Università degli Studi di Modena e Reggio Emilia

DIPARTIMENTO DI SCIENZE FISICHE, INFORMATICHE E MATEMATICHE
Corso di Laurea Magistrale in Matematica

Tesi di laurea magistrale

An approach based on Kolmogorov operators to a Kuramoto model with inertia

Candidato:
Giulio Pecorella

Relatore:
Prof. Sergio Polidoro

Correlatore:
Prof.ssa Cecilia Vernia

Anno Accademico 2021-2022

Contents

Introduction

Kolmogorov operators are a class of hypoelliptic second order differential operators defined in $\mathbb{R}^{N+1}$:

$$\mathscr{L}u(x,t) = \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial^2_{x_i x_j} u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t)$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, $1 \leq m_0 \leq N$, $A_0(z) = (a_{ij}(z))_{i,j=1,\dots,m_0}$ is a symmetric matrix definite positive in $\mathbb{R}^{m_0}$ for all $z \in \mathbb{R}^{N+1}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ a constant matrix with real entries.
If we put $N = 2n$ and we consider A_0 and B in the form:

$$A_0 = \mathcal{I}_n \qquad B = \begin{pmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathcal{I}_n & \mathbb{O}_n \end{pmatrix}$$

where $\mathcal{I}_n$ and $\mathbb{O}_n$ represent $n \times n$ identity and null matrix, we obtain the prototype of Kolmogorov operator:

$$\mathscr{L}u(x,y,t) = \sum_{i=1}^{n} \partial^2_{x_i x_i} u(x,y,t) + \sum_{i=1}^{n} x_i \partial_{y_i} u(x,y,t) - \partial_t u(x,y,t)$$

where $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Equations like this arise in diffusion process, where x represents the velocity of the particle in the position y at the time t.
This class of operators defines a structure in $\mathbb{R}^{N+1}$ different from the Euclidean one: all operators belonging to this class are invariant with respect to a particular translation, and a subclass is invariant with respect to a certain family of dilations, similar in a certain sense to the dilations associated to the heat operator (which belongs to this class, $A_0 = \mathcal{I}_N$ and $B = \mathbb{O}_N$). This structure allows us to define a quasi-metric, from which follows the definition of Hölder continuity. In particular we are interested in considering operators with Hölder continuous coefficients. In this situation we can construct a fundamental solution via parametrix method, although we don't know explicitly its form.

In the first chapter we will study the costant coefficients case, focusing on the geometric structure that this operator defines. These properties will allow us to move to the variable coefficients case. We will see that the regularity we ask is strictly related to the geometry induced by the operator, and this regularity will allow us to define a parametrix method to construct the fundamental solution.

In the last chapter we will see an application of this theory in a Kuramoto model, a mathematical model that describes the spontaneous synchronization phenomenon between coupled

oscillators. Synchronization is a state into which incoherent systems may go, often as it occurs in phase transition. It concerns phenomena belonging to different fields, such as Biology, Physics, Engineering and even Social Sciences.
We will study the following nonlinear Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 \rho}{\partial \omega^2} + \dfrac{\partial}{\partial \omega}\left[(\omega - \Omega - K_\rho(\theta, t))\rho\right] - \omega \dfrac{\partial \rho}{\partial \theta} - \dfrac{\partial \rho}{\partial t} = 0 & \text{in } \mathbb{R}^3 \times (0, \mathrm{T}) \\ \rho(\theta, \omega, \Omega, 0) = \rho_0(\theta, \omega, \Omega) & \text{in } \mathbb{R}^3 \end{cases}$$

where we set:

$$K_\rho(\theta, t) = K \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{2\pi} g(\Omega') \sin(\theta' - \theta) \rho(\theta', \omega', \Omega', t) d\theta' d\omega' d\Omega'$$

This problem was studied in [**Sp2**] using a regularized parabolic equation with bounded coefficients.
We will take advantage of a linearization of the problem to apply the parametrix method, achieving an existence and uniqueness result with less assumptions on the initial datum.

Chapter 1

Kolmogorov operators with constant coefficients

1.1 The class $\mathbb{K}$

In this chapter we start studying Kolmogorov operators with constant coefficients, creating a starting point to study the Hölder continuous coefficients case. We refer to [**AP**] and [**LP**].

We consider the family of Kolmogorov operators of the form:

$$\mathscr{L}u(x,t) = \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j}u(x,t) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) \tag{1.1}$$

$$\mathscr{L} = \mathrm{div}_x(A\nabla_x) + \langle Bx, \nabla_x\rangle - \partial_t$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, $A = (a_{ij})_{i,j=1,\dots,N}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are matrices with real constant coefficients, A symmetric and non negative, $\langle\cdot,\cdot\rangle$ represents the inner product in $\mathbb{R}^N$, $\nabla_x = (\partial_{x_1}, ..., \partial_{x_N})$ and div_x is the usual divergence operator with respect to variable x. An example of an operator that belongs to this family can be found in kinetic theory: if $(W_t)_{t\geq 0}$ denotes a real Brownian motion, the density $p = p(t, v, y, t_0, v_0, y_0)$ of this stochastic process $(V_t, Y_t)_{t\geq 0}$:

$$\begin{cases} V_t = v_0 + \sigma W_t & \sigma \in \mathbb{R}^+ \\ Y_t = y_0 + \int_0^t V_s ds \end{cases}$$

satisfies the following partial differential equation:

$$\frac{1}{2}\sigma\partial^2_{vv}p + v\partial_y p = \partial_t p \qquad (t, v, y) \in \mathbb{R}^+_0 \times \mathbb{R}^2 \tag{1.2}$$

In 1934 Kolmogorov in [**K**] provided us with the explicit expression of the density $p = p(t, v, y, v_0, y_0)$ of the above equation:

$$p(t, v, y, v_0, y_0) = \frac{\sqrt{3}}{\sigma\pi t^2} e^{\left(-\frac{2}{\sigma}\left(\frac{(v-v_0)^2}{t} + 3\frac{(v-v_0)(y-y_0-tv_0)}{t^2} + 3\frac{(y-y_0-ty_0)^2}{t^3}\right)\right)} \qquad t > 0 \tag{1.3}$$

What we want to point out is the regularity of (1.3) despite the strong degeneracy of (1.2). This is due to the particular structure of the operator $\mathscr{L} = \frac{1}{2}\sigma\partial^2_{vv} + v\partial_y - \partial_t$.
Indeed this operator is *hypoelliptic*:

Definition 1.1.1 (Hypoelliptic operator) *Let $\mathscr{L}$ be an operator acting in an open subset $\Omega \subseteq \mathbb{R}^N$. We call this operator hypoelliptic if for every distributional solution $u \in L^1_{\text{loc}}(\Omega)$ to the equation $\mathscr{L}u = f$ we have:*

$$f \in C^\infty(\Omega) \Longrightarrow u \in C^\infty(\Omega)$$

Equivalent conditions for the hypoellipticity of operators (1.1) were proved in [**H**], [**LP**] . We first need to give some definitions:

$$E(t) := \exp(-tB) \qquad \forall\, t \in \mathbb{R} \tag{1.4}$$

$$C(t) := \int_0^t E(s)AE^T(s)ds \qquad \forall\, t \in \mathbb{R} \tag{1.5}$$

We consider then (1.1) in terms of differential geometry. Let's consider these first order differential operators:

$$X_j = \sum_{k=1}^{N} a_{jk}\partial_{x_k}, \quad j = 1, ..., N, \quad Y_1 = \langle Bx, \nabla_x \rangle, \quad Y = Y_1 - \partial_t \tag{1.6}$$

We see that $\mathscr{L} = \sum_{j=1}^N X_j^2 + Y$.
We can identify a generic first order differential operetor $C = \sum_{k=1}^N c_k(y)\partial_{x_k}$ with the vector field $c(y) = (c_1(y), ..., c_N(y))$.
The following conditions are equivalent:

Theorem 1.1.1 *Consider an operator $\mathscr{L}$ in the form* (1.1)*. The following statements are equivalent:*

1. *Hörmander condition: rank Lie$(X_1, ..., X_N, Y_1)(x) = N \quad \forall x \in \mathbb{R}^N$ (thus rank Lie$(X_1, ..., X_N, Y)(x,t) = N + 1 \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$) ;*
2. $\ker(A)$ *does not contain non-trivial subspaces which are invariant for B^T;*
3. *$C(t) > 0 \quad \forall t > 0$;*
4. *for some basis of $\mathbb{R}^N$ the matrices A and B take the following block form:*

$$A = \begin{pmatrix} A_0 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}$$

$$B = \begin{pmatrix} \star & \star & \dots & \star & \star \\ B_1 & \star & \dots & \star & \star \\ \mathbb{O} & B_2 & \dots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \star \end{pmatrix}$$

where A_0 is a square non singular matrix with rank m_0, whereas B_j is a $m_j \times m_{j-1}$ block with rank m_j, $j = 1, ..., \kappa$. Moreover $m_0 \geq m_1 \geq ... \geq m_\kappa \geq 1$ and $N = m_0 + m_1 + ..., m_\kappa$, and the entries $\star$ of the blocks are arbitrary.
When the above conditions are satisfied, then $\mathscr{L}$ is hypoelliptic and its fundamental solution Γ is:

$$\Gamma(x, t, \xi, \tau) = \Gamma(x - E(t - \tau)\xi, t - \tau) \tag{1.7}$$

where

$$\Gamma(x,t,0,0)=\Gamma(x,t)=\begin{cases}\frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det(C(t))}}\exp\left(-\frac{1}{4}\langle C^{-1}(t)x,x\rangle - t(\mathrm{Tr}(B))\right) & t>0\\ 0 & t\leq 0\end{cases} \tag{1.8}$$

Moreover $\Gamma(\cdot,\zeta)=\Gamma(x,t,\xi,\tau)\in C^\infty(\mathbb{R}^{N+1}\setminus\{\zeta\})$.

Proof.
Since A is a positive semidefinite matrix, then $C(t)\geq 0\ \forall t>0$. For the same reason, for every fixed $\xi\in\mathbb{R}^N$, $f(t)=\langle C(t)\xi,\xi\rangle$ is a monotone non-decreasing function. As a consequence, if 3. does not hold, then there exist $t>0$ and $\xi\neq 0$ such that:

$$\langle C(s)\xi,\xi\rangle=0 \quad \forall s\in(0,t]$$

Recalling definition (1.5) this yields:

$$\langle AE^T(s)\xi,E^T(s)\xi\rangle=0 \quad \forall s\in(0,t]$$

Hence for (1.4) we have:

$$\left(\sum_{k=0}^{\infty}(-1)^k\frac{s^k}{k!}A(B^T)^k\right)\xi=0 \quad \forall s\in(0,t]$$

which implies:

$$A(B^T)^k\xi=0 \quad \forall k\in\mathbb{N} \tag{1.9}$$

Vice-versa, if (1.9) holds then $\langle C(t)\xi,\xi\rangle=0\ \ \forall t>0$. This proves the equivalence between 3. and the following proposition:

$$A(B^T)^k\xi=0 \quad \forall k\in\mathbb{N}\Longrightarrow \xi=0 \tag{1.10}$$

On the other hand, since

$$V=\left\{\xi\in\mathbb{R}^N : A(B^T)^k\xi=0,\ \ k\in\mathbb{N}\right\}$$

is the greatest B^T-invariant subspace of $\ker(A)$, (1.10) is equivalent to 2.
To show the equivalence between 1. , 2. and 3. we need first to give some definitions about $Lie(X_1,...,X_N,Y_1)(x)$. For any $k\in\mathbb{N}$ and $j=1,...,N$, we set:

$$X_{(j,0)}=X_j,\quad X_{(j,k+1)}=[X_{(j,k)},Y_1]$$

where $[\cdot,\cdot]$ denotes Lie bracket between vector fields defined in (1.6). Then we have:

$$Lie(X_1,...,X_N,Y_1)=V_k=\mathrm{span}\left\{X_{(j,i)},j=1,...,N,i=1,...,k\right\} \quad \textit{for some } k\in\mathbb{N}$$

Condition 1. means that $V_k=\mathbb{R}^N$.
We observe now that condition 1. is equivalent to (1.10) because:

$$X_{(j,k)}=j\text{-th row of } A(B^T)^k \quad \forall k\in\mathbb{N}, j=1,...,N$$

thus proving the equivalence.
We now prove the equivalence between condition 1. and condition 4.
Let's suppose 1. holds, if we define:

$$\kappa:=\min\{k\in\mathbb{N}:V_k=\mathbb{R}^N\}$$

we know this is well posed. Since $X_{(j,k)}$ is the j-th row of $A(B^T)^k$, then for every $k \in \mathbb{N}$ we have:

$$V_k = \left(\ker(A) \cap \ker(AB^T) \cap ... \cap \ker(A(B^T)^k)\right)^{\perp}$$

Thus

$$V_0 \subset V_1 \subset ... \subset V_\kappa = \mathbb{R}^N$$

and $V_i \neq V_{i-1}$ for every $i \leq \kappa$. In fact, if $V_i = V_{i-1}$ for some $i \geq 1$ then $V_{k-1} = V_k$ for every $k \geq i$.

Let us set

$$m_0 = \dim(V_0), \quad m_i = \dim(V_i) - \dim(V_{i-1}), \quad 1 \leq i \leq \kappa$$

We say that $\{u_1, ..., u_N\}$ is a canonical basis of $\mathbb{R}^N$ for the operator (1.1) if $\{u_1, ..., u_N\}$ is a fan orthonormal basis for the subspaces $V_0, ..., V_\kappa$. We show that this is the basis such that A and B take the form described in 4.

It's easy to see that A takes the form described in 4. Moreover, if we write:

$$B^T = \begin{pmatrix} B^T_{0,0} & B^T_{0,1} & B^T_{0,2} & \cdots & B^T_{0,\kappa} \\ B^T_{1,0} & B^T_{1,1} & B^T_{1,2} & \cdots & B^T_{1,\kappa} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T_{\kappa,0} & B^T_{\kappa,1} & B^T_{\kappa,2} & \cdots & B^T_{\kappa,\kappa} \end{pmatrix}$$

where $B^T_{i,j}$ is a $m_i \times m_j$ block, then

$$AB^T = \begin{pmatrix} A_0B^T_{0,0} & A_0B^T_{0,1} & A_0B^T_{0,2} & \cdots & A_0B^T_{0,\kappa} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{pmatrix}$$

As a consequence, since $X_{(j,1)}$ is the j-th row of AB^T and $X_{(j,1)} \in V_1 = span\{X_{(j,0)}, X_{(j,1)} : j = 1, ...N\}$, it has to be $A_0B^T_{0,2} = 0, ..., A_0B^T_{0,\kappa} = 0$ and $\text{rank}(A_0B^T_{0,1}) = m_1$. Thus, as A_0 is not singular, we have $B^T_{0,2} = ... = B^T_{0,\kappa} = 0$ and $\text{rank}(B^T_{0,1}) = m_1$. Finally, because $B^T_1 := B^T_{0,1}$ is a $m_0 \times m_1$ matrix, we have $m_1 \leq m_0$.

Then the matrices B^T and $A(B^T)^2 = (AB^T)B^T$ can be expressed, respectively, in the following way:

$$B^T = \begin{pmatrix} \star & B^T_1 & \mathbb{O} & \cdots & \mathbb{O} \\ B^T_{1,0} & B^T_{1,1} & B^T_{1,2} & \cdots & B^T_{1,\kappa} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T_{\kappa,0} & B^T_{\kappa,1} & B^T_{\kappa,2} & \cdots & B^T_{\kappa,\kappa} \end{pmatrix}$$

and

$$A(B^T)^2 = \begin{pmatrix} \star & \star & A_0B^T_1B^T_{1,2} & \cdots & A_0B^T_1B^T_{1,\kappa} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{pmatrix}$$

Now, since $X_{(j,2)}$ is the j-th row of the matrix $A(B^T)^2$, arguing as above we can prove that $\text{rank}(A_0B^T_1B^T_{1,2}) = m_2$ and $A_0B^T_1B^T_{1,3} = ... = A_0B^T_1B^T_{1,\kappa} = 0$. As a consequence, since $\text{rank}(B^T_{1,2}) \geq \text{rank}((A_0B^T_1)B^T_{1,2})$ and $B^T_2 := B^T_{1,2}$ is a $m_1 \times m_2$ matrix, we get $\text{rank}(B^T_2) = m_2$ and $m_2 \leq m_1$. Moreover, as $\ker(A_0B^T_1) = \{0\}$ ($A_0B^T_1$ is a $m_0 \times m_1$ matrix of rank $m_1 \leq m_0$)

from the previous equality we obtain $B^T_{1,3} = ... = B^T_{1,\kappa} = 0$. Therefore, the matrix B^T can be written in the following way

$$B^T = \begin{pmatrix} \star & B_1^T & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & B_2^T & \mathbb{O} & \cdots & \mathbb{O} \\ B^T_{2,0} & B^T_{2,1} & B^T_{2,2} & B^T_{2,3} & \cdots & B^T_{1,\kappa} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T_{\kappa,0} & B^T_{\kappa,1} & B^T_{\kappa,2} & B^T_{\kappa,3} & \cdots & B^T_{\kappa,\kappa} \end{pmatrix}$$

The proof of the first part of the equivalence follows by iterating the previous arguments. Let's prove now the vice-versa: if A and B are in the form 4. let's show that $\{e_1, ..., e_N\}$ with

$$e_i = (0, ..., \underset{i}{1}, ..., 0), \qquad 1 \le i \le N$$

is a canonical basis of $\mathbb{R}^N$ for the operator $\mathscr{L}$, thus implying its hypoellipticity. If A and B are in that form then:

$$A(B^T)^i = \begin{pmatrix} \star_0 & \star_1 & \cdots & \star_{i-1} & C_i & \mathbb{O} & \cdots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \end{pmatrix}$$

where $\star_j$ denotes a $m_0 \times m_j$ block, and $C_i = A_0 B_1^T ... B_i^T$ is a $m_0 \times m_i$ matrix of rank m_i (if $i = 0$ we agree to let $C_i = A_0$).
By definition, since $X_{(j,k)}$ is the j-th row of $A(B^T)^k$, we get:

$$V_k = \text{span}\left\{ e_h : 0 \le h \le \sum_{i=0}^{k} m_i \right\}, \; 0 \le k \le \kappa$$

In other words, $\{e_1, ..., e_N\}$ is a fan orthonormal basis for $V_0, ..., V_r$.
The proof of the equivalence between condition 1. and hypoellipticity and the construction of the fundamental solution were proved by Hörmander in [**H**]. □

Remark 1.1.1 *From the proof of Theorem 1.1.1 it easily follows that condition 3. is equivalent to the following ones:*

- *there exists $t > 0$ such that $C(t) > 0$;*
- $C(t) < 0 \;\; \forall t < 0$;
- *there exists $t < 0$ such that $C(t) < 0$;*

We denote by $\mathbb{K}$ the class of Kolmogorov operators (1.1) satisfying one (thus all) condition in Theorem 1.1.1. We will now study an important subclass of $\mathbb{K}$, that defines a structure in $\mathbb{R}^{N+1}$. This structure is important to extend the theory to operators in non divergence form.

1.2 Lie group

In this section we will also refer to [**BLU**].
When we study a differential operator it is very useful to define its associated structure, a non Euclidean structure that define a quasi-metric.
Let's consider $\mathscr{L} \in \mathbb{K}$, and suppose the basis of $\mathbb{R}^N$ is such that the constant matrices A and B have the form 4.

Definition 1.2.1 (Lie group on $\mathbb{R}^N$) *Let $\circ$ be a given group law on $\mathbb{R}^N$, and suppose that the map:*

$$\begin{aligned}\mathbb{R}^N \times \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ (x,y) &\longmapsto y^{-1} \circ x\end{aligned}$$

is smooth. Then $\mathbb{G} = (\mathbb{R}^N, \circ)$ is called Lie group on $\mathbb{R}^N$.

