tau} = 0$ on ∂G. Thus, π satisfies

$$\begin{cases} \Delta' \, \pi = 0 \quad \text{in} \quad G \\ \dfrac{\partial \pi}{\partial \nu} = 0 \quad \text{on} \quad \partial G \end{cases}$$

and therefore $\pi = 0$ up to an additive constant.

This concludes the proof of the counter-example.

4. SOME APPROXIMATE CONTROLLABILITY RESULTS AND FURTHER COMMENTS

4.1. Proof of Theorem 2

As we said in the Introduction, from Theorem 1 one can obtain easily Theorem 2. This can be done by means of Hahn-Banach Theorem or by the direct method introduced in [L2] and [FaPZ1,2]. We are going to briefly sketch the proof of Theorem 2 following the latter.

Let G be a smooth, bounded domain of $\mathbb{R}^2$ such that the uniqueness property (1.2) holds for all $L > 0$ and $T > 0$.

Given any $L > 0$ we consider the three-dimensional cylinder $\Omega = G \times (0, L)$ and we fix any $T > 0$.

We first observe that due to the linearity and the well posedness of the Stokes system in $L^2_{\mathrm{div}}(\Omega)$, it is sufficient to prove the density of $R(y^0; T)$ in $L^2_{\mathrm{div}}(\Omega)$ with $y^0 \equiv 0$.

Given any $y^1 \in L^2_{\mathrm{div}}(\Omega)$ and $\varepsilon > 0$ we are going to construct a scalar control $v \in L^2(\Omega \times (0, T))$ such that the solution of (1.3) satisfies

$$\|y(T) - y^1\|_{L^2_{\mathrm{div}}(\Omega)} \leq \varepsilon. \tag{4.1}$$

We consider the adjoint Stokes system

$$\begin{cases} -\varphi_t - \Delta\varphi = \nabla q & \text{in} \quad \Omega \times (0, T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0, T) \\ \operatorname{div} \varphi = 0 & \text{in} \quad \Omega \times (0, T) \\ \varphi(T) = \varphi^0 & \text{in} \quad \Omega \end{cases} \tag{4.2}$$

with $\varphi^0 \in L^2_{\mathrm{div}}(\Omega)$.

We then define on $L^2_{\mathrm{div}}(\Omega)$ the functional

$$J(\varphi^0) = \frac{1}{2} \int_{\omega \times (0,T)} |\varphi_3|^2 \, dx \, dt + \varepsilon \|\varphi^0\|_{L^2_{\mathrm{div}}(\Omega)} - \int_\Omega y^1 \cdot \varphi^0 \, dx. \tag{4.3}$$

If is easy to see that J is continous in $L^2_{\mathrm{div}}(\Omega)$ and strictly convex.

On the other hand, proceeding as in [FaPZ1,2] we deduce easily that the uniqueness property (1.2) implies that

$$\liminf_{\|\varphi^0\|_{L^2_{\mathrm{div}}(\Omega)} \to \infty} \frac{J(\varphi^0)}{\|\varphi^0\|_{L^2_{\mathrm{div}}(\Omega)}} \geq \varepsilon. \tag{4.4}$$

Therefore, J is coercive in $L^2_{\rm div}(\Omega)$ and there is a unique $\hat\varphi^0 \in L^2_{\rm div}(\Omega)$ such that

$$J(\hat\varphi^0) = \min_{\varphi^0 \in L^2_{\rm div}(\Omega)} J(\varphi^0). \tag{4.5}$$

If is then easy to see that, if $\hat\varphi$ is the solution of (4.2) and we set

$$v = \hat\varphi_3 \tag{4.6}$$

then, (4.1) holds.

If is also easy to see that $\hat\varphi_0 \equiv 0$ (and therefore $\hat\varphi_3 \equiv 0$) if and only if $\|y^1\|_{L^2_{\rm div}(\Omega)} \le \varepsilon$.

4.2. Bang-bang controls

Following the arguments of [FaPZ1,2,3] it can be proved that the approximate controllability holds with “quasi bang-bang controls”.