The operators $\mathscr{L} \in \mathbb{K}$ have the remarkable property of being invariant with respect to the left translations of a Lie group on $\mathbb{R}^{N+1}$. In fact, if we define

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \qquad (x,t),(\xi,\tau) \in \mathbb{R}^N \times \mathbb{R} \tag{1.11}$$

it's easy to show that $(\mathbb{R}^{N+1}, \circ)$ is a group with identity element $(0,0)$, and inverse

$$(x,t)^{-1} = (-E(-t)x, -t)$$

If, for every $\zeta \in \mathbb{R}^{N+1}$, we denote by l_ζ the left translation defined as

$$\begin{aligned}\mathbb{R}^{N+1} &\longrightarrow \mathbb{R}^{N+1} \\ z &\longmapsto l_\zeta(z) = \zeta \circ z\end{aligned} \tag{1.12}$$

then every $\mathscr{L} \in \mathbb{K}$ is left invariant with respect to (1.12):

$$\mathscr{L} \circ l_\zeta = l_\zeta \circ \mathscr{L} \Longleftrightarrow \mathscr{L}(u(\zeta \circ z)) = (\mathscr{L}u)(\zeta \circ z)$$

We note also that (1.7) can be interpreted in the following way:

$$\Gamma(x,t,\xi,\tau) = \Gamma((\xi,\tau)^{-1} \circ (x,t))$$

We focus now on dilations: let us call dilations group in $\mathbb{R}^{N+1}$ a family $\mathscr{G} = (D(r))_{r \geq 0}$ where $D(r)$ is a symmetric and positive definite matrix in $\mathbb{R}^{N+1}$. We say that the operator $\mathscr{L}$ is $\mathscr{G}$-invariant (or that $\mathscr{L}$ commutes with the dilations of $\mathscr{G}$) if the following identity holds

$$\mathscr{L} \circ D(r) = r^2(D(r) \circ \mathscr{L})$$

or equivalently we have

$$\mathscr{L}(u(D(r)z)) = r^2 \mathscr{L}u(D(r)z)$$

for every $u \in C_0^\infty(\mathbb{R}^{N+1})$, $r > 0$ and $z \in \mathbb{R}^{N+1}$.
We denote by $\mathbb{K}_0$ the subclass of operators $\mathscr{L} \in \mathbb{K}$ which are invariant with respect to some dilations group. We show now when $\mathscr{L} \in \mathbb{K}_0$:

Theorem 1.2.1 *The operator $\mathscr{L} \in \mathbb{K}$ is invariant with respect to a dilations group $\mathscr{G} = (D(r))_{r \geq 0}$ if and only if all the $\star$ blocks of the canonical form (4.) of B are zeros matrices. In this case we have:*

$$D(r) = \operatorname{diag}(r\mathcal{I}_{m_0}, r^3\mathcal{I}_{m_1}, ..., r^{2\kappa+1}\mathcal{I}_{m_\kappa}, r^2) \tag{1.13}$$

where $\mathcal{I}_k$ denote the identity matrix $k \times k$.

Proof.
Let $\mathscr{G} = (D(r))_{r \geq 0}$ be a dilations group on $\mathbb{R}^{N+1}$ commuting with $\mathscr{L}$ and let $(m_{ij}(r))_{i,j=1,...,N+1}$ be the matrix $D(r)$. If we put, for every $p, r \in \mathbb{R}$

$$u(z) = u(x,t) = e^{pt}, \quad v(z) = u(D(r)z)$$

then

$$(\mathscr{L}u)(D(r)z) = -pu(D(r)z) = -pv(z)$$

and

$$\mathscr{L}v(z) = \left(p^2 \sum_{i,j=1}^{N} a_{ij} m_{N+1,i}(r) m_{N+1,j}(r) + p \sum_{i,j=1}^{N} b_{ij} z_i m_{N+1,j}(r) - p m_{N+1,N+1}(r)\right) v(z)$$

As a consequence, since the invariance of $\mathscr{L}$ with respect to $\mathscr{G}$ yields $\mathscr{L}(u(D(r)z)) = r^2(\mathscr{L}u)(D(r)z)$ for every $r > 0$ and for every $z_i \in \mathbb{R}, 1 \leq i \leq N$, the following conditions hold:

$$\sum_{i,j=1}^{N} a_{ij} m_{N+1,i}(r) m_{N+1,j}(r) = 0$$

$$\sum_{i,j=1}^{N} b_{ij} z_i m_{N+1,j}(r) = 0$$

$$m_{N+1,N+1}(r) = r^2$$

the first equation and the form of A (4.) give

$$m_{N+1,i}(r) = 0 \quad for \;\; 1 \leq i \leq m_0, \;\; r > 0$$

Using this relation in the second equation give

$$\sum_{j=1}^{N} b_{ij} m_{N+1,j}(r) = 0 \quad for \;\; m_0 + 1 \leq i \leq N, \;\; r > 0$$

Since the matrix B is in form 4., using this last relation we obtain

$$B_1^T(m_{m_0+1,N+1}, ..., m_{m_0+m_1,N+1}) = (0, ..., 0)^T$$

hence, because B_1^T is a $m_0 \times m_1$ matrix of rank $m_1 \leq m_0$,

$$m_{N+1,i}(r) = 0 \quad for \;\; m_0 + 1 \leq i \leq m_0 + m_1, \;\; r > 0$$

By iterating this argument we finally obtain

$$m_{N+1,i}(r) = 0 \quad for \;\; 1 \leq i \leq N, \;\; r > 0$$

Now we put

$$U(z) = e^{pz_i + qz_j}, \quad V(z) = U(D(r)z)$$

where $p, q, r \in \mathbb{R}$ and $1 \leq i, j \leq N$. Then

$$(\mathscr{L}U)(D(r)z) = \left(p^2 a_{ii} + 2pq a_{ij} + q^2 a_{jj} + \sum_{h=1}^{N} (p b_{hi} + q b_{hj})(D(r)z)_h\right) V(z)$$

and

$$\mathscr{L}V(z) = (p^2 \sum_{h,k=1}^{N} a_{hk} m_{ih}(r) m_{ik}(r) + 2pq \left(\sum_{h,k=1}^{N} a_{hk} m_{ih}(r) m_{ik}(r)\right) \left(\sum_{h,k=1}^{N} a_{hk} m_{ih}(r) m_{jk}(r)\right)$$
$$+ q^2 \sum_{h,k=1}^{N} a_{hk} m_{ih}(r) m_{ik}(r) + \sum_{h,k=1}^{N} b_{hk} z_k (p m_{ih}(r) + q m_{ik}(r)) V(z) \tag{1.14}$$

Since $\mathscr{L}$ is invariant with respect to $\mathscr{G}$, then $(\mathscr{L}V)(z) = r^2(\mathscr{L}U)(D(r)z)$ for every $r > 0$ and for every $z \in \mathbb{R}^{N+1}$. Consequently, since p, q, i, j in (1.14) are arbitrary, the following equations hold:

$$\begin{aligned} r^2 A &= M(r)AM(r) \\ r^2 M(r)B^T &= B^T M(r) \end{aligned} \tag{1.15}$$

where $M(r)$ denotes the matrix $(m_{ij}(r))_{i,j=1,...,N}$. Now, we break up the matrices $M(r)$ and B^T into $m_i \times m_j$ blocks $M_{ij}(r)$ and B^T_{ij}, respectively, $i, j = 0, ..., \kappa$. From the first of this last equations and the form of A we have:

$$\begin{aligned} &r^2 A_0 = M_{00}(r)A_0 M_{00}(r) \\ &M_{i0}(r)A_0 M_{00}(r) = 0, \qquad 1 \le i \le \kappa \end{aligned}$$

Then, since $M(r)$(and so $M_{00}(r)$) is positive definite,

$$\begin{aligned} &M_{00}(r) = r\mathcal{I}_{m_0} \\ &M_{i0}(r) = 0, \qquad 1 \le i \le \kappa \end{aligned}$$

On the other hand considering the form of B^T we have $B^T_{j-1,j} = B^T_j$ if $1 \le j \le \kappa$ and $B^T_{i,i+1} = 0$ if $0 \le i \le \kappa - 1$.
From the second of (1.15) we then obtain:

$$r^3 B^T_{00} = r B^T_{00} \quad \forall r > 0$$

hence $B^T_{00} = 0$. But from the same relation we also have

$$r^3 B^T_{0i} = B^T_{01} M_{1,i}(r) \quad for\ 1 \le i \le \kappa$$

and so, since $B^T_{01} = B^T_1$ has maximum rank and $B^T_{0i} = 0$ for $i \ge 2$,

$$\begin{aligned} &M_{11}(\lambda) = \lambda^3 \mathcal{I}_{m_1} \\ &M_{1i}(\lambda) = 0 \quad 2 \le i \le \kappa \end{aligned}$$

By iterating these arguments and by using (1.15) for the (i, j) blocks, $i \ge 2$ and $j \ge 0$, we finally get

$$D(r) = \mathrm{diag}(r\mathcal{I}_{m_0}, r^3\mathcal{I}_{m_1}, ..., r^{2\kappa+1}\mathcal{I}_{m_\kappa}, r^2)$$

and $B^T_{ij} = 0$ when $i \ge 0$ and $j \le i$. This proves the "only if" part. The "if" part is an easy exercise. □

From now on we will denote by $D_0(r)$ the "restriction" of $D(r)$ to $\mathbb{R}^N$:

$$D_0(r) = \mathrm{diag}(r\mathcal{I}_{m_0}, r^3\mathcal{I}_{m_1}, ..., r^{2\kappa+1}\mathcal{I}_{m_\kappa})$$

Definition 1.2.2 (Homogeneous dimension of $\mathbb{R}^{N+1}$) *We define homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $(D(r))_{r \ge 0}$ the natural number:*

$$Q + 2 = \log_r(\det(D(r))) = \log_r(\det(D_0(r))) + 2$$

The homogeneous dimension is usually considered when we define a dilations group associated to an operator, for example the dilations associated to the laplacian are the Euclidean one, and the homogeneous dimension of $\mathbb{R}^N$ with respect to this dilations is $Q = N$.
We will prove now a relation between the fundamental solution defined in Theorem 1.1.1 and the homogeneous dimension that has an equivalent with respect to laplacian and other operators.

Theorem 1.2.2 *Let's suppose* $\mathscr{L} \in \mathbb{K}_0$*, its fundamental solution* Γ *is homogeneous of degree* $-Q$ *with respect to* $(D(r))_{r\geq 0}$ *dilations.*

Proof.
We can consider the fundamental solution with pole $\zeta = 0$. If we denote with δ the Dirac measure with support in 0 we have:

$$\mathscr{L}(\Gamma \circ D(r)) = r^2 D(r)(\mathscr{L}\Gamma) = -r^2 D(r)\delta = -r^{-Q}\delta$$

were we used the following property of the fundamental solution: $\mathscr{L}\Gamma = -\delta$.
Therefore $\Gamma \circ D(r) = r^{-Q}\Gamma$. □

When $\mathscr{L} \in \mathbb{K}_0$, the matrix $C(t)$ defined in (1.5) takes a particular form, as stated in the following theorem (for more details see [**K1**] and [**K2**]):

Theorem 1.2.3 *If* $\mathscr{L} \in \mathbb{K}_0$ *then*

$$C(t) = D_0(\sqrt{t})C(1)D_0(\sqrt{t}), \quad \forall t > 0 \tag{1.16}$$

Moreover, if Γ *denotes the fundamental solution of* $\mathscr{L}$ *with pole at* $(0,0)$*, then*

$$\Gamma(x,t) = \frac{c_N}{t^{\frac{Q}{2}}} \exp\left(-\frac{1}{4}\langle C^{-1}(1)D_0\left(\frac{1}{\sqrt{t}}\right)x, D_0\left(\frac{1}{\sqrt{t}}\right)x\rangle\right) \tag{1.17}$$

where c_N *is the following positive constant*

$$c_N = (4\pi)^{-\frac{N}{2}}(\det(C(1)))^{-\frac{1}{2}} \tag{1.18}$$

Proof,
Since Γ is $D(r)$ homogeneous of degree $-Q$, then for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ we have $\Gamma(D(r)(x,t)) = r^{-Q}\Gamma(x,t)$, thus

$$r^{-Q} = \left(\frac{\det(C(t))}{\det(C(r^2 t))}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{4}\langle C^{-1}(r^2 t)D_0(r)x, D_0(r)x\rangle + \frac{1}{4}\langle C^{-1}(t)x, x\rangle\right)$$

this yields

$$D_0(r)C^{-1}(r^2 t)D_0(r) = C^{-1}(t)$$

which implies (1.16) by choosing $r = \frac{1}{\sqrt{t}}$.
Now, recalling $\det(D_0(r)) = r^Q$, (1.17) and (1.18) follow from (1.8). □

If $\mathcal{L} \in \mathbb{K}_0$ the matrix B is nilpotent and

$$E(t) = \sum_{k=0}^{\kappa}(-1)^k \frac{t^k}{k!}B^k \tag{1.19}$$

Using this identity keeping in mind Theorem 1.1.1 and Theorem 1.2.1 we get:

$$E(r^2 t) = D_0(r)E(t)D_0\left(\frac{1}{r}\right) \tag{1.20}$$

In fact, one needs to prove that

$$E(r^2 t)D_0(r) = D_0(r)E(t) \tag{1.21}$$

From (1.19) this equivalence holds if and only if

$$r^{2k}B^kD_0(r) = D_0(r)B^k, \qquad 0 \leq k \leq \kappa$$

For $k = 0$ this identity holds. A direct computation shows that it also holds true also for $k = 1$. As a consequence

$$\begin{aligned} r^4B^2D_0(r) &= r^2B(r^2BD_0(r)) = r^2B(D_0(r)B) = (r^2BD_0(r))B = \\ &= (D_0(r)B)B = D_0(r)B^2 \end{aligned}$$

Then it holds also for $k = 2$. The thesis follows iterating these steps.
We can use this relation to prove the following proposition:

Proposition 1.2.1 *Let* $\circ$ *and* $D(r)$ *be the left translation and the dilation defined in* (1.11) *and* (1.13)*. Then the following "distributive" property holds:*

$$D(r \circ \zeta) = (D(r)\zeta) \circ (D(r)\zeta)$$

and moreover

$$D(r)z^{-1} = (D(r)z)^{-1}$$

Proof.
Since $D(r)D(\sqrt{t}) = D(\sqrt{r^2t})$, from (1.20) we obtain

$$\begin{aligned} D(r \circ \zeta) &= D(r)(\xi + E(\tau)x, t + \tau) = (D_0(r)\xi + D_0(r)E(\tau)x, r^2t + r^2\tau) = \\ &= (D_0(r)\xi + E(r^2\tau)D_0(r)x, r^2t + r^2\tau) = (D(r)\zeta) \circ (D(r)\zeta) \end{aligned}$$

Moreover

$$\begin{aligned} D(r)z^{-1} &= D(r)(-E(t)x, -t) = (-D_0(r)E(t)x, -r^2t) = \\ &= (-E(r^2t)D_0(r)x, -r^2t) = (D(r)z)^{-1} \end{aligned}$$

□

We can now consider particular Lie groups:

Definition 1.2.3 (Homogeneous Lie group on $\mathbb{R}^N$) *Let* $\mathbb{G} = (\mathbb{R}^N, \circ)$ *be a Lie group on* $\mathbb{R}^N$ *(according to Definition 1.2.1). We say that* $\mathbb{G}$ *is a homogeneous Lie group on* $\mathbb{R}^N$ *if the following property holds:*

- *There exists an* N*-tuple of real numbers* $\sigma = (\sigma_1, ..., \sigma_N)$*, with* $1 \leq \sigma_1 \leq ... \leq \sigma_N$*, such that the dilation*

$$\begin{aligned} \delta_r : \qquad & \mathbb{R}^N \longrightarrow \mathbb{R}^N \\ & x \longmapsto (r^{\sigma_1}x_1, ..., r^{\sigma_N}x_N) \end{aligned}$$

  is an automorphism of the group $\mathbb{G}$ *for every* $r > 0$ *.*

We will denote by $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_r)$ *the datum of a homogeneous Lie group on* $\mathbb{R}^N$ *with composition law* $\circ$ *and dilation group* $(\delta_r)_{r>0}$*.*

If we consider the dilations defined in (1.13), we can define a suitable homogeneous Lie group associated to the subclass $\mathbb{K}_0$ if we define $\delta_r(x, t) = D(r)(x, t)$.

Definition 1.2.4 (Homogeneous Lie group for subclass $\mathbb{K}_0$) *If the matrix* B *is in the form described in Theorem 1.2.1, the following structure:*

$$\mathbb{G}_0 = \left(\mathbb{R}^{N+1}, \circ, D(r)\right)$$

(where $\circ$ *and* $D(r)$ *are defined in* (1.11) *and* (1.13)*) is a homogeneous Lie group.*

Definition 1.2.5 *A measurable function u on $\mathbb{G}_0$ will be called homogeneous of degree $\alpha \in \mathbb{R}$ if*

$$u(D(r)z) = r^{\alpha} u(z) \qquad \forall z \in \mathbb{R}^{N+1}$$

A differential operator X will be called homogeneous of degree $\beta \in \mathbb{R}$ with respect to $D(\lambda)$ if

$$Xu(D(r)z) = r^{\beta}(Xu)(D(r)z) \qquad \forall z \in \mathbb{R}^{N+1}$$

and for every sufficiently smooth function u. Note that, if u is homogeneous of degree α and X is homogeneous of degree β, then Xu is homogeneous of degree $\alpha - \beta$.

As far as we are concerned with the vector fields of the Kolmogorov operators as defined in (1.6), we have that $X_1, ..., X_N$ are homogeneous of degree 1, Y and Y_1 are homogeneous of degree 2 and $\mathscr{L} = \sum_{i=1}^{N} X_i^2 + Y$ is homogeneous of degree 2 with respect to $D(r)$.

We now introduce an homogeneous semi-norm of degree 1 with respect to the family of dilations $(D(r))_{r \geq 0}$, and a quasi-distance which is invariant with respect to the group operation $\circ$.

Definition 1.2.6 *For every $z = (x, t) \in \mathbb{R}^{N+1}$ we set*

$$\|z\| = |t|^{\frac{1}{2}} + \sum_{j=1}^{N} |x_j|^{\frac{1}{q_j}} \tag{1.22}$$

where the numbers q_j are associated to the dilation group $D(r)$ as follows:

$$D(r) = \text{diag}(r^{q_1}, ..., r^{q_N}, r^2)$$

this semi-norm is homogeneous of degree 1, that means

$$\|D(r)z\| = r\|z\| \qquad \forall r > 0, \; z \in \mathbb{R}^{N+1}$$

Because every norm is equivalent to any other in $\mathbb{R}^{N+1}$, other definitions have been used in the literature. For instance in [**M**] it is chosen the following one: for every $z = (x, t) \in \mathbb{R}^N \times \mathbb{R}$ the norm of z is the unique positive solution r to the following equation:

$$\frac{x_1^2}{r^{2q_1}} + \frac{x_2^2}{r^{2q_2}} + ... + \frac{x_N^2}{r^{2q_N}} + \frac{t^2}{r^4} = 1$$

Note that, if we choose this definition the set $\{z \in \mathbb{R}^{N+1} : \|z\| = r\}$ is a smooth manifold for every $r > 0$, while is not the case for (1.22).
The semi-norm (1.22) induces a quasi-distance $d : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}_0^+$:

Definition 1.2.7 *For every $z, w \in \mathbb{R}^{N+1}$, we define a quasi-distance $d(z, w)$ invariant with respect to the translation group G_0 defined in Definition 1.2.3 as follows*

$$d(z, w) = \|z^{-1} \circ w\| \tag{1.23}$$

and we denote by $B_r(z)$ the d-ball of center z and radius r.

The following properties hold:

1. $d(z, w) = 0$ if and only if $z = w$ for every $z, w \in \mathbb{R}^{N+1}$;

2. for every compact subset K of $\mathbb{R}^{N+1}$, there exists a positive constant $C_K \geq 1$ such that

$$d(z,w) \leq C_K d(w,z);$$
$$d(w,z) \leq C_K(d(z,\zeta)+d(\zeta,w)) \qquad \forall z,w,\zeta \in K$$

We can now state a proper definition for Hölder continuous function associated to this quasi-metric:

Definition 1.2.8 (Hölder continuous function) *Let α be a positive constant such that $\alpha \leq 1$, and let Ω be an open subset of $\mathbb{R}^{N+1}$. We say a function $f : \Omega \longrightarrow \mathbb{R}$ is Hölder continuous with exponent α in Ω with respect to the homogeneous Lie group $\mathbb{G}_0$ defined in Definition 1.2.3 (in short, Hölder continuous with exponent α, $f \in C^\alpha(\Omega)$) if there exists a positive constant $k > 0$ such that*

$$|f(z) - f(\zeta)| \leq k d(z,\zeta)^\alpha \qquad \forall z,\zeta \in \Omega$$

To every bounded function $f \in C^\alpha(\Omega)$ we associate the norm

$$|f|_{\alpha,\Omega} = \sup_{\Omega} |f| + \sup_{z,\zeta\in\Omega, z\neq\zeta} \frac{|f(z)-f(\zeta)|}{d(z,\zeta)^\alpha}$$

Moreover, we say a function f is locally Hölder continuous, and we write $f \in C^\alpha_{\text{loc}}(\Omega)$, if $f \in C^\alpha(\Omega')$ for every compact subset $\Omega' \subset \Omega$.

Remark 1.2.1 *Let Ω be a bounded subset of $\mathbb{R}^{N+1}$. If f is a Hölder continuous function of exponent α in the usual Euclidean sense, then f is Hölder continuous of exponent α, because it can be shown that Euclidean metric is continuous with respect to the following one. Vice versa if $f \in C^\alpha(\Omega)$ then f is a $\beta-$Hölder continuous in the Euclidean sense, where $\beta = \frac{\alpha}{2\kappa+1}$ and κ is the constant appearing in 4.*

1.3 Principal part operator

Definition 1.3.1 *Let's consider the canonical form 4. of the matrix B related to some operator $\mathscr{L} \in \mathbb{K}$, $\mathscr{L} = \operatorname{div}_x(A\nabla_x) + \langle Bx, \nabla_x\rangle - \partial_t$. We denote by B_0 the matrix obtained by annihilating every $\star$ block. We define by principal part of $\mathscr{L}$ the operator obtained by substituting the matrix B with matrix B_0, that means:*

$$\mathscr{L}_0 = \operatorname{div}_x(A\nabla_x) + \langle B_0 x, \nabla_x\rangle - \partial_t$$

Theorem 1.2.1 shows us that the invariance with respect to $D(r)$ dilations is related to the matrix B, this means that $\mathscr{L}_0 \in \mathbb{K}_0$.

In the following we will state the equivalence between $\mathscr{L}$ and its principal part $\mathscr{L}_0$ on the level set $\left\{z \in \mathbb{R}^{N+1} : \Gamma_0(z) \geq K\right\}$ (for more details see [**LP**]).

Theorem 1.3.1 *Let $\mathscr{L} \in \mathbb{K}$ and let $\mathscr{L}_0$ be its principal part. Then for every $K > 0$ there exists a positive constant $\varepsilon > 0$ such that*

$$(1-\varepsilon)\Gamma_0(z) \leq \Gamma(z) \leq (1+\varepsilon)\Gamma_0(z)$$

for every $z \in \mathbb{R}^{N+1}$ such that $\Gamma_0(z) \geq K$. Moreover, $\varepsilon = \varepsilon(K) \to 0$ as $K \to +\infty$.

We observe that this relation does not hold, in general, for the fundamental solutions $\Gamma(z;\zeta)$ and $\Gamma_0(z;\zeta)$ when $\zeta \neq 0$.

1.4 Mean value formulas

In this section we will first show a mean value formula which demonstration uses the divergence theorem, then starting from this formula we will derive mean value formulas with more regular kernel, by the Hadamard's descent method.

For every $r > 0$ and for every $z_0 \in \mathbb{R}^{N+1}$ let us denote by $\Omega_r(z_0)$ the following level set of the fundamental solution Γ of the operator $\mathscr{L}$:

$$\Omega_r(z_0) = \left\{ z \in \mathbb{R}^{N+1} : \Gamma(z_0; z) > \frac{1}{r} \right\}$$

Since $\Gamma(z_0; z) = \Gamma(z^{-1} \circ z_0; 0) = \Gamma(0; z_0^{-1} \circ z)$ we have

$$\Omega_r(z_0) = z_0 \circ \Omega_r(0) \tag{1.24}$$

where $z_0 \circ \Omega_r(0)$ denotes the z_0-left translated of $\Omega_r(0)$:

$$z_0 \circ \Omega_r(0) = \{ z_0 \circ w : w \in \Omega_r(0) \}$$

When $\mathscr{L} \in \mathbb{K}_0$, if $\mathscr{G} = (D(r))_{r \geq 0}$ is the dilations group related to $\mathscr{L}$ and Q is the spatial homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to $\mathscr{G}$, then $r\Gamma(0; z) = \Gamma\left(0; D\left(r^{-\frac{1}{Q}}\right) z\right)$; therefore:

$$\Omega_r(0) = D\left(r^{\frac{1}{Q}}\right) \Omega_1(0) \tag{1.25}$$

By (1.24) and (1.25) we can easily argue that , if $\mathscr{L} \in \mathbb{K}_0$, then $\overline{\Omega_r(z_0)}$ is a compact set for every $r > 0$. In general, if $\mathscr{L}$ is an operator of $\mathbb{K}$ not belonging to the subclass $\mathbb{K}_0$, this statement fails, as the following example shows.

Example 1.4.1 *If $\mathscr{L} = \partial^2_{xx} - x\partial_x - \partial_t$, where $(x, t) \in \mathbb{R}^2$, then*

$$\Gamma(x, t) = (2\pi(1 - e^{-2t}))^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{x^2}{e^{2t} - 1} \right), \qquad t > 0$$

Whereas, for $t < 0$,

$$\Gamma((0, 0); (x, t)) = (2\pi(1 - e^{-2t}))^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{x^2}{1 - e^{2t}} \right)$$

Then

$$\Gamma((0, 0); (0, t)) = (2\pi(1 - e^{-2t}))^{-\frac{1}{2}} > (2\pi)^{-\frac{1}{2}}$$

Consequently, for every $r \geq \sqrt{2\pi}$, $\Omega_r((0, 0))$ is an unbounded set, since it contains the half-line $\{(0, t) : t < 0\}$.

Proposition 1.4.1 *Let Ω be an open set of $\mathbb{R}^{N+1}$ and let $u \in C^\infty(\Omega)$ be a solution of the equation $\mathscr{L}u = 0$, with $\mathscr{L} \in \mathbb{K}$. Then, for every $z_0 \in \Omega$ and for every $r > 0$ such that $\overline{\Omega_r(z_0)}$ is a compact subset of Ω, we have*

$$u(z_0) = \frac{1}{r} \int_{\Omega_r(z_0)} M(z_0, z) u(z) dz \tag{1.26}$$

where

$$M(z_0, z) = \frac{\langle A \nabla_x \Gamma(z_0; z), \nabla_x \Gamma(z_0; z) \rangle}{\Gamma^2(z_0; z)} \tag{1.27}$$

The proof of the following proposition follows from an appropriate definition of Green's identity associated to $\mathscr{L}$ and standard arguments.

Starting from (1.26), by the Hadamard's descent method, we can derive some mean value formulas with more regular kernels. The descent method relies on the following remark: if $u = u(x,t)$ is a solution of $\mathscr{L}u = 0$, then, for every $m \in \mathbb{N}$, the function $\tilde{u}(y,x,t) = u(x,t)$, $y \in \mathbb{R}^m$, is a solution of

$$\widetilde{\mathscr{L}}\tilde{u} = (\Delta_y + \mathscr{L})\tilde{u} = 0 \tag{1.28}$$

This operator belongs to $\mathbb{K}$, and can be written in the form (1.1), with matrices A and B replaced by

$$\widetilde{A} = \begin{pmatrix} \mathcal{I}_m & \mathbb{O} \\ \mathbb{O} & A \end{pmatrix} \qquad \widetilde{B} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B \end{pmatrix}$$

If Γ is the fundamental solution of $\mathscr{L}$, then a straightforward computations shows that the fundamental solution $\widetilde{\Gamma}$ of $\widetilde{\mathscr{L}}$ is given by

$$\widetilde{\Gamma}(y_0, x_0, t_0; y, x, t) = \Gamma(x_0, t_0; x, t) K_m(y_0, t_0; y, t)$$

where $K_m(y_0, t_0; y, t)$ denotes the fundamental solution of the heat equation in $\mathbb{R}^{m+1}$:

$$K_m(y_0, t_0; y, t) = \begin{cases} \left(\frac{1}{4\pi(t_0-t)}\right)^{\frac{m}{2}} e^{-\frac{1}{4}\frac{|y_0-y|^2}{(t_0-t)}} & t < t_0 \\ 0 & t \geq t_0 \end{cases}$$

As a consequence the kernel $\widetilde{M}$ in (1.27) corresponding to the operator $\widetilde{\mathscr{L}}$ becomes

$$\widetilde{M}(y_0, x_0, t_0; y, x, t) = M(x_0, t_0; x, t) + \frac{1}{4}\frac{|y_0 - y|^2}{(t_0 - t)^2}$$

As shown in [**LP**], the level set

$$\widetilde{\Omega}_r(y_0, z_0) = \left\{ (y, z) \in \mathbb{R}^{m+N+1} : \widetilde{\Gamma}(y_0, z_0; y, z) > \frac{1}{r} \right\}$$

is bounded for every $r > 0$.
Now, let u be a solution of $\mathscr{L}u = 0$ in the open subset Ω. For every $z_0 \in \Omega$ and $r > 0$ such that the closure of $\widetilde{\Omega}_r(y_0, z_0)$ is contained in $\mathbb{R}^m \times \Omega$, keeping in mind (1.28), formula (1.26) gives:

$$u(z_0) = \frac{1}{r} \int_{\widetilde{\Omega}_r(0,z_0)} \widetilde{M}(0, z_0; y, z) u(z) dy dz$$

If we integrate with respect to the variable y we obtain

$$u(z_0) = \frac{1}{r} \int_{\Omega_r^{(m)}(z_0)} \widetilde{M}_r^{(m)}(z_0; z) u(z) dz \tag{1.29}$$

where

$$\widetilde{M}_r^{(m)} = \omega_m N_r^m \left(M + \frac{m}{m+2} \frac{N_r^2}{4(t_0 - t)^2} \right) \tag{1.30}$$

$$N_r^2(z_0; z) = 4(t_0 - t) \ln \left(\frac{r\Gamma(z_0; z)}{(4\pi(t_0 - t))^{\frac{m}{2}}} \right)$$

$$\Omega_r^{(m)}(z_0) = \left\{ z \in \mathbb{R}^{N+1} : \frac{\Gamma(z_0; z)}{(4\pi(t_0 - t))^{\frac{m}{2}}} > \frac{1}{r} \right\}$$

In (1.30) ω_m denotes the measure of the unity ball in $\mathbb{R}^m$.
We state then this mean value formula, that does not need the to check if the level set is bounded, because we can prove that $\overline{\Omega_r^{(m)}(z_0)}$ is compact.

Proposition 1.4.2 *Let Ω be an open set of $\mathbb{R}^{N+1}$ and let $u \in C^\infty(\Omega)$ be a solution of the equation $\mathscr{L}u = 0$, with $\mathscr{L} \in \mathbb{K}$. Then, for every $z_0 \in \Omega$ and for every $r > 0$ such that the closure of the set $\Omega_r^{(m)}(z_0)$ is contained in Ω, we have*

$$u(z_0) = \frac{1}{r} \int_{\Omega_r^{(m)}(z_0)} \widetilde{M}_r^{(m)}(z_0; z) u(z) dz \tag{1.31}$$

1.5 Harnack inequality

In this section we will give some Harnack inequalities for the non-negative solutions of the equation $\mathscr{L}u = 0$, which are invariant with respect to $D(r)$. These proof rely on mean value formulas (1.26) and (1.31).

Theorem 1.5.1 *Let $\mathscr{L} \in \mathbb{K}_0$ and let Ω be an open subset of $\mathbb{R}^{N+1}$.*
We define:

$$K_r(z_0, \varepsilon) = \Omega_r(z_0) \cap \left\{(x,t) \in \mathbb{R}^{N+1} : t \leq t_0 - \varepsilon r^{\frac{2}{Q}}\right\}, \qquad \varepsilon > 0$$

Then for every $\varepsilon \in \left(0, c_N^{\frac{2}{Q}}\right]$ there exists a constant $c = c(\varepsilon) > 0$ such that

$$\sup_{K_r(z_0,\varepsilon)} u \leq c u(z_0)$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω and for every $z_0 \in \Omega$ and $r > 0$ such that $\overline{\Omega_{2r}(z_0)} \subset \Omega$ (here c_N is the constant defined in (1.18)*).*

Theorem 1.5.2 *Let $\mathscr{L} \in \mathbb{K}$ and let Ω be an open subset of $\mathbb{R}^{N+1}$. Let us fix an integer $m \geq 3$.*
We define:

$$K_r^{(m)}(z_0, \varepsilon) = \Omega_r^{(m)}(z_0) \cap \left\{(x,t) \in \mathbb{R}^{N+1} : t \leq t_0 - \varepsilon r^{\frac{2}{Q+m}}\right\}, \qquad \varepsilon > 0$$

Then for every $\varepsilon \in \left(0, c_N^{\frac{2}{Q}}\right]$ and $r_0 > 0$, there exists a constant $c = c(\varepsilon, r_0) > 0$ such that

$$\sup_{K_r^{(m)}(z_0,\varepsilon)} u \leq c u(z_0)$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω and for every $z_0 \in \Omega$, $r \in (0, r_0]$ such that $\overline{\Omega_{2r}(z_0)} \subset \Omega$.

We can easily obtain an Harnack inequality on the following "cylinders":

$$H_r(z_0) = z_0 \circ \left\{(D_0(r)x, r^2 t) : |x| \leq 1, -1 \leq t \leq 0\right\}$$

Theorem 1.5.3 *Let $\mathscr{L} \in \mathbb{K}$ and let Ω be an open subset of $\mathbb{R}^{N+1}$.*
We define:

$$H_r^-(z_0) = z_0 \circ \left\{(x,t) \in H_r(0) : t = -r^2\right\}$$

Then there exist three positive constant $\theta = \theta(\mathscr{L})$, $c = c(\mathscr{L})$ and $r_0 = r_0(\mathscr{L})$ such that

$$\sup_{H_{\theta r}^-(z_0)} u \leq c u(z_0)$$

for every non-negative solution u of $\mathscr{L} u = 0$ in Ω and for every $z_0 \in \Omega$ and $r \in (0, r_0]$ such that $H_r(z_0) \subset \Omega$.

This last theorem restores the analogy between results presented in Theorem 1.5.1, Theorem 1.5.2 and cylindric geometry of the classical parabolic Harnack inequality.

Chapter 2

Kolmogorov operators with variable coefficients

In this chapter we will refer to [**AP**] and [**P**].

We consider Kolmogorov operators in non-divergence form in $\mathbb{R}^{N+1}$:

$$\begin{aligned}\mathscr{L}u(x,t) &= \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial_{x_i x_j} u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) \\ \mathscr{L} &= \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial_{x_i x_j} + \langle Bx, \nabla_x \rangle - \partial_t\end{aligned} \tag{2.1}$$

with continuous coefficients $a_{ij}(x,t)$. As in the parabolic case, the classical theory for degenerate Kolmogorov operators is developed for spaces of Hölder continuous functions introduced in Definition 1.2.8. We remark that this definition relies on the Lie group $\mathbb{G}$ (1.11), that is an invariant structure for the constant coefficients operators. Even though the non-constant coefficients operators in (2.1) are not invariant with respect to $\mathbb{G}$, we will rely on the Lie group invariance of the model operator

$$\sum_{j=1}^{m_0} \partial^2_{x_j x_j} + \langle Bx, \nabla_x \rangle - \partial_t$$

associated to $\mathscr{L}$. Indeed, this is a standard procedure in the study of uniformly parabolic operators. We next list the standing assumptions of this chapter:

[H.1] *The matrix $B = (b_{ij})_{i,j=1,\dots,N}$ is a real constant matrix of the form 4. where all $\star$ blocks are zero matrices.*

[H.2] *The matrix $A(x,t) = (a_{ij}(x,t))_{i,j=1,\dots,N}$ is a symmetric matrix of the form 4. Moreover, it is positive definite in $\mathbb{R}^{m_0}$ and there exists a positive constant λ such that*

$$\frac{1}{\lambda}\sum_{i=1}^{m_0} |\xi_i|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \le \lambda \sum_{i=1}^{m_0} |\xi_i|^2 \tag{2.2}$$

for every $(\xi_1, \dots, \xi_{m_0}) \in \mathbb{R}^{m_0}$ and $(x,t) \in \mathbb{R}^{N+1}$.

[H.3] *There exists $\alpha \in (0,1]$ and $M > 0$ such that*

$$|a_{ij}(z) - a_{ij}(\zeta)| \le M d(z,\zeta)^\alpha$$

where $d(\cdot,\cdot)$ is the quasi-distance defined in Definition 1.2.7.

[H.4] *For every $i,j = 1,...,m_0$ there exist the derivatives $\partial_{x_i} a_{ij}(z)$ and they are bounded and Hölder continuous functions, with exponent α.*

Hypothesis H.1 and Hypothesis H.2 are related to hypoellipticity, in fact if all coefficients are constant then operators (2.1) belong to $\mathbb{K}$. In [**P**] is made the assumption of $\star$ blocks equal to zero, this hypothesis was later removed in [**DP**]. The statement of Hypothesis H.2 is linked to uniformly parabolic operators: if $m_0 = N$, the operator $\mathscr{L}$ is uniformly parabolic and $B = \mathbb{O}$. Hypothesis H.3 implies that $a_{ij}(z)$ are Hölder continuous functions. Hypothesis H.4 allow us to extend the results about fundamental solution to operators in divergence form:

$$\mathscr{L} = \operatorname{div}(A(z)\nabla_x) + \langle Bx, \nabla_x\rangle - \partial_t \tag{2.3}$$

We give now a definition of classical solution to the equation $\mathscr{L}u = f$ under minimal regularity assumption on u. We need first to give the definition of Lie differentiability with respect to the vector field Y defined in (1.6):

Definition 2.0.1 (Lie differentiable function) *A function u is Lie differentiable with respect to the vector field X at the point z if there exists and is finite the following limit:*

$$Xu(z) = \lim_{s\to 0} \frac{u(\gamma(s)) - u(\gamma(0))}{s}$$

where γ is the integral curve of X from z.

In particular, if Y is the vector field defined in (1.6), its integral curve is $\gamma(s) = (E(-s)x, t-s)$, with $z = (x,t)$. Clearly, if $u \in C^1(\Omega)$ where Ω is an open subset of $\mathbb{R}^{N+1}$, then $Yu(x,t)$ agrees with $\langle Bx, \nabla_x u(x,t)\rangle - \partial_t u(x,t)$ considered as a linear combination of the derivatives of u. We can now give a proper definition of classical solution to $\mathscr{L}u = 0$:

Definition 2.0.2 *Let $f \in C(\Omega)$. A function u is solution to the equation $\mathscr{L}u = f$ in a domain Ω of $\mathbb{R}^{N+1}$ if there exist the Euclidean derivatives ∂_{x_i}, $\partial_{x_i x_j} \in C(\Omega)$ for $i,j = 1,...,m_0$, the Lie derivative $Yu \in C(\Omega)$ and the equation*

$$\sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial_{x_i x_j} u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) = f(x,t)$$

is satisfied at any point $(x,t) \in \Omega$.