Let us go back to the proof of Theorem 2 and let us consider instead of (4.3) the functional

$$J(\varphi^0) = \frac{1}{2}\left(\int_{\omega\times(0,T)} |\varphi_3|\, dx\, dt\right)^2 + \varepsilon\, \|\varphi^0\|_{L^2_{\rm div}(\Omega)} - \int_\Omega y^1 \cdot \varphi^0 dx. \tag{4.7}$$

The functional J: $L^2_{\rm div}(\Omega) \longrightarrow \mathbb{R}$ is continuous and strictly convex and also satisfies

$$\liminf_{\|\varphi^0\|_{L^2_{\rm div}(\Omega)}\to\infty} \frac{J(\varphi^0)}{\|\varphi^0\|_{L^2_{\rm div}(\Omega)}} \ge \varepsilon \tag{4.8}$$

Thus J achieves its minimum value at some $\hat\varphi^0 \in L^2_{\rm div}(\Omega)$. As it was shown in [FaPZ1,2,3] there exists then some

$$\lambda(x,t) \in \left(\int_{\omega\times(0,T)} |\hat\varphi_3|\, dx\, dt\right) \operatorname{sgn}(\hat\varphi_3(x,t))$$

such that the solution of (1.3) with $v = \lambda$ satisfies

$$\|y(T) - y^1\|_{L^2_{\rm div}}(\Omega) \le \varepsilon.$$

By sgn we have denoted the sign function:

$$\operatorname{sgn}(z) = \begin{cases} 1 & \text{if} \quad z > 0 \\ [-1,1] & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0. \end{cases}$$

Rigorously speaking, to ensure that the control obtained in this way is of bang-bang form we would need to show that the set $\{(x,t) \in \Omega\times(0,T)\ ;\ \hat\varphi_3(x,t) = 0\}$ has zero Lebesgue measure. This is done in the following Theorem.

Theorem 3

Generically with respect to the smooth, bounded cross-section G, the following uniqueness property holds for all $L > 0$, $T > 0$ and for any measurable set $E \subset \Omega = G \times (0, L)$ with $\mathrm{meas}(E) > 0$:

$$\text{If } u \text{ is a solution of (1.1) such that } u_3 = 0 \text{ in } E \text{ then } u \equiv 0 \text{ in } \Omega \times (0, T). \tag{4.9}$$

When this uniqueness property holds, then system (1.3) is approximately controllable with bang-bang controls supported on E and oriented in the direction of e_3.

Proof of Theorem 3.

It is sufficent to prove the uniqueness result (4.9). The generic approximate controllability result then holds immediately, as above, by considering the functional:

$$J\left(\varphi^0\right) = \frac{1}{2}\left(\int_E |\varphi_3|\, dx\, dt\right)^2 + \varepsilon \left\|\varphi^0\right\|_{L^2_{\mathrm{div}}(\Omega)} - \int_\Omega y^1 \cdot \varphi^0 dx. \tag{4.10}$$

Let (u, p) be a solution of (1.1) with initial data in $L^2_{\mathrm{div}}(\Omega)$ such that

$$u_3 = 0 \quad \text{in } E.$$

(We may consider as well, equivalently, solutions (φ, q) of the adjoint system (4.2)). Then there exists a set $E_1 \subset (0, T)$ with $\mathrm{meas}(E_1) > 0$, such that

$$\text{if } t \in E_1,\ u_3(\cdot, t) = \partial u_3/\partial t = \Delta u_3 = 0 \text{ on } F_t \subset \Omega, \qquad \mathrm{meas}(F_t) > 0. \tag{4.11}$$

On the other hand,

$$\Delta p(\cdot, t) = 0 \quad \text{ in } \Omega,\ \forall t$$

and by virtue of (4.11),

$$\frac{\partial p(\cdot, t)}{\partial x_3} = 0 \text{ on } F_t,\ \forall t.$$

Then, since the zero set of non-trivial harmonic functions in Ω is of zero Lebesgue measure, we deduce that

$$\frac{\partial p(\cdot, t)}{\partial x_3} = 0 \text{ in } \Omega,\ \forall t \in E_1. \tag{4.12}$$