The natural functional setting for the study of classical solutions is the space:

$$C^{2,\alpha}(\Omega) = \left\{u \in C^\alpha(\Omega) : \partial_{x_i}, \partial_{x_i x_j}, Yu \in C^\alpha(\Omega), \quad \text{for } i,j = 1,...,m_0\right\}$$

This functional setting is related to the quasi-metric defined in (1.23). In fact, if we define the following norm for $u \in C^{2,\alpha}(\Omega)$:

$$|u|_{2+\alpha,\Omega} := |u|_{\alpha,\Omega} + \sum_{i=1}^{m_0} |\partial_{x_i} u|_{\alpha,\Omega} + \sum_{i,j=1}^{m_0} |\partial^2_{x_i x_j} u|_{\alpha,\Omega} + |Yu|_{\alpha,\Omega}$$

we can state fundamentals results in classical regularity theory: Schauder estimates. We state the following one, proved in [**DPo**].

Theorem 2.0.1 *Let us consider an operator* $\mathscr{L}$ *of the type* (2.1) *satisfying assumptions* ***(H1),(H2),(H3)*** *with* $\alpha < 1$. *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$, $f \in C^{\alpha}_{\mathrm{loc}}(\Omega)$ *and let* u *be a classical solution to* $\mathscr{L}u = f$ *in* Ω. *Then for every* $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *there exists a positive constant* C *such that*

$$|u|_{2+\alpha,\Omega'} \le C \left(\sup_{\Omega''} |u| + |f|_{\alpha,\Omega''} \right)$$

With a suitable adaptation of the Levi's method of parametrix, we shall construct the fundamental solution of operators (2.1) verifying the first three hypothesis, and the fundamental solution of operators (2.3) verifying all four hypothesis. In particular we will refer to [**P**].

2.1 Fundamental solution

In this section we will construct the fundamental solution via parametrix method, according to [**P**]. We define for every $\overline{z} \in \mathbb{R}^{N+1}$:

$$\mathscr{L}_{\overline{z}} = \sum_{i,j=1}^{m_0} a_{ij}(\overline{z})\partial^2_{x_i x_j} + \langle Bx, \nabla_x \rangle - \partial_t$$

We denote by $Z_{\overline{z}}(z;\zeta)$ the fundamental solution of $\mathscr{L}_{\overline{z}}$. To simplify the notation, when $\overline{z} = \zeta$ we shall write $Z_\zeta(z;\zeta) = Z(z;\zeta)$. According to Levi's method, we seek the fundamental solution Γ of $\mathscr{L}$ by using the parametrix $Z(z;\zeta)$. We put

$$\Gamma(z;\zeta) = Z(z;\zeta) + J(z;\zeta) \tag{2.4}$$

and we require that $\Gamma(\cdot;\zeta)$ is a solution of the differential equation (2.1), for $z \neq \zeta$:

$$0 = \mathscr{L}\Gamma(z;\zeta) = \mathscr{L}Z(z;\zeta) + \mathscr{L}J(z;\zeta) \tag{2.5}$$

Suppose that the function J can be written as

$$J(x,t;\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^N} Z(x,t;y,s)G(y,s;\xi,\tau)dyds \tag{2.6}$$

for some unknown function G.
Assuming $J(z;\zeta)$ can be differentiated under the integral sign, we obtain:

$$\mathscr{L}J(z;\zeta) = \int_\tau^t \int_{\mathbb{R}^N} \mathscr{L}Z(x,t;y,s)G(y,s;\xi,\tau)dyds - G(z;\zeta) \tag{2.7}$$

then condition (2.5) can be written as

$$G(z;\zeta) = \int_\tau^t \int_{\mathbb{R}^N} \mathscr{L}Z(x,t;y,s)G(y,s;\xi,\tau)dyds + (\mathscr{L}Z)(z;\zeta) \tag{2.8}$$

It then follows that the differential equation $\mathscr{L}\Gamma(z;\zeta) = 0$ is transformed into the integral equation (2.8), where the unknown function is G. This function can be determined by means of the successive approximation method, which yields:

$$G(z;\zeta) = \sum_{k=1}^{\infty} (\mathscr{L}Z)_k(z;\zeta) \tag{2.9}$$

where

$$(\mathscr{L}Z)_1(z;\zeta) = (\mathscr{L}Z)(z;\zeta)$$

$$(\mathscr{L}Z)_{k+1} = \int_\tau^t \int_{\mathbb{R}^N} (\mathscr{L}Z)(z;y,s)(\mathscr{L}Z)_k(y,s;\zeta)dyds$$

In order to give some results about G (and then about the fundamental solution), we give some estimates, whose demonstration can be found in [**P**]. We only remark that they are based on **(H.1)**,**(H.2)**,**(H.3)**, and **(H.4)** for the divergence form operators.

Proposition 2.1.1 *Fix $\varepsilon > 0$, put $\widetilde{A}_0 = (\lambda + \varepsilon)\mathcal{I}_{m_0}$, where λ is the constant appearing in **(H.2)**. Let denote by $\widetilde{\Gamma}$ the fundamental solution corresponding to*

$$\widetilde{\mathscr{L}} = \operatorname{div}(\widetilde{A}_0 \nabla_x) + \langle Bx, \nabla_x \rangle - \partial_t \tag{2.10}$$

Then there exists a constant $\tilde{c} > 0$ such that, for every $i, j = 1, ..., m_0$ and for every $z, \zeta \in \mathbb{R}^{N+1}$, we have

$$|Z(z;\zeta)| \leq \tilde{c}\widetilde{\Gamma}(z;\zeta) \tag{2.11}$$

$$|\partial_{x_i} Z(z;\zeta)| \leq \frac{\tilde{c}}{\sqrt{t-\tau}}\widetilde{\Gamma}(z;\zeta)$$

$$|\partial^2_{x_i x_j} Z(z;\zeta)| \leq \frac{\tilde{c}}{(t-\tau)}\widetilde{\Gamma}(z;\zeta)$$

Lemma 2.1.1 *There exists an operator $\widetilde{\mathscr{L}}$ and a constant $\tilde{c} > 0$ such that, if $\widetilde{\Gamma}$ is the fundamental solution of $\widetilde{\mathscr{L}}$, then*

$$|\mathscr{L} Z(z;\zeta)| \leq \frac{\tilde{c}}{(t-\tau)^{1-\frac{\alpha}{2}}}\widetilde{\Gamma}(z;\zeta), \qquad \forall z \neq \zeta$$

*where α is the Hölder continuity constant defined in **(H.3)**.*

Under **(H.4)** we have the analogous for divergence form operators.

Lemma 2.1.2 *Let M be the divergence form operator defined in* (2.3)*, verifying Hypothesis **(H.1)-(H.4)**. Then there exists an operator $\widetilde{\mathscr{L}}$ such that, for every bounded interval $I \subset \mathbb{R}$, there exists a constant $\tilde{c}_I > 0$ such that, if $\widetilde{\Gamma}$ denotes the fundamental solution of $\widetilde{\mathscr{L}}$, then*

$$|MZ(z;\zeta)| \leq \frac{\tilde{c}_I}{(t-\tau)^{1-\frac{\alpha}{2}}}\widetilde{\Gamma}(z;\zeta)$$

Corollary 2.1.1 *For every $k \in \mathbb{N}$ we have*

$$|(\mathscr{L} Z)_k(z;\zeta)| \leq \frac{c_k}{(t-\tau)^{1-\frac{\alpha k}{2}}}\widetilde{\Gamma}(z\ \zeta), \qquad \forall z, \zeta \in \mathbb{R}^{N+1} \tag{2.12}$$

where

$$c_k = \tilde{c}^k \frac{\Gamma_{eul}(\frac{\alpha}{2})^k}{\Gamma_{eul}(\frac{k\alpha}{2})} \tag{2.13}$$

Here $\tilde{c}$ and $\widetilde{\Gamma}$ denote the constant and the function of Lemma 2.1.1, and Γ_{eul} denotes the Euler's Gamma function.

We can now state this proposition, that gives us all information we need about G.

Proposition 2.1.2 *There exists $k_0 \in \mathbb{N}$ such that, for every interval $I \subset \mathbb{R}$:*

1. *$(\mathscr{L} Z)_k$ is a bounded function in $S_I := \mathbb{R}^N \times I$, for every $k \geq k_0$;*
2. *the series*

$$\sum_{k=k_0}^{\infty} (\mathscr{L} Z)_k(z;\zeta)$$

   converges uniformly on S_I;

3. *the function G defined in* (2.9) *satisfies the integral equation* (2.8) *for every* $\zeta \in \mathbb{R}^{N+1}$ *and for every* $z \neq \zeta$.

Proof.

1. follows from Corollary 2.1.1 and from the explicit expression of $\widetilde{\Gamma}(x,t;y,s)$, for $k_0 \in \mathbb{N}$ and $k_0 \geq \frac{Q+2}{\alpha}$;

2. follows from (2.12), noting that the power series

$$\sum_{j=1}^{\infty} c_{k_0+j} t^j$$

where c_k is defined in (2.13), has radius of convergence equal to infinity;

3. using (2.12), (2.13) and the *reproduction property* of the function $\widetilde{\Gamma}$:

$$\int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\widetilde{\Gamma}(y,s;\zeta)dy = \widetilde{\Gamma}(z;\zeta)$$

we obtain, for every $z \neq \zeta$:

$$\int_\tau^t \int_{\mathbb{R}^N} (\mathscr{L}Z)(z;y,s)G(y,s;\zeta)dyds = \sum_{k=1}^{\infty} \int_\tau^t \int_{\mathbb{R}^N} (\mathscr{L}Z)(z;y,s)(\mathscr{L}Z)_k(y,s;\zeta)dyds =$$
$$= \sum_{k=1}^{\infty} (\mathscr{L}Z)_{k+1}(z;\zeta) = G(z;\zeta) - (\mathscr{L}Z)(z;\zeta)$$

□

Result (3) ensures that the function Γ defined in (2.4) is a solution of $\mathscr{L}\Gamma(z;\zeta) = 0$, for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $z \neq \zeta$, once we prove we can differentiate J under the integral sign.
As a consequence of Lemma 2.1.1 we have these results:

Corollary 2.1.2 *For every bounded interval $I \subset \mathbb{R}$ there exists a constant $\tilde{k}_I > 0$ such that*

$$|G(z;\zeta)| \leq \frac{\tilde{k}_I}{(t-\tau)^{1-\frac{\alpha}{2}}} \widetilde{\Gamma}(z;\zeta) \qquad \forall z, \zeta \in S_I,\ z \neq \zeta$$

Corollary 2.1.3 *For every bounded interval $I \subset \mathbb{R}$ there exists a constant $c_I > 0$ such that*

$$\Gamma(z;\zeta) \leq c_I \widetilde{\Gamma}(z;\zeta) \qquad \forall z, \zeta \in S_I,\ z \neq \zeta$$

We can now prove (2.7).

Proposition 2.1.3 *Let $J(z;\zeta)$ be the function defined in* (2.6). *Then, for $i, j = 1, ..., m_0$ the functions $\partial_{x_i} J$, $\partial^2_{x_i x_j} J$ and the Lie derivative YJ exist and are continuous. Moreover for every $z, \zeta \in \mathbb{R}^{N+1}$ such that $z \neq \zeta$ we have:*

$$\mathscr{L}J(z;\zeta) = \int_\tau^t \int_{\mathbb{R}^N} \mathscr{L}Z(x,t;y,s)G(y,s;\xi,\tau)dyds - G(z;\zeta)$$

Proof.
We start showing that, for $i=1,...,m_0$, $\partial_{x_i}J(z;\zeta)$ is continuous and

$$\partial_{x_i}J(z;\zeta)=\int_\tau^t\int_{\mathbb{R}^N}\partial_{x_i}Z(z;y,s)G(y,s;\zeta)dyds \tag{2.14}$$

We first note this integral converges since, by Proposition 2.1.1 and Corollary 2.1.2 there exists a positive constant $\tilde{c}$ such that

$$\int_\tau^t\int_{\mathbb{R}^N}|\partial_{x_i}Z(z;y,s)G(y,s;\zeta)|dyds\le\int_\tau^t\int_{\mathbb{R}^N}\widetilde{\Gamma}(z;y,s)\widetilde{\Gamma}(y,s;\zeta)(t-s)^{-\frac{1}{2}}(s-\tau)^{-1+\frac{\alpha}{2}}dyds=$$
$$=\widetilde{\Gamma}(z;\zeta)\int_\tau^t(t-s)^{-\frac{1}{2}}(s-\tau)^{-1+\frac{\alpha}{2}}ds<\infty$$

Let φ be a function of class $C^2(\mathbb{R})$, such that $0\le\varphi(t)\le1$ for every $t\ge0$, $\varphi(t)=1$ for every $t\le\frac{1}{2}$, and $\varphi(t)=0$ for every $t\ge1$. For any fixed $\varepsilon>0$, put

$$\eta_\varepsilon(z;w)=1-\varphi\left(\left\|D\left(\frac{1}{\varepsilon}\right)(w^{-1}\circ z)\right\|\right)$$

and

$$J_\varepsilon(z;\zeta)=\int_{\mathbb{R}^N\times(\tau,t)}Z(z;w)\eta_\varepsilon(z;w)G(w;\zeta)dw$$

We first show that for every $z=(x,t)$, $\zeta=(\xi,\tau)\in\mathbb{R}^{N+1}$, with $t>\tau$, it holds

$$\partial_{x_i}J_\varepsilon(z;\zeta)=\int_{\mathbb{R}^N\times(\tau,t)}\partial_{x_i}(Z\eta_\varepsilon)(z;w)G(w;\zeta)dw \tag{2.15}$$

Note that, for every $\varepsilon>0$, functions η_ε, $\partial_{x_i}\eta_\varepsilon$ are bounded. From Proposition 2.1.1 it follows that there exists $c_\varepsilon>0$ such that

$$|\partial_{x_i}(Z\eta_\varepsilon)(z;w)|\le\frac{c_\varepsilon}{\sqrt{t-s}}\widetilde{\Gamma}(z;w)$$

On the other hand, if we set

$$B(z,\rho)=\left\{\zeta\in\mathbb{R}^{N+1}:\|\zeta^{-1}\circ z\|<\rho\right\}$$

then $(Z\eta_\varepsilon)(z;w)=0$ for every $w\in B(z,\frac{\varepsilon}{2})$, from which

$$|\partial_{x_i}(Z\eta_\varepsilon)(z;w)|\le c_\varepsilon\sup_{w\in\mathbb{R}^{N+1}\setminus B(z,\frac{\varepsilon}{2})}\frac{\widetilde{\Gamma}(z;w)}{\sqrt{t-s}}:=\tilde{c}_\varepsilon<\infty$$

Hence, from Corollary 2.1.2, the bound

$$|\partial_{x_i}(Z\eta_\varepsilon)(z;w)G(w;\zeta)|\le\frac{\tilde{c}_\varepsilon\tilde{k}_I}{(s-t)^{1-\frac{\alpha}{2}}}\widetilde{\Gamma}(w;\zeta)$$

holds uniformly with respect to the z variable, where the constant $\tilde{k}_I$ in Corollary 2.1.2 corresponds to the interval $I=(\tau,t)$. Since this upper bound is absolutely integrable, (2.15) follows by Lebesgue's Theorem, for any fixed $\varepsilon>0$.
Fix $\zeta=(\xi,\tau)\in\mathbb{R}^{N+1}$, and for any $T_0,T_1\in\mathbb{R},\tau<T_0<T_1$, let $S_I=\mathbb{R}^N\times(T_0,T_1)$.
We claim that:

$$J_\varepsilon(z;\zeta)\xrightarrow[\epsilon\to0]{}J(z;\zeta)\qquad\forall z\in S_I \tag{2.16}$$

and, for every $i = 1, ..., m_0$, $z \in S_I$:

$$\partial_{x_i} J_\varepsilon(z;\zeta) \underset{\varepsilon\to 0}{\Longrightarrow} \int_{\mathbb{R}^N\times(\tau,t)} \partial_{x_i} Z(z;w)G(w;\zeta)dw \tag{2.17}$$

(where $\underset{\varepsilon\to 0}{\Longrightarrow}$ denotes the uniform convergence). If (2.16) and (2.17) hold, function $J(\cdot;\zeta)$ has derivatives $\partial_{x_i} J(\cdot;\zeta)$ on S_I. Moreover these derivatives are continuous functions and, for every $z \in S_I$:

$$\partial_{x_i} J(z;\zeta) = \int_{\mathbb{R}^N\times(\tau,t)} \partial_{x_i} Z(z;w)G(w;\zeta)dw$$

The arbitrariness on the choice of $I = (T_0, T_1)$, with $\tau < T_0 < T_1$, yields the result we wanted to show.
We are thus left with the proof of (2.16) and (2.17).
Without loss of generality, we can suppose $0 < \varepsilon \le \varepsilon_0$, with $\varepsilon_0 = \sqrt{\frac{T_0-\tau}{2}}$. Then

$$J_\varepsilon(z;\zeta) - J(z;\zeta) = \int_{\mathbb{R}^N\times(\tau,t)} Z(z;w)[\eta_\varepsilon(z;w) - 1]G(w;\zeta)dw$$

From the definition of the function η_ε, we have

$$\begin{aligned} &|\eta_\varepsilon(z;w) - 1| \le 1 && \forall w \in \mathbb{R}^{N+1} \\ &\eta_\varepsilon(z;w) - 1 = 0 && \forall w \in \mathbb{R}^{N+1} \setminus B(z,\varepsilon) \end{aligned} \tag{2.18}$$

From (2.11), (2.18) and Corollary 2.1.2 it follows

$$|J_\varepsilon(z;\zeta) - J(z;\zeta)| \le \int_{\substack{B(z,\varepsilon)\\ s<t}} \tilde{c}\tilde{k}_I \widetilde{\Gamma}(z;w) \frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}} dw$$

Note that, since $z \in S_I$, $w \in B(z,\varepsilon)$ and $\varepsilon \le \varepsilon_0 = \sqrt{\frac{T_0-\tau}{2}}$, then $s - \tau \ge \varepsilon_0^2$, and

$$\frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}} \le \tilde{c}_N \varepsilon_0^{\alpha-Q-2} \tag{2.19}$$

Hence there exists a constant $c_S > 0$, depending only on the set S_I, such that

$$|J_\varepsilon(z;\zeta) - J(z;\zeta)| \le c_S \int_{\substack{B(z,\varepsilon)\\ s<t}} \widetilde{\Gamma}(z;w)dw \tag{2.20}$$

Recalling that $\widetilde{\Gamma}(z;w) = \widetilde{\Gamma}(w^{-1}\circ z;0)$ and that

$$\widetilde{\Gamma}(D(r)z;0) = r^{-Q}\widetilde{\Gamma}(z;0); \qquad \det(D(r)) = r^{Q+2}$$

Hence, by setting $w' = D\left(\frac{1}{\varepsilon}\right)\left(w^{-1}\circ z\right)$ in (2.20), we obtain

$$|J_\varepsilon(z;\zeta) - J(z;\zeta)| \le \varepsilon^2 c_S \int_{\substack{B(0,1)\\ s<0}} \widetilde{\Gamma}(w';0)dw' = \varepsilon^2 k_S$$

from which (2.16) follows.
In order to prove (2.17) we again assume $0 < \varepsilon \le \varepsilon_0 = \sqrt{\frac{T_0 - \tau}{2}}$. From (2.15)

$$\begin{aligned} &\partial_{x_i} J_\varepsilon(z;\zeta) - \int_{\mathbb{R}^N \times (\tau,t)} \partial_{x_i} Z(z;w) G(w;\zeta) dw = \\ &= \int_{\mathbb{R}^N \times (\tau,t)} \partial_{x_i} Z(z;w)[\eta_\varepsilon(z;w) - 1] G(w;\zeta) dw + \\ &+ \int_{\mathbb{R}^N \times (\tau,t)} Z(z;w) \partial_{x_i} \eta_\varepsilon(z;w) G(w;\zeta) dw = I_\varepsilon(z;\zeta) + II_\varepsilon(z;\zeta) \end{aligned} \tag{2.21}$$

We next evaluate the quantity I_ε and II_ε. Using (2.18), Proposition 2.1.1 and Corollary 2.1.2 we get:

$$|I_\varepsilon(z;\zeta)| \le \int_{\substack{B(z;\varepsilon)\\ s<t}} \frac{\tilde{c}\tilde{k}_I}{\sqrt{t-s}} \widetilde{\Gamma}(z;w) \frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}} dw \le c'_S \int_{\substack{B(z,\varepsilon)\\ s<t}} \frac{\widetilde{\Gamma}(z;w)}{\sqrt{t-s}} dw$$

Last inequality follows from (2.19). Substituting again $w' = D\left(\frac{1}{\varepsilon}\right)(w^{-1} \circ z)$ we finally obtain

$$|I_\varepsilon(z;\zeta)| \le \varepsilon c_S \int_{\substack{B(0,1)\\ s<0}} \frac{\widetilde{\Gamma}(w';0)}{\sqrt{-s}} dw' = \varepsilon k'_S \tag{2.22}$$

To evaluate $II_\varepsilon(z;\zeta)$, we note at first that

$$\partial_{x_i} \eta_\varepsilon(z;w) = \frac{1}{\varepsilon} \varphi' \left(\left\| D\left(\frac{1}{\varepsilon}\right) \left(w^{-1} \circ z\right) \right\| \right)$$

Then, letting $m = \sup_{\mathbb{R}} \varphi'$,

$$\begin{aligned} |\partial_{x_i} \eta_\varepsilon(z;w)| &\le \frac{m}{\varepsilon} & &\forall w \in \mathbb{R}^{N+1} \\ \partial_{x_i} \eta_\varepsilon(z;w) &= 0 & &\forall w \in \mathbb{R}^{N+1} \setminus B(z,\varepsilon) \end{aligned}$$

By using these relations II_ε can be treated as I_ε in the previous case. We have:

$$\begin{aligned} |II_\varepsilon(z;\zeta)| &\le \frac{m\tilde{c}\tilde{k}_I}{\varepsilon} \int_{\substack{B(z,\varepsilon)\\ s<t}} \widetilde{\Gamma}(z;w) \frac{\widetilde{\Gamma}(w;\zeta)}{(s-\tau)^{1-\frac{\alpha}{2}}} dw \le \\ &\le \frac{mc'_S}{\varepsilon} \int_{\substack{B(z,\varepsilon)\\ s<t}} \widetilde{\Gamma}(z;w) dw \le mc'_S \varepsilon \int_{\substack{B(0,1)\\ s<0}} \widetilde{\Gamma}(w';0) dw' = mk_S \varepsilon \end{aligned} \tag{2.23}$$

Then (2.17) follows combining (2.21), (2.22) and (2.23).

To show analogous relations for $\partial^2_{x_i x_j}$ and the Lie derivative with respect to Y, we need these two estimates (whose proof can be found in [**P**]):

1. For every bounded interval I there exist three constant $c > 0, \gamma, \gamma' \in (0,1)$ such that

$$|G(x,t;\zeta) - G(x',t;\zeta)| \le c \frac{|x - x'|_B^{\gamma'}}{(t-\tau)^{1-\frac{\gamma}{2}}} \left[\widetilde{\Gamma}(x,t;\zeta) + \widetilde{\Gamma}(x',t;\zeta) \right] \tag{2.24}$$

   for every $\zeta = (\xi,\tau) \in \mathbb{R}^{N+1}$, for every $t, \tau \in I, t > \tau$, and for every $x, x' \in \mathbb{R}^N$.
   Here $|x - x'|_B = \sum_{j=1}^N |x_j - x'_j|^{\frac{1}{q_j}}$ represents the part related to x in (1.22).

2. There exists a constant $k > 0$ such that

$$|\partial^2_{x_i x_j} Z_\zeta(z;w) - \partial^2_{x_i x_j} Z_{\zeta'}(z;w)| \leq \frac{k}{t-s} \|\zeta^{-1} \circ \zeta'\|^\alpha \widetilde{\Gamma}(z;w) \tag{2.25}$$

for every $z, w, \zeta, \zeta' \in \mathbb{R}^{N+1}$ and for every $i, j = 1, ..., m_0$.

We can now prove that, for every fixed $\zeta \in \mathbb{R}^{N+1}$ and for every $i, j = 1, ..., m_0$, the derivatives $\partial^2_{x_i x_j} J(\cdot;\zeta)$ exist and are continuous functions. Moreover:

$$\partial^2_{x_i x_j} J(z;\zeta) = \int_\tau^t \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z(z;y,s) G(y,s;\zeta) dyds \tag{2.26}$$

We observe we cannot assert that this integrating function is absolutely integrable on $\mathbb{R}^N \times (\tau, t)$. However, we can consider the integral in (2.26) as a "repeated integral". Indeed, since the function G is absolutely integrable, it is sufficient to prove that, for every fixed $t_0 \in (\tau, t)$, the integral

$$\int_{t_0}^t \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z(z;y,s) G(y,s;\zeta) dyds$$

converges. For every fixed $s \in (\tau, t)$ and for every $y' \in \mathbb{R}^N$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z(z;y,s) G(y,s;\zeta) dyds = \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z(z;y,s) \left[G(y,s;\zeta) - G(y',s;\zeta)\right] dyds + \\
&+ G(y',s;\zeta) \int_{\mathbb{R}^N} \partial^2_{x_i x_j} \left[Z_{(y,s)} - Z_{(y',s)}\right](z;y,s)dy + G(y',s;\zeta) \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z_{(y',s)}(z;y,s)dy = \\
&= I'(z;\zeta;y',s) + II''(z;\zeta;y',s) + III'''(z;\zeta;y',s)
\end{aligned} \tag{2.27}$$

We next give some estimates for the three addend above. Let $y' = E(s-t)x$. From Proposition 2.1.1 and from (2.24) we have

$$|I'(z;\zeta;y',s)| \leq \int_{\mathbb{R}^N} \frac{\tilde{c}}{t-s} \widetilde{\Gamma}(z;y,s) c \frac{|y-y'|_B^{\gamma'}}{(s-\tau)^{1-\frac{\gamma}{2}}} \left[\widetilde{\Gamma}(y,s;\zeta) + \widetilde{\Gamma}(y',s;\zeta)\right] dy$$

If $s \in (t_0, t)$, the explicit expression of $\widetilde{\Gamma}$ allows one to derive the existence of $k = k(t_0) > 0$ such that

$$\frac{\widetilde{\Gamma}(y,s;\zeta)}{(s-\tau)^{t-\frac{\gamma}{2}}} \leq k \qquad \forall (y,s) \in \mathbb{R}^{N+1} \ : \ s \geq t_0$$

Moreover, using (1.21) we can prove that there exists a constant $M = M(T_0, T_1) > 0$ such that

$$|y-y'|_B^{\gamma'} = |y - E(s-t)x|_B^{\gamma'} \leq M(t-s)^{\frac{\gamma'}{2}} \left| D\left((t-s)^{-\frac{1}{2}}\right)(x - E(t-s)y)\right|_B$$

Therefore, from the last three inequalities it follows that there exists a constant $c'(t_0) > 0$ such that

$$|I'(z;\zeta;E(s-t)x,s)| \leq \frac{c'(t_0)}{(t-s)^{1-\frac{\gamma'}{2}}} \int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)dy = \frac{c'(t_0)}{(t-s)^{1-\frac{\gamma'}{2}}} \tag{2.28}$$

Arguing as above, using (2.25) we can show that there exists a constant $c''(t_0) > 0$ such that

$$|I''(z;\zeta;E(s-t)x,s)| \leq \frac{c''(t_0)}{(t-s)^{1-\frac{\alpha}{2}}} \int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)dy = \frac{c''(t_0)}{(t-s)^{1-\frac{\alpha}{2}}} \tag{2.29}$$

To evaluate I''', note that, for every $\overline{z} \in \mathbb{R}^{N+1}$ we have

$$\int_{\mathbb{R}^N} Z_{\overline{z}}(z; y, s) dy = 1$$

Then, for every $i, j = 1, ..., m_0$, we have

$$\partial^2_{x_i x_j} \int_{\mathbb{R}^N} Z_{\overline{z}}(z; y, s) dy = 0$$

On the other hand, using the Lebesgue's Theorem, for every $s \in (\tau, t)$ it holds

$$\partial^2_{x_i x_j} \int_{\mathbb{R}^N} Z_{\overline{z}}(z; y, s) dy = \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z_{\overline{z}}(z; y, s) dy$$

Setting $\overline{z} = (E(t-s)x, s)$, we obtain $I''' = 0$. This proves the existence of integral (2.26).

Now we put our focus in taking the second derivative inside the integral. Arguing exactly as in the first part of the proof, it can be shown that

$$\partial^2_{x_i x_j} J_\varepsilon(z; \zeta) = \int_{\mathbb{R}^N \times (\tau, t)} \partial^2_{x_i x_j} (Z\eta_\varepsilon)(z; w) G(w; \zeta) dw \tag{2.30}$$

For every fixed $\zeta = (\xi, \tau)$, for every $T_0, T_1 \in \mathbb{R}$, with $\tau < T_0 < T_1$, let $S_I = \mathbb{R}^N \times (T_0, T_1)$. If we prove that, for every $z \in S_I$ it holds

$$\partial^2_{x_i x_j} J_\varepsilon(z; \zeta) \underset{\varepsilon \to 0}{\rightrightarrows} \int_\tau^t \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z(z; y, s) G(y, s; \zeta) dy ds \tag{2.31}$$

then from (2.16) and (2.17) it follows (2.26).
Without loss of generality, we can suppose that $0 < \varepsilon \leq \varepsilon_0 = \sqrt{\frac{T_0 - \tau}{2}}$. From (2.30) we have

$$\begin{aligned} &\partial^2_{x_i x_j} J_\varepsilon(z; \zeta) - \int_\tau^t \int_{\mathbb{R}^N} \partial^2_{x_i x_j} Z(z; y, s) G(y, s; \zeta) dy ds = \\ &= \int_\tau^t \int_{\mathbb{R}^N} \partial^2_{x_i x_j} [(\eta_\varepsilon - 1) Z](z; y, s) G(y, s; \zeta) dy ds = \\ &= \int_\tau^t \left(\widetilde{I}'_\varepsilon(z; \zeta; y', s) + \widetilde{I}''_\varepsilon(z; \zeta; y', s) + \widetilde{I}'''_\varepsilon(z; \zeta; y', s) \right) ds \end{aligned} \tag{2.32}$$

Functions $\widetilde{I}'_\varepsilon, \widetilde{I}''_\varepsilon$ and $\widetilde{I}'''_\varepsilon$ are obtained as in (2.27) with Z replaced by $(\eta_\varepsilon - 1)Z$.
In order to prove (2.31) it is sufficient to find bounds independent of ε, for $\widetilde{I}'_\varepsilon$ and $\widetilde{I}''_\varepsilon$, as in (2.28) and (2.29). This requirement can be met by showing that

$$|\partial^2_{x_i x_j} [(\eta_\varepsilon - 1) Z](x, t; y, s)| \leq \frac{k(\varepsilon_0)}{t - s} \widetilde{\Gamma}(x, t; y, s) \tag{2.33}$$

for every $x, y \in \mathbb{R}^N, s \in (\tau, t), \varepsilon \in (0, \varepsilon_0)$. As mentioned above

$$|\partial_{x_i} \eta_\varepsilon(z; w)| \leq \frac{m}{\varepsilon} \qquad \forall w \in \mathbb{R}^{N+1}$$

$$\partial_{x_i} \eta_\varepsilon(z; w) = 0 \qquad \forall w \in B\left(z, \frac{\varepsilon}{2}\right)$$

Moreover

$$\|w^{-1} \circ z\| = \sum_{j=1}^N |(x - E(t-s)y)_j|^{\frac{1}{q_j}} + \sqrt{t-s} \leq \frac{\varepsilon}{2} \Longrightarrow \frac{1}{\varepsilon} \leq \frac{1}{2\sqrt{t-s}}$$

Then

$$|\partial_{x_i}\eta_\varepsilon(z;w)| \leq \frac{m}{2\sqrt{t-s}} \ ; \quad \forall w,z\in\mathbb{R}^{N+1}$$

Analogously

$$|\partial^2_{x_ix_j}\eta_\varepsilon(z;w)| \leq \frac{m'}{4(t-s)} \ ; \quad \forall w,z\in\mathbb{R}^{N+1}$$

Using Proposition 2.1.1 the estimate (2.33) follows. Since the integrating function on the right hand side of (2.32) is zero for every $(y,s)\in\mathbb{R}^{N+1}$ satisfying $s\geq t-\varepsilon^2$, estimates analogous to (2.28) and (2.29) guarantee that there exist two constant $\beta\in(0,1)$ and $c(\varepsilon_0)>0$ such that

$$\begin{aligned}
&\left|\partial^2_{x_ix_j}J_\varepsilon(z;\zeta)-\int_\tau^t\int_{\mathbb{R}^N}\partial^2_{x_ix_j}Z(z;y,s)G(y,s;\zeta)dyds\right|\leq\\
&\leq\int_{t-\varepsilon^2}^t\left|\int_{\mathbb{R}^N}\partial^2_{x_ix_j}[(\eta_\varepsilon-1)Z](z;y,s)G(y,s;\zeta)dy\right|ds\leq\\
&\leq\int_{t-\varepsilon^2}^t\frac{c(\varepsilon_0)}{(t-s)^{1-\beta}}ds=c'(\varepsilon_0)\varepsilon^\beta
\end{aligned}$$

This proves (2.31), and thus (2.26).

We now work on the Lie derivative, showing that for every $\zeta\in\mathbb{R}^{N+1}$ the derivative $YJ(z;\zeta)=\langle Bx,\nabla_xJ(z;\zeta)\rangle-\partial_tJ(z;\zeta)$ exists and it is a continuous function with respect to z. Moreover

$$YJ(z;\zeta)=\int_\tau^t\int_{\mathbb{R}^N}YZ(z;y,s)G(y,s;\zeta)dyds-G(z;\zeta) \tag{2.34}$$

Here, for every $\varepsilon>0$, we set

$$J_\varepsilon(z;\zeta)=\int_\tau^{t-\varepsilon}\int_{\mathbb{R}^N}Z(z;y,s)G(y,s;\zeta)dyds$$

Similarly to (2.16), it can be shown that, for $z,\zeta\in\mathbb{R}^{N+1},\zeta\neq z$, we have

$$J_\varepsilon(z;\zeta)\xrightarrow[\varepsilon\to0]{}J(z;\zeta)$$

Let $z=(x,t)\in\mathbb{R}^{N+1},\delta>0$ and define the path $\gamma:(-\delta,\delta)\to\mathbb{R}^{N+1}$:

$$\gamma(s)=(x(s),t(s))=(E(s)x,t+s) \tag{2.35}$$

From the definition of $E(s)$, it follows

$$\gamma(0)=(x,t),\qquad\dot\gamma(s)=(-Bx(s),1)$$

Let $\zeta=(\xi,\tau)\in\mathbb{R}^{N+1}$ be given, with $\tau<t$ and set $\varepsilon>0$ such that $\varepsilon\leq\varepsilon_0=\frac{t-\tau}{2}$.
We next show that

$$YJ_\varepsilon(z:\zeta)=\int_\tau^{t-\varepsilon}\int_{\mathbb{R}^N}YZ(z;y,s)G(y,s;\zeta)dyds-\int_{\mathbb{R}^N}Z(z;y,t-\varepsilon)G(y,t-\varepsilon;\zeta)dy \tag{2.36}$$

Consider the path defined in (2.35) with $\delta=\frac{\varepsilon}{2}$. We have

$$\begin{aligned}
\frac{J_\varepsilon(\gamma(\sigma);\zeta)-J_\varepsilon(\gamma(0);\zeta)}{\sigma}&=\int_\tau^{t-\varepsilon}\int_{\mathbb{R}^N}\frac{Z(\gamma(\sigma);y,s)-Z(\gamma(0);y,s)}{\sigma}G(y,s;\zeta)dyds+\\
&+\frac{1}{\sigma}\int_{t-\varepsilon}^{t+\sigma-\varepsilon}\int_{\mathbb{R}^N}Z(\gamma(\sigma);y,s)G(y,s;\zeta)dyds
\end{aligned} \tag{2.37}$$

Being $Z(z;y,s)$ the fundamental solution of $L_{(y,s)}$, it follows from (2.35) that there exists $\sigma^\star \in (-|\sigma|, |\sigma|)$ satisfying

$$\frac{Z(\gamma(\sigma);y,s) - Z(\gamma(0);y,s)}{\sigma} = \frac{d}{d\sigma} Z(\gamma(\sigma);y,s)_{|\sigma=\sigma^\star} =$$
$$= -YZ(\gamma(\sigma^\star);y,s) = \sum_{i,j=1}^{m_0} a_{ij}(y,s)\partial^2_{x_ix_j} Z(\gamma(\sigma^\star);y,s)$$

Using the fact that a_{ij} are bounded functions and that $t+\sigma^\star > t - \frac{\varepsilon}{2}$, Proposition 2.1.1 gives that the function $\sum_{i,j=1}^{m_0} a_{ij}(y,s)\partial^2_{x_ix_j} Z(\gamma(\sigma^\star);y,s)$ is bounded on $\mathbb{R}^N \times (\tau, t-\varepsilon)$.
The summability of G yields

$$\begin{aligned}&\lim_{\sigma\to 0} \int_\tau^{t-\varepsilon} \int_{\mathbb{R}^N} \frac{Z(\gamma(\sigma);y,s) - Z(\gamma(0);y,s)}{\sigma} G(y,s;\zeta)dyds = \\ &= -\int_\tau^{t-\varepsilon} \int_{\mathbb{R}^N} YZ(z;y,s)G(y,s;\zeta)dyds\end{aligned} \tag{2.38}$$

Hence

$$\begin{aligned}&\int_{\mathbb{R}^N} Z(z;y,t-\varepsilon)G(y,t-\varepsilon;\zeta)dy - \frac{1}{\sigma}\int_{t-\varepsilon}^{t+\sigma-\varepsilon} \int_{\mathbb{R}^N} Z(\gamma(\sigma);y,s)G(y,s;\zeta)dyds = \\ &\left(\text{setting } r = \frac{s-t+\varepsilon}{\sigma}\right) \\ &= \int_0^1 \int_{\mathbb{R}^N} [Z(z;y,t-\varepsilon) - Z(\gamma(\sigma);y,t-\varepsilon+r\sigma)]G(y,t-\varepsilon;\zeta)dydr + \\ &+ \int_0^1 \int_{\mathbb{R}^N} Z(\gamma(\sigma);y,t-\varepsilon+r\sigma)[G(y,t-\varepsilon;\zeta) - G(y,t-\varepsilon+r\sigma;\zeta)]dydr = \\ &= \hat{I}'_\sigma(z;\zeta) + \hat{I}''_\sigma(z;\zeta)\end{aligned} \tag{2.39}$$