But $t \to u(t)$ is analytic from $t > 0 \to V = H^1_{0,\mathrm{div}}(\Omega)$, so that

$$t > 0 \to \nabla p(\cdot, t) \quad \text{ is real analytic from } t > 0 \to V'$$

which together with (4.12) implies that

$$\frac{\partial p(\cdot, t)}{\partial x_3} = 0 \text{ in } \Omega \times (0, T). \tag{4.13}$$

(By $V = H^1_{0,\mathrm{div}}(\Omega)$ we denote the space $V = (H^1_0(\Omega))^3 \cap L^2_{\mathrm{div}}(\Omega)$ and by V' its dual.) Therefore,

$$\frac{\partial u_3}{\partial t} - \Delta u_3 = 0 \quad \text{in } \Omega \times (0,T), \quad u_3 = 0 \quad \text{in } \Gamma \times (0,T)$$

and since u_3 satisfies (4.11) it follows that (see, for instance F. H. Lin [Li])

$$u_3 = 0 \quad \text{in } \Omega \times (0,T).$$

We are in a situation in which Theorem 1 applies so that (4.9) follows.

4.3 Relaxing the assumption of simple spectrum

Our proof of Theorem 1 shows that given a cross-section G (not necessarily smooth) the uniqueness property (1.2) holds for all $L > 0$ and $T > 0$ simultaneously if and only if there is no eigenfunction of the Stokes system in G with Dirichlet boundary conditions such that the corresponding pressure vanishes, i.e. if $z = (z_1, z_2)$ satisfies

$$\begin{cases} -\Delta' z = \lambda z & \text{in} \quad G \\ z = 0 & \text{on} \quad \partial G \\ \mathrm{div}' z = 0 & \text{in} \quad G \end{cases} \tag{4.14}$$

then, necessarily, z has to be identically zero.

A sufficient condition to guarantee that this holds is to request all eigenvalues of the Laplacian in $H^1_0(G)$ to be simple. In view of the counterexample of section 3 this condition turns to be sharp in general.

The simplicity of the Dirichlet spectrum for the Laplacian is a generic property among smooth domains. We have then decided to formulate our uniqueness result as a generic property. However, it is clear that for a particular given domain G, even if it is not smooth, as soon as the spectrum of the Laplacian is simple we can conclude that the uniqueness property holds.

Furthermore, is some particular geometries, even if the spectrum of the Laplacian is not simple we can conclude, analyzing (4.14), that (1.2) holds.

Let G be the square $G = (0,\pi)^2 \subset \mathbb{R}^2$. It is well known that the spectrum of the Laplacian in $H^1_0(G)$ is not simple since there may exist two couples (n,m) and (n',m') in $\mathbb{N}^2$ such that

$$n^2 + m^2 = (n')^2 + (m')^2.$$

In particular we may take $(n',m') = (m,n)$ when $n \neq m$.

Let us consider two such couples and the corresponding eigenfunctions of the Laplacian:

$$\varphi(x_1,x_2) = \sin n x_1 \sin m x_2$$

$$\psi(x_1,x_2) = \sin n' x_1 \sin m' x_2.$$

It is clear that these functions (φ, ψ), or any linear combination of them, cannot be such that

$$\text{div}'\,(\varphi, \psi) = 0.$$

Indeed

$$\text{div}'\,(\varphi, \psi) = n\,\cos\,n\,x_1\,\sin\,m\,x_2 + m'\,\sin\,n'\,x_1\,\cos\,m'\,x_2$$

and it is clear that $\text{div}'\,(\varphi, \psi)$ can not be identically zero since since at $x_1 = 0$, $\text{div}'\,(\varphi, \psi) = n \sin\,m\,x_2$.

Thus, in this particular case, even if the spectrum of the Dirichlet Laplacian is not simple the uniqueness property (1.2) holds. **Acknowledgements** The work of the second author was supported by Project PB93-1203 of the DGICYT (Spain) and Projects SC1*-CT91-0732 and CHRX-CT94-0471 of the European Community.

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