Since $\delta = \frac{\varepsilon}{2}$, it follows that $Z(z;y,t-\varepsilon) - Z(\gamma(\sigma);y,t-\varepsilon+r\sigma)$ is a bounded function, while $(y,r) \to G(y,t-\varepsilon;\zeta)$ is an absolutely integrable function on $\mathbb{R}^N \times (0,1)$. Hence, from (2.35) and the Lebesgue's Theorem,

$$\lim_{\sigma\to 0} \hat{I}'_\sigma(z;\zeta) = 0 \tag{2.40}$$

Applying on $\hat{I}''_\sigma(z;\zeta)$ the change of variable

$$\eta = D\left(\frac{1}{\sqrt{\varepsilon + (1-r)\sigma}}\right)(E(\sigma)x - E(\varepsilon + (1-r)\sigma)y)$$

with an obvious meaning of notations we have

$$|\hat{I}''_\sigma(z;\zeta)| \le \tilde{c} \int_0^1 \int_{\mathbb{R}^N} \exp\left(\langle \widetilde{C}^{-1}(1)\eta, \eta\rangle\right) |G(y(\eta), t-\varepsilon;\zeta) - G(y(\eta), t-\varepsilon+r\sigma;\zeta)| d\eta dr$$

Since $G(y(\eta),t-\varepsilon;\zeta) - G(y(\eta),t-\varepsilon+r\sigma;\zeta)$ is a bounded function, from Lebesgue's Theorem it holds

$$\lim_{\sigma\to 0} \hat{I}''_\sigma(z;\zeta) = 0 \tag{2.41}$$

Then, from (2.39), (2.40) and (2.41) it follows

$$\lim_{\sigma\to 0} \frac{1}{\sigma} \int_{t-\varepsilon}^{t+\sigma-\varepsilon} \int_{\mathbb{R}^N} Z(\gamma(\sigma);y,s)G(y,s;\zeta)dyds = \int_{\mathbb{R}^N} Z(z;y,t-\varepsilon)G(y,t-\varepsilon;\zeta)dy \tag{2.42}$$

Hence (2.37), (2.38) and (2.42) yield (2.36).
Fix now $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$, and, for every $T_0, T_1 \in \mathbb{R}$ with $\tau < T_0 < T_1$, put $S_I = \mathbb{R}^N \times (T_0, T_1)$. Let us proves that, for every $z \in S_I$

$$YJ_\varepsilon(z;\zeta) \underset{\varepsilon\to 0}{\rightrightarrows} \int_\tau^t \int_{\mathbb{R}^N} YZ(z;y,s)G(y,s;\zeta)dyds - G(z;\zeta) \tag{2.43}$$

Without loss of generality, we can assume $0 < \varepsilon \leq \varepsilon_0 = \frac{T_0-\tau}{2}$. As above,

$$\int_{t-\varepsilon}^t \int_{\mathbb{R}^N} YZ(z;y,s)G(y,s;\zeta)dyds = -\sum_{i,j=1}^{m_0} \int_{t-\varepsilon}^t \int_{\mathbb{R}^N} a_{ij}(y,s)\partial^2_{x_i x_j} Z(z;y,s)G(y,s;\zeta)dyds \tag{2.44}$$

Using Hypothesis **(H.3)** and (2.24), we have that $a_{ij}(y,s)G(y,s;\zeta)$ is uniformly Hölder continuous with respect to y. Following the lines used for second derivatives (see (2.27)), we can show that there exist two constant $c(\varepsilon_0) >$ and $\beta \in (0,1)$ such that

$$\left| \int_{t-\varepsilon}^t \int_{\mathbb{R}^N} a_{ij}(y,s)\partial^2_{x_i x_j} Z(z;y,s)G(y,s;\zeta)dyds \right| \leq \frac{c(\varepsilon_0)}{(t-s)^{1-\frac{\beta}{2}}}$$

for every $z \in S_I, s \in (t-\varepsilon, t)$. Hence, from (2.44), for all $z \in S_I$

$$\int_\tau^{t-\varepsilon} \int_{\mathbb{R}^N} YZ(z;y,s)G(y,s;\zeta)dyds \underset{\varepsilon\to 0}{\rightrightarrows} \int_\tau^t \int_{\mathbb{R}^N} YZ(z;y,s)G(y,s;\zeta)dyds \tag{2.45}$$

Since $G(\cdot\,;\zeta)$ is a continuous and bounded function on S_I, we obtain for all $z \in S_I$ (see the proof of Proposition 2.1.2)

$$\int_{\mathbb{R}^N} Z(z;y,t-\varepsilon)G(y,t-\varepsilon;\zeta)dy \underset{\varepsilon\to 0}{\rightrightarrows} G(z;\zeta) \tag{2.46}$$

From (2.36), (2.45) and (2.46) relation (2.43) follows, and therefore (2.34).
The thesis follows from (2.14), (2.26) and (2.34). □

We have shown that the function Γ we built via parametrix method is a solution of the equation $\mathscr{L}\Gamma = 0$ in $\mathbb{R}^{N+1} \setminus \{\zeta\}$, for any $\zeta \in \mathbb{R}^{N+1}$. In order to prove that Γ is the fundamental solution of $\mathscr{L}$, we only need to prove the following:

Theorem 2.1.1 *For every function $f \in C_0(\mathbb{R}^N)$ we have*

$$\lim_{t\to\tau^+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)f(\xi)d\xi = f(x)$$

for every $x \in \mathbb{R}^N, \tau \in \mathbb{R}$.

Proof.
Recall that Γ was defined as

$$\Gamma(z;\zeta) = Z(z;\zeta) + J(z;\zeta)$$

Since $Z_{\overline{z}}$ is the fundamental solution of $\mathscr{L}_{\overline{z}}$, for every $x \in \mathbb{R}^N$,

$$\lim_{t\to\tau^+} \int_{\mathbb{R}^N} Z_{(x,\tau)}(x,t;\xi,\tau)f(\xi)d\xi = f(x) \tag{2.47}$$

Moreover, since, from Hypothesis **(H.3)**, it follows

$$\lim_{w \to w'} \int_{\mathbb{R}^N} |Z_w(z;\xi,\tau) - Z_{w'}(z,\xi,\tau)| \, d\xi =$$
$$= \lim_{w \to w'} \int_{\mathbb{R}^N} |Z_w(0,0;\xi,\tau-t) - Z_{w'}(0,0,\xi,\tau-t)| \, d\xi = 0$$

and f is a bounded function, we can easily derive

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} \left[Z_{(\xi,\tau)}(z;\xi,\tau) - Z_{(x,\tau)}(z;\xi,\tau) \right] f(\xi) d\xi = 0 \tag{2.48}$$

Finally, applying Corollary 2.1.2 and Proposition 2.1.1 to the definition of J in (2.6), we get

$$|J(z;\zeta)| \leq \int_\tau^t \frac{\tilde{c}\tilde{k}_I}{(s-\tau)^{1-\frac{\alpha}{2}}} \int_{\mathbb{R}^N} \widetilde{\Gamma}(z;y,s)\widetilde{\Gamma}(y,s;\zeta) dyds = c'(t-s)^{\frac{\alpha}{2}} \widetilde{\Gamma}(z;\zeta) \tag{2.49}$$

from which

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} J(z;\xi,\tau) f(\xi) d\xi = 0 \tag{2.50}$$

The result follows from (2.47), (2.48) and (2.50). □

We can see this theorem from the following point of view.
Let's consider the following Cauchy problem:

$$\begin{cases} \mathscr{L}u = 0 & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \\ u(x,0) = f(x) & x \in \mathbb{R}^N \end{cases} \tag{2.51}$$

then

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0) f(\xi) d\xi$$

is a classical solution of (2.51). In fact, from the boundedness of f we can easily show that

$$\mathscr{L}u(x,t) = \int_{\mathbb{R}^N} (\mathscr{L}\Gamma)(x,t;\xi,0) f(\xi) d\xi = 0 \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+$$

Moreover, we take the initial condition in the limit sense, *i.e.*:

$$\lim_{t \to 0} u(x,t) = f(x)$$

We prove now the following theorem about the regularity of Γ.

Theorem 2.1.2 *Let* Γ *be the fundamental solution of* (2.1) *built via the parametrix method. Then* $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{\zeta\})$.

Proof.
The continuity derives from the definition of Γ (2.4) and from the property of G stated in Proposition 2.1.2.
From (2.11) and (2.49) we have, for every $K \subset\subset \mathbb{R}^{N+1}$:

$$\int_K |\Gamma(x,t;\xi,\tau)| dxdt = \int_K |Z(x,t;\xi,\tau) + J(x,t;\xi,\tau)| dxdt \leq$$
$$\leq \int_K |Z(x,t;\xi,\tau)| dxdt + \int_K |J(x,t;\xi,\tau)| dxdt \leq$$
$$\leq \tilde{c} \int_K |\widetilde{\Gamma}(x,t;\xi,\tau)| dxdt + c' \int_K (t-s)^{\frac{\alpha}{2}} \widetilde{\Gamma}(x,t;\xi,\tau) dxdt < +\infty$$

These last two integrals are both finite because $\widetilde{\Gamma}$ is the fundamental solution associated to (2.10), and there exists a constant M_K, depending on the compact K, such that $(t-s)^{\frac{\alpha}{2}} \leq M_K$. □

We give now two estimates for the fundamental solution and its derivatives.

Theorem 2.1.3 *Let $\mathscr{L}$ be an operator as in* (2.1) *veryfing Hypothesis **(H.1),(H.2),(H.3)**, or be as in* (2.3) *veryfing Hypothesis **(H.1),(H.2),(H.3),(H.4)**. Then, for every $\varepsilon > 0$, there exists $K > 0$ such that*

$$(1-\varepsilon)Z(z;\zeta) \leq \Gamma(z;\zeta) \leq (1+\varepsilon)Z(z;\zeta)$$

for any $(z;\zeta) \in \mathbb{R}^{N+1}$ such that $Z(z;\zeta) \geq K$.

Theorem 2.1.4 *There exist two positive constant c_0, K such that, for every $z, \zeta \in \mathbb{R}^{N+1}$ satisfying $\Gamma(z;\zeta) \geq K_0$, and for every $i = 1, ..., m_0$,*

$$|\partial_{x_i}\Gamma(z;\zeta)| \leq c\left[\frac{|D((t-\tau)^{-\frac{1}{2}}(x-E(t-\tau)\xi)|}{\sqrt{t-\tau}} + 1\right]\Gamma(z;\zeta)$$

2.2 Cauchy problem

We will give now some results concerning the Cauchy problem associated to the operator:

$$\mathscr{L} = \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial_{x_i x_j} + \sum_{i=1}^{m_0} a_i(x,t)\partial_{x_i} + \langle Bx, \nabla_x\rangle + c(x,t) - \partial_t \tag{2.52}$$

that is a generalization of (2.1) containing low order terms.
We assume that:

[H.1] *The matrix $B = (b_{ij})_{i,j=1,...,N}$ is a real constant matrix of the form 4.*

[H.2] *The matrix $A(x,t) = (a_{ij}(x,t))_{i,j=1,...,N}$ is a symmetric matrix of the form 4. Moreover, it is strictly positive in $\mathbb{R}^{m_0}$ and there exists a positive constant λ such that*

$$\frac{1}{\lambda}\sum_{i=1}^{m_0}|\xi_i|^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i\xi_j \leq \lambda\sum_{i=1}^{m_0}|\xi_i|^2 \tag{2.53}$$

for every $(\xi_1, ..., \xi_{m_0}) \in \mathbb{R}^{m_0}$ and $(x,t) \in \mathbb{R}^{N+1}$.

[H.3] *The coefficients $a_{ij}(x,t)$, $a_i(x,t)$, $c(x,t)$ are bounded and belong to $C^\alpha(\mathbb{R}^{N+1})$.*

We summarize the results obtained in [**DP**], that are a generalization of the results stated in [**P**].

Theorem 2.2.1 *Assume that $\mathscr{L}$ in* (2.52) *verifies hypotheses **(H1),(H2),(H3)**. Then there exists a fundamental solution Γ to $\mathscr{L}$ with the following properties:*

1. *$\Gamma(\cdot;\zeta) \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{\zeta\})$ for every $\zeta \in \mathbb{R}^{N+1}$;*
2. *$\Gamma(\cdot;\zeta)$ is a classical solution to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1} \setminus \{\zeta\}$ for every $\zeta \in \mathbb{R}^{N+1}$;*

3. *if $g \in C(\mathbb{R}^N)$ is such that:*

$$|g(x)| \leq C_0 \mathrm{e}^{C_0|x|^2}, \qquad \forall x \in \mathbb{R}^N \tag{2.54}$$

for some positive constant C_0 ; then there exists:

$$\lim_{t \to \tau^+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau) g(\xi) d\xi = g(x), \qquad \forall x \in \mathbb{R}^N, \tau \in \mathbb{R}$$

4. *the reproduction property holds:*

$$\int_{\mathbb{R}^N} \Gamma(x,t;y,s)\Gamma(y,s;\xi,\tau)dy = \Gamma(x,t;\xi,\tau), \qquad \forall x,\xi \in \mathbb{R}^N, \tau < s < t$$

5. *if $c(x,t) \equiv c$ is constant, then:*

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau))d\xi = \mathrm{e}^{-c(t-\tau)}, \qquad \forall x \in \mathbb{R}^N, \tau < t \tag{2.55}$$

6. *if Γ^ε denotes the fundamental solution of the following operator:*

$$\mathscr{L}^\varepsilon = (\lambda + \varepsilon)\Delta_{\mathbb{R}^{m_0}} + \langle Bx, \nabla_x \rangle - \partial_t \tag{2.56}$$

where $\varepsilon > 0$, λ is defined as in (2.2) and $\Delta_{\mathbb{R}_{m_0}}$ denotes the Laplacian in $\mathbb{R}^{m_0}$ (in particular $\mathscr{L}^\varepsilon \in \mathbb{K}_0$); then for every positive ε and T , there exists a constant $\overline{C}$, only dependent on λ, B, ε, and T , such that:

$$\begin{aligned}
&\Gamma(z;\zeta) \leq \overline{C}\Gamma^\varepsilon(z;\zeta) \\
&|\partial_{x_i}\Gamma(z;\zeta)| \leq \frac{\overline{C}}{\sqrt{t-\tau}}\Gamma^\varepsilon(z;\zeta) \\
&|\partial^2_{x_i x_j}\Gamma(z;\zeta)| \leq \frac{\overline{C}}{t-\tau}\Gamma^\varepsilon(z;\zeta) \\
&|Y\Gamma(z;\zeta)| \leq \frac{\overline{C}}{t-\tau}\Gamma^\varepsilon(z;\zeta)
\end{aligned}$$

for any $i,j = 1,...,m_0$ and $z,\zeta \in \mathbb{R}^{N+1}$ with $0 < t - \tau < T$.

Let's consider the following Cauchy problem:

$$\begin{cases} \mathscr{L}u(x,t) = f(x,t) & (x,t) \in \mathbb{R}^N \times (T_0,T_1) \\ u(x,T_0) = g(x) & x \in \mathbb{R}^N \end{cases}$$

where $g \in C(\mathbb{R}^N)$ is such that:

$$|g(x)| \leq C_0 \mathrm{e}^{C_0|x|^2}, \qquad \forall x \in \mathbb{R}^N, C_0 \in \mathbb{R}^+ \tag{2.57}$$

and $f \in C(\mathbb{R}^N \times (T_0,T_1))$ is such that:

$$|f(x,t)| \leq C_1 \mathrm{e}^{C_1|x|^2}, \qquad \forall (x,t) \in \mathbb{R}^N \times (T_0,T_1) \tag{2.58}$$

for some positive constant C_0, and for any compact subset M of $\mathbb{R}^N$ there exists a positive constant C and $\beta \in (0,1)$ such that:

$$|f(x,t) - f(y,t)| \leq Cd((x,t),(y,t))^\beta \qquad \forall x,y \in M, t \in (T_0,T_1)$$

where d is the quasi-distance defined in (1.23)*.*
Then there exists $T \in (T_0, T_1)$ such that the function:

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,T_0) g(\xi) d\xi - \int_{T_0}^t \int_{\mathbb{R}^N} \Gamma(x,t;\xi.\tau) f(\xi,\tau) d\xi d\tau \tag{2.59}$$

is solution to the Cauchy problem:

$$\begin{cases} \mathscr{L}u(x,t) = f(x,t) & (x,t) \in \mathbb{R}^N \times (T_0,T) \\ u(x,T_0) = g(x) & x \in \mathbb{R}^N \end{cases}$$

Moreover if u is a solution to the Cauchy problem with null f and g, and verifies estimate (2.58)*, then $u \equiv 0$; in particular, the function in* (2.59) *is the unique solution to problem verifying estimate* (2.58)*.*

These results follow using the same argument we used in the last section.

We state now two propositions about the function $G(z;\zeta)$ defined in the parametrix method, proved in [**DP**], that we will use in the sequel.

Proposition 2.2.1 *The function G satisfies the following estimate: for any $\zeta \in \mathbb{R}^{N+1}$, for any $\varepsilon > 0$ there exists a positive constant C_1 (only depending on $\varepsilon, T^*, \lambda$ and B) such that:*

$$|G(z;\zeta)| \leq C_1 \frac{\Gamma^\varepsilon(z;\zeta)}{(t-\tau)^{1-\frac{\alpha}{2}}}, \qquad \forall z = (x,t) \in \mathbb{R}^N \times (\tau, T^*)$$

Proposition 2.2.2 *For every $\varepsilon > 0$ and $T^* > 0$ there exists a positive constant C_2, such that:*

$$|G(x,t;\xi,\tau) - G(y,t;\xi,\tau)| \leq C_2 \frac{|x-y|_B^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{4}}} (\Gamma^\varepsilon(x,t;\xi,\tau) + \Gamma^\varepsilon(y,t;\xi,\tau))$$

for any $(\xi,\tau) \in \mathbb{R}^{N+1}$, $t \in (\tau, \tau + T^*]$ and $x, y \in \mathbb{R}^N$.
Here $|\cdot|_B$ represents the component in the the homogeneous norm associated to the first N variables, that means:

$$\|z\| = |x|_B + |t|^{\frac{1}{2}} \qquad \forall z = (x,t) \in \mathbb{R}^{N+1}$$

Chapter 3

Kuramoto model

One of the most fascinating phenomena in nature is the tendency to synchronization. It pervades nature at every scale from the nucleus to the cosmos, and even our heart exhibits this phenomenon. A cluster of about $10,000$ cells, called the sinoatrial node, generates the electrical rhythm that commands the rest of the heart to beat, and it must do so reliably, minute after minute, for about three billion beats in a lifetime. These cells are a collection of oscillators, i.e. entities that cycle automatically, and repeat themselves over and over again at more or less regular time intervals. The cells have to coordinate their rhythm, reaching some kind of synchronization: as a matter of fact if they send mixed signals the heart becomes deranged.
Huygens was one of the pioneers in the study of synchronization. In February of 1665 the Dutch physicist was confined to his bedroom for several days, and observed a curious phenomenon: the two pendulum clocks in the room, separated by two feet, kept oscillating together without any variation; moreover, mixing up the swings of the pendulums the synchronization returned after some time. Intrigued by this phenomenon, he carried out several experiments, finding out that the synchronization could take place only if they could communicate in some way.
In the following centuries many mathematicians studied the problem, coming up with different models: some of these describe specific problems, while others have been created with the aim of being general enough to describe the dynamics of different systems, even concerning very different areas such as Biology, Medicine, Neuroscience, Chemistry, Physics, Engineering and Social Sciences as well. One of the most famous model that belongs to the latter group is the one proposed by Kuramoto, nowadays a landmark in the field of synchronization.

The Kuramoto model is a system of ordinary differential equations introduced in the late decades of the last century in [**Ku**] to describe at least qualitatively many synchronization phenomena: indeed it describes the collective behavior of a population that can be seen as a group of oscillators.
Let's consider a population of N oscillators. We denote as $\theta_i(t)$ the phase related to the i-th oscillator, and as ω_i its natural frequency, which is an intrinsic parameter of the i-th oscillator. We suppose that these natural frequencies are distributed according to an unimodal distribution $g(\omega)$ such that:

- $g \in \mathrm{L}^1(\mathbb{R})$;
- g is normalized, that means:
$$\int_{\mathbb{R}} g(\omega)d\omega = 1$$

- g has mean frequency Ω, that means:

$$\Omega = \int_{\mathbb{R}} \omega g(\omega) d\omega$$

- g is symmetric with respect to its mean frequency Ω, that means:

$$g(\Omega - \omega) = g(\Omega + \omega), \qquad \forall \omega \in \mathbb{R},$$

The governing equation is given by the following system of ordinary differential equations:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \qquad i = 1, ..., N \tag{3.1}$$

where K is a real positive constant sizing the coupling strength between the oscillators.
We observe that this is a mean-field coupling, where each oscillator exerts the same influence on all the others. There are also more general models, called hierarchicals:

$$\dot{\theta}_i = \omega_i + \frac{1}{N} \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \qquad i = 1, ..., N$$

In this kind of models the coupling between each oscillator may vary.
The coupling is nonlinear, thus the ensuing phenomena may be expected to be rather complicated.
Let us define the following complex number, called *order parameter*:

$$r_N(t) \mathrm{e}^{i\psi_N(t)} = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{i\theta_j(t)} \tag{3.2}$$

The phase $\psi_N(t)$ is the mean phase of the system. The modulus $r_N(t)$ represents the coherence of the system: when $r_N(t) = 0$ the system is in a full incoherence state and no synchronization occurs, while when it increases the system enters in a state of partial synchronization where some oscillators are synchronized. In the full coherence state ($r_N(t) = 1$) every oscillator is synchronized, and the system can be treated as one oscillator with phase $\psi_N(t)$.
We can restate (3.1) using the order parameter:

$$\dot{\theta}_i = \omega_i + K r_N \sin(\psi_N - \theta_i), \qquad i = 1, ..., N \tag{3.3}$$

which is the equation of an overdamped pendulum with torque ω_i and restoring force proportional to $K r_N$. This formulation simplifies the analytical treatment.
Although requiring this type of coupling may seem restrictive and unrealistic, it was shown that (3.3) describes some important physical phenomena such as some Josephson array and the interaction of quasi-optical oscillators with a cavity, and it gives us a starting point to study synchronization in generic oscillators.
Kuramoto model shows how synchronization in a population of many coupled oscillators may occur, and the type of synchronization:

- frequency synchronization, where each oscillator completes its cycle at the same time;
- phase synchronization, where each oscillator is at the same point in the cycle;

Kuramoto based his studies on a self consistency argument, reaching the following conclusions:

- the oscillators that synchronize have natural frequency ω_i "close" to the mean frequency Ω:

$$|\omega_i - \Omega| \leq Kr$$

- the oscillators will only synchronize in frequency generally;
- the oscillators that do not synchronize form a stationary distribution which does not affect the coherence $r_N(t)$;
- full coherence state ($r_N(t) = 1$) is in general impossible for finite coupling K;
- for coupling K lower than a threshold K_c no synchronization occurs (phase transition phenomenon).

Equations (3.1) represent a molecular dynamics type model, typical in the case of a population of oscillators made up of a finite number of elements. We can extend the Kuramoto model to a population of infinitely many oscillators: this was described by Strogatz in [**S**].
Let's consider a continuum of oscillators whose natural frequency distribution is $g(\omega)$.
Let $\rho = \rho(\theta, \omega, t)$ be a density function describing the fraction of oscillators at phase θ with natural frequency ω at the time t, a nonnegative function that is normalized:

$$\int_0^{2\pi} \rho(\theta, \omega, t) d\theta = 1, \qquad \forall\ (\omega, t) \in \mathbb{R} \times [0, +\infty)$$

Moreover $\rho(\theta, \omega, t)$ is a function 2π periodic in θ.
The evolution of ρ is described by the following continuity equation, an hyperbolic partial differential equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho v)$$

which expresses conservation of oscillators of frequency ω. Here the velocity $v(\theta, \omega, t)$ is interpreted in an Eulerian sense as the instantaneous velocity of an oscillator at position θ, given that it has natural frequency ω.
From (3.3) that velocity is:

$$v(\theta, \omega, t) = \omega + Kr \sin(\psi - \theta)$$

where $r(t)$ and $\psi(t)$ follow from the law of large numbers applied to (3.2):

$$r\mathrm{e}^{\mathrm{i}\psi} = \int_0^{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta} \rho(\theta, \omega, t) g(\omega) d\omega d\theta$$

Combining these equations we obtain the following nonlinear partial integro-differential equation for the density ρ:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}\left[\rho\left(\omega + K\int_0^{2\pi}\int_{\mathbb{R}} \sin(\theta' - \theta)\rho(\theta', \omega', t) g(\omega') d\omega' d\theta'\right)\right] \tag{3.4}$$

Both (3.3) and (3.4) describe deterministic phenomena. We can extend these models adding noise to describe stochastic phenomena. A first generalization has been studied by Sakaguchi in [**Sa**], who extended the model to allow rapid stochastic fluctuations in the natural frequencies.
The governing equations for N oscillators take now the form of a system of Langevin equations:

$$\dot{\theta}_i = \omega_i + \xi_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \qquad i = 1, ..., N \tag{3.5}$$

where the variables $\xi_i = \xi_i(t)$ are independent white noise processes that satisfy:

$$\begin{aligned} &\langle \xi_i(t) \rangle = 0 \\ &\langle \xi_i(s)\xi_j(t) \rangle\rangle = 2D\delta_{ij}\delta(s-t) \end{aligned} \tag{3.6}$$

where $D \geq 0$ is the noise strength, δ_{ij} is the Kronecker delta, $\delta(t)$ is the Dirac delta and the angular brackets denote in this case an average over realizations of the noise.
Sakaguchi argued intuitively that since (3.5) is a system of Langevin equations with mean-field coupling, as $N \to \infty$ the density $\rho(\theta, \omega, t)$ should satisfy the following Fokker–Planck equation:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\rho\left(\omega + K\int_0^{2\pi}\int_{\mathbb{R}} \sin(\theta' - \theta)\rho(\theta', \omega', t)g(\omega')d\omega' d\theta'\right)\right] \tag{3.7}$$

The previous equation is a nonlinear parabolic partial integro-differential equation, that reduces to (3.4) when $D = 0$.

Linear stability of the incoherent state, concerning both the deterministic and the stochastic case, was studied by Strogatz and Mirollo in [**SM**]. They showed that in the continuum limit, the incoherent solution is stable for all coupling not strong enough ($K < K_c$) and becomes unstable for stronger coupling ($K > K_c$), meaning that a partially synchronized state exists. Moreover, they showed that if some noise affects the system then the incoherent solution ($r(t) = 0$) is unique whenever $K < K_c$, while there are infinitely many solutions in the absence of noise.

One of the most fascinating phenomena where we observe some kind of synchronization is that of the southeast Asia fireflies: thousands of fireflies that orchestrate their flashings so precisely that they become one pulsating light. One of the most famous and studied species is *Pteroptyx malaccae*. This particular species has the ability to alter its firing frequency up to 15%, realizing synchronization with very small phase lags.
In 1991 Ermentrout, in his studies on this fireflies, introduced in [**E**] a model with inertia, that is a model with a mass-type element, formulated as a system of second-order ordinary differential equations for phases. This was an "adaptive frequency" model, which implies that the natural frequencies of all oscillators are allowed to vary with time. This step towards a second order model was done to cross the obstacle that is in phase synchronization: indeed, a second order model leads now to have almost a phase synchrony, where the phase shift between synchronized oscillators is inversely proportional to the mass.
Based on the model proposed by Ermentrout, Tanaka and al. in [**TLO**] elaborated a second order variation of Kuramoto model:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \qquad i = 1, ..., N \tag{3.8}$$

where $m > 0$ is the inertial term, and now we denote as Ω_i the natural frequency related to the i-th oscillator, which oscillates at frequency $\omega_i = \dot{\theta}_i$.
Through (3.2) we can restate (3.8):

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + Kr_N \sin(\psi_N - \theta_i), \qquad i = 1, ..., N$$

which is the governing equation for a single damped driven pendulum with torque Ω_i.
The analytical treatment is now more complex, also for the dependence on initial conditions.

Following the same argument used by Kuramoto, it was shown that the system can exhibit hysteretic synchrony depending on the initial conditions: the coupling needed to bring a completely incoherent system to a partial synchronization state must be higher than a threshold $K_{c,low}$, while the coupling needed to make incoherent a coherent system must be lower than a threshold $K_{c,up}$ such that $K_{c,up} < K_{c,low}$.

Starting from such contributions, in 1998 Acebroan and Spigler formulated in [**AS**] an extension of (3.8) adding white noise, obtaining a system of second-order Langevin equations that consider an inertial term:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + Kr_N \sin(\psi_N - \theta_i) + \xi_i \qquad i = 1, ..., N \tag{3.9}$$

where $\xi_i = \xi_i(t)$ is the function that takes into account noise, whose properties are defined in (3.6).
If we set $\dot{\theta}_i = \omega_i$, reasoning as in (3.7) we get the following nonlinear ultraparabolic partial integro-differential equation:

$$\frac{D}{m^2}\frac{\partial^2 \rho}{\partial \omega^2} + \frac{1}{m}\frac{\partial}{\partial \omega}[(\omega - \Omega - K_\rho(\theta, t))\rho] - \omega\frac{\partial \rho}{\partial \theta} - \frac{\partial \rho}{\partial t} = 0 \tag{3.10}$$

where we set:

$$K_\rho(\theta, t) = K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\sin(\theta' - \theta)\rho(\theta', \omega', \Omega', t)d\theta' d\omega' d\Omega' \tag{3.11}$$

Here $g(\Omega')$ denotes the natural frequency distribution of the oscillators, and it has the same property described for the first order model.
Many results about existence, uniqueness and estimates for (3.7) and (3.10) are included in [**Sp**], [**Sp1**], [**Sp2**].

We end this section by mentioning the uncertainty principle introduced in [**Sp**].
The synchronization phenomena deal with phase and frequencies synchronization. It was shown that, as in Quantum Mechanics, we can't determine with arbitrary accuracy both phase and velocity in the noise models. Indeed, denoting as $\Delta\theta$ and Δv the spreads in phase and in velocity, for large value of the coupling parameter K we have:

$$\Delta\theta\Delta v \sim D$$

for equation (3.5), while for equation (3.9) we have:

$$\Delta\theta\Delta v \sim \frac{D}{\sqrt{mK}}$$

and we observe that a large mass m and a strong coupling K lower this uncertainty, while the noise D tends to make the uncertainty more significant.

3.1 Existence and uniqueness of classical solutions

In this section we study the existence and uniqueness of classical solutions for the non linear equation arising in the model with inertial term.
We will consider normalized parameters, that is $D = m = 1$.

Let define $S_T = \mathbb{R}^3 \times (0, T)$: we are looking for a function $\rho = \rho(\theta, \omega, \Omega, t)$ such that satisfies the following Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 \rho}{\partial \omega^2} + \dfrac{\partial}{\partial \omega}\left[(\omega - \Omega - K_\rho(\theta, t))\rho\right] - \omega \dfrac{\partial \rho}{\partial \theta} - \dfrac{\partial \rho}{\partial t} = 0 & \text{in } \mathrm{S_T} \\ \rho(\theta, \omega, \Omega, 0) = \rho_0(\theta, \omega, \Omega) & \text{in } \mathbb{R}^3 \end{cases} \tag{3.12}$$

where we set:

$$K_\rho(\theta, t) = K \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^{2\pi} g(\Omega') \sin(\theta' - \theta) \rho(\theta', \omega', \Omega', t) d\theta' d\omega' d\Omega' \tag{3.13}$$

We define a property that will be use often in the sequel.

Definition 3.1.1 *We say that a function $f(\theta, \omega, \Omega, t) : \Lambda \subseteq \mathbb{R}^4 \to \mathbb{R}$ has an exponential decay in ω if there exist two positive constant M, C such that:*

$$f(\theta, \omega, \Omega, t) \leq C\mathrm{e}^{-M\omega^2} \quad \forall (\theta, \omega, \Omega, t) \in \Lambda \tag{3.14}$$

We will say that f verifies property **(E)** if (3.14) holds.

We list the assumptions on the initial datum ρ_0:

1. $\rho_0 \in C(\mathbb{R}^3)$;

2. ρ_0 is strictly positive and verifies **(E)**;

3. ρ_0 is 2π periodic in θ;

4. for every $\Omega \in \mathbb{R}$ we have:
$$\int_{\mathbb{R}} \int_0^{2\pi} \rho_0(\theta, \omega, \Omega) d\theta d\omega = 1$$

Remark 3.1.1 *We notice that $K_\rho(\theta, t)$ is 2π periodic with respect to θ, in accordance with the periodicity of Kuramoto model. This hints to search for periodic solutions.*

Let us now make some observations:

- this equation is similar to the one associated to operator (2.52), so we are looking for a continuous function ρ classical solution to the equation with $\partial_\omega \rho$, $\partial^2_{\omega\omega} \rho$ and $Y\rho = \omega \partial_\omega \rho - \omega \partial_\theta \rho - \partial_t \rho$ continuous functions;

- the function $g(\Omega)$ denotes the natural frequency distribution of the oscillators: we suppose that $g \in \mathrm{L}^1(\mathbb{R})$;

- we observe that Ω appears in the equation as a constant and as integration parameter, but no derivatives in Ω appear;

- the positive parameter K represents the strength of the coupling between the oscillators, and it is a constant parameter;

- we observe that point 4. means that ρ_0 is normalized: this is because we want a solution ρ that is a density function.

In the sequel we will use the distance defined in (1.23), where the matrix related to the left translations is:

$$E(t) = \begin{pmatrix} \mathrm{e}^{-\mathrm{t}} & 0 \\ 1-\mathrm{e}^{-\mathrm{t}} & -1 \end{pmatrix}$$

while the matrix related to the dilations, that defines the homogeneous norm, is:

$$D(r) = \mathrm{diag}(r, r^3, r^2)$$

We prove now a lemma that will be useful in our proof, based on the link between (3.12) and (2.52).

Lemma 3.1.1 *If a classical solution* ρ *to* (3.12) *exists and verifies* ***(E)****, then:*

1. *ρ derivatives verify **(E)** for every fixed $t > 0$, in particular there exist two positive constant M, C such that:*

   $$\sup_{\theta,\Omega} |\partial\rho(\theta,\omega,\Omega,t)| \leq \frac{C}{\sqrt{t^i}}\mathrm{e}^{-M\omega^2}.$$

   where ∂ stands for ∂_ω $(i = 1)$, $\partial^2_{\omega\omega}$ or Lie derivative Y $(i = 2)$;

2. *ρ is 2π periodic in θ;*

3. *for every $\Omega \in \mathbb{R}$ and for every $t \in [0, T)$ we have:*

   $$\int_{\mathbb{R}}\int_0^{2\pi} \rho(\theta,\omega,\Omega,t)d\theta d\omega = 1$$

4. *ρ is strictly positive, that is $\rho(\theta,\omega,\Omega,t) > 0$ for every $t \geq 0$.*

Proof.
In the sequel we will study equation (3.12) for fixed $\Omega \in \mathbb{R}$. We set $S^*_T := \mathbb{R}^2 \times (0, T)$.
First we note that if a solution ρ exists, then the operator associated to the equation in (3.12) is in the form (2.52). Indeed the equation becomes:

$$\mathscr{L}\rho = \partial^2_{\omega\omega}\rho + \omega\partial_\omega\rho - \omega\partial_\theta\rho + \Phi_\rho(\theta,\Omega,t)\partial_\omega\rho + \rho - \partial_t\rho = 0 \tag{3.15}$$

where $\Phi_\rho(\theta,\Omega,t) = -\Omega - K_\rho(\theta,t)$.
We have:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$a(\omega,\theta,t) = \Phi_\rho(\theta,\Omega,t) \qquad c(\omega,\theta,t) = 1$$

Moreover, if we write explicitly $\Phi_\rho(\theta,\Omega,t)$:

$$\Phi_\rho(\theta,\Omega,t) = -\Omega - K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\sin(\theta'-\theta)\rho(\theta',\omega',\Omega',t)d\theta' d\omega' d\Omega'$$

we claim that for every $(\theta,\omega,\Omega,t), (\tilde\theta,\tilde\omega,\Omega,t) \in \mathbb{R}^3 \times (0,T)$, we have:

$$|\Phi_\rho(\theta,\Omega,t) - \Phi_\rho(\tilde\theta,\Omega,t)| \leq \tilde{L}\ \mathrm{d}((\theta,\omega,t),(\tilde\theta,\tilde\omega,t))$$

where $\tilde{L}$ is a positive constant.
Through trigonometric formulas, we have:

$$\begin{aligned}
&|\Phi_\rho(\theta,\Omega,t)-\Phi_\rho(\tilde{\theta},\Omega,t)| = \left|K_\rho(\theta,t)-K_\rho(\tilde{\theta},t)\right| \leq \\
\leq &K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\left|\sin(\theta'-\theta)\rho(\theta',\omega',\Omega',t)-\sin(\theta'-\tilde{\theta})\rho(\theta',\omega',\Omega',t)\right| d\theta' d\omega' d\Omega' \leq \\
\leq &K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\left|\sin(\theta')\right|\left|\cos(\theta)\rho(\theta',\omega',\Omega',t)-\cos(\tilde{\theta})\rho(\theta',\omega',\Omega',t)\right| d\theta' d\omega' d\Omega' + \\
+ &K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\left|\cos(\theta')\right|\left|\sin(\theta)\rho(\theta',\omega',\Omega',t)-\sin(\tilde{\theta})\rho(\theta',\omega',\Omega',t)\right| d\theta' d\omega' d\Omega' \leq \\
\leq &K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\left|\cos(\theta)-\cos(\tilde{\theta})\right|\left|\rho(\theta',\omega',\Omega',t)\right| d\theta' d\omega' d\Omega' + \\
+ &K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\left|\sin(\theta)-\sin(\tilde{\theta})\right|\left|\rho(\theta',\omega',\Omega',t)\right| d\theta' d\omega' d\Omega'
\end{aligned}$$

By the regularity of $\cos(\theta)$ and $\sin(\theta)$, there exists $C_1 \in \mathbb{R}^+$ such that:

$$\left|\cos(\theta)-\cos(\tilde{\theta})\right| \leq C_1 \ \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},t))$$
$$\left|\sin(\theta)-\sin(\tilde{\theta})\right| \leq C_1 \ \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},t))$$

Then for property **(E)** we have:

$$\begin{aligned}
&|\Phi_\rho(\theta,\Omega,t)-\Phi_\rho(\tilde{\theta},\Omega,t)| \leq 2K_1 \ \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},t)) \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\mathrm{e}^{-M(\omega')^2} d\theta' d\omega' d\Omega' = \\
= &\tilde{L} \ \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},t))
\end{aligned} \tag{3.16}$$

where K_1 is a positive constant.
Furthermore Φ_ρ is bounded, as a direct consequence of property **(E)**.

These properties allow us to use the parametrix method, this means that the operator associated to the equation in (3.15) is in the form (2.52), with Hölder exponent $\alpha = 1$, implying that ρ is a classical solution of the following Cauchy problem:

$$\begin{cases} \mathscr{L}\rho = \partial^2_{\omega\omega}\rho + \omega\partial_\omega\rho - \omega\partial_\theta\rho + \Phi_\rho\theta,\Omega,t)\partial_\omega\rho + \rho - \partial_t\rho = 0 & \text{in } \mathrm{S}^*_{\mathrm{T}} \\ \rho(\theta,\omega,\Omega,0) = \rho_0(\theta,\omega,\Omega) & \text{in } \mathbb{R}^2 \end{cases}$$

We can apply then the results stated in Theorem 2.2.1: let Γ_Ω be the fundamental solution of (3.15), the representation formula (2.59) holds:

$$\rho(\theta,\omega,\Omega,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\eta,\xi,0)\rho_0(\eta,\xi,\Omega) d\eta d\xi \tag{3.17}$$

To prove that $\partial\rho$ verifies property **(E)** we first note that:

$$\partial\rho(\theta,\omega,\Omega,t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \partial\Gamma_\Omega(\theta,\omega,t;\eta,\xi,0)\rho_0(\eta,\xi,\Omega) d\eta d\xi$$

Then by point 6 of Theorem 2.2.1 we have:

$$|\partial\rho(\theta,\omega,\Omega,t)| \leq \frac{\overline{C}}{\sqrt{t^i}} \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma^\varepsilon_\Omega(\theta,\omega,t;\eta,\xi,0)\rho_0(\eta,\xi,\Omega) d\eta d\xi$$

where $i = 1$ for ∂_ω, otherwise $i = 2$.
Property **(E)** follows then from the exponential decay of the fundamental solution and the exponential decay of the initial datum.
Indeed, the following change of variables:

$$(x, y) = (\eta, \xi) - E^{-1}(-t)(\omega, \theta)^T$$

leads to:

$$|\partial\rho(\theta,\omega,\Omega,t)| \leq \frac{\overline{C}}{\sqrt{t^i}} \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma^{\varepsilon}_{\Omega}(0,0,t;x,y,0)\rho_0(\mathrm{e}^{-t}\omega + x, y - \mathrm{e}^{-t}\omega + \omega + \theta, \Omega)dxdy \leq$$
$$\leq \frac{\overline{C}}{\sqrt{t^i}} C_0 \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma^{\varepsilon}_{\Omega}(0,0,t;x,y,0)\mathrm{e}^{-M(\mathrm{e}^{-t}\omega+x)^2}dxdy \leq$$
$$\leq \frac{\overline{C}}{\sqrt{t^i}} C_0 \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma^{\varepsilon}_{\Omega}(0,0,t;x,y,0)\mathrm{e}^{-M\mathrm{e}^{-2t}\omega^2}\mathrm{e}^{-2M(\mathrm{e}^{-t}\omega x)}dxdy \leq \frac{C}{\sqrt{t^i}}\mathrm{e}^{-M\omega^2}$$

The proof of the periodicity follows from an uniqueness argument. Indeed, let define $v(\theta,\omega,\Omega,t) := \rho(\theta + 2\pi,\omega,\Omega,t)$: from the 2π periodicity of (3.13) and of the initial datum $\rho_0(\theta,\omega,\Omega)$, we have that $v(\theta,\omega,\Omega,t)$ is a solution of the same Cauchy problem.
Then $u(\theta,\omega,\Omega,t) := \rho(\theta,\omega,\Omega,t) - v(\theta,\omega,\Omega,t)$ is a solution of:

$$\begin{cases} \mathscr{L}u = 0 \\ u(\theta,\omega,\Omega,0) = 0 \end{cases}$$

Moreover $u(\theta,\omega,\Omega,t)$ verifies property **(E)**, this means we can apply the result stated in the last part of Theorem 2.2.1, concluding that $u \equiv 0$, then $\rho(\theta,\omega,\Omega,t) = \rho(\theta + 2\pi,\omega,\Omega,t)$.

For the proof of the third point we integrate (3.15) in θ and in ω, and we get:

$$\int_{\mathbb{R}}\int_0^{2\pi} \partial_t\rho d\theta d\omega = \int_{\mathbb{R}}\int_0^{2\pi} \partial_\omega(\partial_\omega\rho + (\omega - \Omega - K_\rho(\theta,t))\rho)d\theta d\omega - \int_{\mathbb{R}}\int_0^{2\pi} \omega\partial_\theta\rho d\theta d\omega$$

For the periodicity in θ we have:

$$\int_{\mathbb{R}}\int_0^{2\pi} \omega\partial_\theta\rho d\theta d\omega = \int_{\mathbb{R}} \omega\, [\rho]_0^{2\pi}\, d\omega = 0$$

Thanks to the exponential decay in ω for every fixed $t \in (0,T)$, we can apply Fubini Theorem, then the first integral in the right side vanishes:

$$\int_0^{2\pi}\int_{\mathbb{R}} \partial_\omega(\partial_\omega\rho + (\omega - \Omega - K_\rho(\theta,t))\rho)d\omega d\theta = \int_0^{2\pi} \lim_{R\to\pm\infty} [\partial_\omega\rho + (\omega - \Omega - K_\rho(\theta,t))\rho)]_{-R}^{+R} d\theta = 0$$

Moreover the same arguments lead to the following relations:

$$\partial_t \int_{\mathbb{R}}\int_0^{2\pi} \rho(\theta,\omega,\Omega,t)d\theta d\omega = \partial_t \int_{\mathbb{R}}\int_0^{2\pi} \rho(\theta,\omega,\Omega,t)d\theta d\omega - \omega\partial_\omega \int_{\mathbb{R}}\int_0^{2\pi} \rho(\theta,\omega,\Omega,t)d\theta d\omega +$$
$$+\omega\partial_\theta \int_{\mathbb{R}}\int_0^{2\pi} \rho(\theta,\omega,\Omega,t)d\theta d\omega = -Y \int_{\mathbb{R}}\int_0^{2\pi} \rho(\theta,\omega,\Omega,t)d\theta d\omega = -\int_{\mathbb{R}}\int_0^{2\pi} Y\rho(\theta,\omega,\Omega,t)d\theta d\omega =$$
$$= \int_{\mathbb{R}}\int_0^{2\pi} \partial_t\rho(\theta,\omega,\Omega,t)d\theta d\omega - \int_{\mathbb{R}}\int_0^{2\pi} \omega\partial_\omega\rho(\theta,\omega,\Omega,t)d\theta d\omega + \int_{\mathbb{R}}\int_0^{2\pi} \omega\partial_\theta\rho(\theta,\omega,\Omega,t)d\theta d\omega =$$
$$= \int_{\mathbb{R}}\int_0^{2\pi} \partial_t\rho(\theta,\omega,\Omega,t)d\theta d\omega$$

Then we have:

$$\int_{\mathbb{R}}\int_0^{2\pi} \partial_t \rho d\theta d\omega = \partial_t \int_{\mathbb{R}}\int_0^{2\pi} \rho d\theta d\omega = 0$$

Thus for every $t \in (0, T)$ we have:

$$\int_{\mathbb{R}}\int_0^{2\pi} \rho d\theta d\omega = c(\Omega)$$

where $c(\Omega)$ is a constant that doesn't depend on t. The thesis follows then from the normalization of the initial datum.

The proof of the fourth point is a consequence of the representation formula (3.17): indeed both Γ_Ω and ρ_0 are positive functions. □

From this lemma we have the following:

Corollary 3.1.1 *For any function $f(\theta, \omega, t, \theta', \omega', \Omega')$ such that:*

$$|f(\theta, \omega, t, \theta', \omega', \Omega')| \leq 1$$

we have:

$$\left| \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} f(\theta, \omega, t, \theta', \omega', \Omega') g(\Omega') \rho(\theta', \omega', \Omega', t) d\theta' d\omega' d\Omega' \right| \leq A$$

where

$$A = \int_{\mathbb{R}} g(\Omega') d\Omega' \tag{3.18}$$

We can now state our result:

Theorem 3.1.1 *Let $\rho_0(\theta, \omega, \Omega)$ and $g(\Omega)$ two functions such that:*

1. *$\rho_0 \in C(\mathbb{R}^3)$;*
2. *for every $\Omega \in \mathbb{R}$ we have:*

$$\int_{\mathbb{R}}\int_0^{2\pi} \rho_0(\theta, \omega, \Omega) d\theta d\omega = 1$$

3. *ρ_0 is strictly positive and verifies property **(E)**;*
4. *$g \in \mathrm{L}^1(\mathbb{R})$.*

Then there exists $T > 0$ such that there exists a unique classical solution $\rho = \rho(\theta, \omega, \Omega, t)$ to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 \rho}{\partial \omega^2} + \dfrac{\partial}{\partial \omega}[(\omega - \Omega - K_\rho(\theta, t))\rho] - \omega \dfrac{\partial \rho}{\partial \theta} - \dfrac{\partial \rho}{\partial t} = 0 & \text{in } \mathrm{S_T} = \mathbb{R}^3 \times (0, \mathrm{T}) \\ \rho(\theta, \omega, \Omega, 0) = \rho_0(\theta, \omega, \Omega) & \text{in } \mathbb{R}^3 \end{cases}$$

where:

$$K_\rho(\theta, t) = K \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega') \sin(\theta' - \theta) \rho(\theta', \omega', \Omega', t) d\theta' d\omega' d\Omega' \tag{3.19}$$

*Moreover ρ verifies property **(E)**, it is continuous with respect to Ω, and Lipschitz continuous for every $(\theta, \omega) \in \mathbb{R}^2 \cap \{t \geq \delta\}$ for every $\delta > 0$, for every fixed $\Omega \in \mathbb{R}$.*

Proof.
We construct a sequence of solutions to a linear Cauchy problem and prove its convergence by a compactness argument.
In the sequel we fix $\Omega \in \mathbb{R}$.
We set $\rho_0(\theta,\omega,\Omega,t) = \rho_0(\theta,\omega,\Omega)$ and we define by induction $\{\rho_n\}_{n\in\mathbb{N}}$ as the solutions to:

$$\begin{cases} \partial^2_{\omega\omega}\rho_n + \partial_\omega[(\omega - \Omega - K_{\rho_n}(\theta,t))\rho_n] - \omega\partial_\theta\rho_n - \partial_t\rho_n = 0 \\ \rho(\theta,\omega,\Omega,0) = \rho_0(\theta,\omega,\Omega) \end{cases} \tag{3.20}$$

where we set:

$$K_{\rho_n}(\theta,t) = K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\sin(\theta'-\theta)\rho_{n-1}(\theta',\omega',\Omega',t)d\theta' d\omega' d\Omega'$$

As $\rho_0 \in C(\mathbb{R}^3)$ and verifies property **(E)**, arguing as in Lemma 3.1.1 we can show that for every $n \in \mathbb{N}$ there exists a classical solution in $\mathbb{R}^2 \times (0,T_n)$.
Moreover, if we define $\Gamma^+(z;\zeta)$ the fundamental solution of:

$$\mathscr{L}^+ = \lambda\Delta_{\mathbb{R}^{m_0}} + \langle Bx, \nabla_x\rangle - \partial_t$$

where λ is the constant appearing in (2.53) (which is $\lambda = 1$ in this case), we notice that Γ^+ is the same for every $n \in \mathbb{N}$. This means that $T_n = T_1 = T \ \ \forall n \in \mathbb{N}$ (for more details see the chapter "Potential estimates" in [**DP**]).
We have that:

- ρ_n is normalized $\forall\, t \in [0,T)$:

$$\int_{\mathbb{R}}\int_0^{2\pi} \rho_n(\theta,\omega,\Omega,t)d\theta d\omega = 1$$

- ρ_n is strictly positive for every $t \geq 0$;
- ρ_n is 2π periodic in θ.

Moreover, if we denote by $\Gamma_{\Omega,n}$ the fundamental solution of (3.20), point 6 of Theorem 2.2.1 give us:

$$\begin{aligned} &\Gamma_{\Omega,n}(\theta,\omega,t;\xi,\eta,0) \leq \overline{C}\Gamma^\varepsilon(\theta,\omega,t;\xi,\eta,0) \\ &|\partial_\omega\Gamma_{\Omega,n}(\theta,\omega,t;\xi,\eta,0)| \leq \frac{\overline{C}}{\sqrt{t}}\Gamma^\varepsilon(\theta,\omega,t;\xi,\eta,0) \\ &|\partial^2_{\omega\omega}\Gamma_{\Omega,n}(\theta,\omega,t;\xi,\eta,0)| \leq \frac{\overline{C}}{t}\Gamma^\varepsilon(\theta,\omega,t;\xi,\eta,0) \\ &|Y\Gamma_{\Omega,n}(\theta,\omega,t;\xi,\eta,0)| \leq \frac{\overline{C}}{t}\Gamma^\varepsilon(\theta,\omega,t;\xi,\eta,0) \end{aligned} \tag{3.21}$$

where the constant $\overline{C} > 0$ depends only on λ, B, ε, and T, which are the same for all $n \in \mathbb{N}$ and for all $\Omega \in \mathbb{R}$. This means that, arguing as in the proof of Lemma 3.1.1, we can show that $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\partial\rho_n\}_{n\in\mathbb{N}}$ have an uniform exponential decay in ω, and there exist four functions $Q_i = Q_i(t)$, $i = 1,2,3,4$, bounded for every $t > 0$, such that for every $n \in \mathbb{N}$ we have:

$$\begin{cases} \sup\limits_{\theta,\omega} \rho_n \leq Q_1(t) = Q_1 \\ \sup\limits_{\theta,\omega} |\partial\rho_n| \leq Q_i(t), \qquad i = 2,3,4 \end{cases}$$

where ∂ stand for $\partial_\omega, \partial^2_{\omega\omega}$ or Y. This means that $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\partial\rho_n\}_{n\in\mathbb{N}}$ are uniformly bounded successions in $\mathbb{R}^2 \times [\mu, T-\mu]$, for every $\mu > 0$.
These successions are also uniformly continuous in $\mathbb{R}^2 \times [\mu, T-\mu]$, indeed we have:

$$\begin{aligned} &\partial\rho_n(\theta_1, \omega_1, \Omega, t_1) - \partial\rho_n(\theta_2, \omega_2, \Omega, t_2) = \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\partial\Gamma_{\Omega,n}(\theta_1, \omega_1, t_1; \eta, \xi, 0) - \partial\Gamma_{\Omega,n}(\theta_2, \omega_2, t_2; \eta, \xi, 0)\right] \rho_0(\eta, \xi, \Omega) d\eta d\xi \end{aligned}$$

We recall that $\Gamma_{\Omega,n}$ is built via parametrix method:

$$\partial\Gamma_{\Omega,n}(\theta, \omega, t; \xi, \eta, 0) = \partial Z_{\Omega,n}(\theta, \omega, t; \xi, \eta, 0) + \int_0^t \int_{\mathbb{R}^2} \partial Z_{\Omega,n}(\theta, \omega, t; y, s) G_{\Omega,n}(y, s; \xi, \eta, 0) dy ds$$

where the parametrix $Z_{\Omega,n}$ is the fundamental solution of:

$$\mathscr{L} = \partial_{\omega\omega} + \omega\partial_\omega - \omega\partial_\theta - \partial_t$$

Then $Z_{\Omega,n} = Z_\Omega$ for all $n \in \mathbb{N}$.
We suppose without loss of generality that $t_1 > t_2$, then we have:

$$\begin{aligned} &\partial\Gamma_{\Omega,n}(\theta_1, \omega_1, t_1; \xi, \eta, 0) - \partial\Gamma_{\Omega,n}(\theta_2, \omega_2, t_2; \xi, \eta, 0) = \\ &= \partial Z_\Omega(\theta_1, \omega_1, t_1; \xi, \eta, 0) - \partial Z_\Omega(\theta_2, \omega_2, t_2; \xi, \eta, 0) + \\ &+ \int_0^{t_2} \int_{\mathbb{R}^2} \left[\partial Z_\Omega(\theta_1, \omega_1, t_1; y, s) - \partial Z_\Omega(\theta_2, \omega_2, t_2; y, s)\right] G_{\Omega,n}(y, s; \xi, \eta, 0) dy ds + \\ &+ \int_{t_2}^{t_1} \int_{\mathbb{R}^2} \partial Z_\Omega(\theta_1, \omega_1, t_1; y, s) G_{\Omega,n}(y, s; \xi, \eta, 0) dy ds \end{aligned}$$

We show that there is an uniform bound on $G_{\Omega,n}$, then, arguing as in [**DP**] we can show that the integrals vanish uniformly with respect to n as $t_2 \to t_1$.
For this purpose we will use Proposition 2.2.1 and Proposition 2.2.2:

$$|G_{\Omega,n}(y, t; \xi, \eta, 0)| \le C_1 \frac{\Gamma^\varepsilon(y, t; \xi, \eta, 0)}{(t)^{1-\frac{\alpha}{2}}}$$

$$|G_{\Omega,n}(x, t; \xi, \eta, 0) - G_{\Omega,n}(y, t; \xi, \eta, 0)| \le C_2 \frac{|x-y|_B^{\frac{\alpha}{2}}}{(t)^{1-\frac{\alpha}{4}}} (\Gamma^\varepsilon(x, t; \xi, \eta, 0) + \Gamma^\varepsilon(y, t; \xi, \eta, 0))$$

Here Γ^ε is the fundamental solution of (2.56), and α is the Hölder continuity order of (2.52). C_1 and C_2 depend on $\varepsilon, t_1, \lambda, B$ (which are the same for all $n \in \mathbb{N}$), moreover C_2 depends on the following upper bounds:

- $|\Phi_{\rho,n}(\theta, \omega, \Omega, t) - \Phi_{\rho.n}(\tilde\theta, \tilde\omega, \Omega, t)| = \left|K_{\rho_n}(\theta, t) - K_{\rho_n}(\tilde\theta, t)\right| \le$
$$\begin{aligned} &\le K \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega') \left|\cos(\theta) - \cos(\tilde\theta)\right| |\rho_{n-1}(\theta', \omega', \Omega', t)| d\theta' d\omega' d\Omega' + \\ &+ K \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega') \left|\sin(\theta) - \sin(\tilde\theta)\right| |\rho_{n-1}(\theta', \omega', \Omega', t)| d\theta' d\omega' d\Omega' \le \tilde L \, \mathrm{d}((\theta, \omega, t), (\tilde\theta, \tilde\omega, t)) \end{aligned}$$
- $|\Phi_{\rho,n}(\theta, \omega, \Omega, t)| = |-\Omega - K_{\rho_n}(\theta, t)| =$
$$= \left|-\Omega - K \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega') \sin(\theta' - \theta) \rho_{n-1}(\theta', \omega', \Omega', t) d\theta' d\omega' d\Omega'\right| \le \widetilde{C}$$

$\tilde{L}$ and $\tilde{C}$ are positive constant that don't depend on n because we have uniform exponential decay, as we saw earlier:

$$\rho_{n-1}(\theta,\omega,\Omega,t) \le C\mathrm{e}^{-M\omega^2} \qquad \forall n \in \mathbb{N}$$

Then we have uniform bounds on $G_{\Omega,n}$ that imply uniform continuity.

Moreover $\{\rho_n\}_{n\in\mathbb{N}}$ is uniformly bounded in Lipschitz norm for every $t \ge \mu > 0$: indeed we can show that there is a positive constant $L = L(\mu)$ such that:

$$|\rho_n(\theta,\omega,\Omega,t) - \rho_n(\tilde{\theta},\tilde{\omega},\Omega,\tilde{t})| \le L\,\mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t})), \qquad \forall n \in \mathbb{N} \tag{3.22}$$

In order to show this relation we need to connect (θ,ω,Ω,t) and $(\tilde{\theta},\tilde{\omega},\Omega,\tilde{t})$ through the geometry induced by the operator: we rely on the results stated in [**PP**], noting that from the boundedness we can assume that these points are "close", that means there exists $\sigma < 1$ such that:

$$|\theta - \tilde{\theta}| < \sigma$$
$$|\omega - \tilde{\omega}| < \sigma$$
$$|t - \tilde{t}| < \sigma$$

First we show that we can connect (θ,ω,Ω,t) and $(\tilde{\theta},\omega,\Omega,t)$. In the sequel we will use the same notation used in [**PP**], then variables will be listed as (ω,θ,t), reminding Ω is fixed.
Let define:

$$\gamma^{(0)}_{v,\delta}(\omega,\theta,t) = (\omega+\delta v,\theta,t), \qquad \delta, v \in \mathbb{R}$$
$$\mathrm{e}^{\delta Y}(\omega,\theta,t) = (\omega\mathrm{e}^{\delta}, \omega(1-\mathrm{e}^{\delta})+\theta, t-\delta), \qquad \delta \in \mathbb{R}$$

These are the integral curves associated to ∂_ω and Y.
We define:

$$\begin{aligned}
&(0) = (\omega,\theta,t)\\
&(1) = \gamma^{(0)}_{v,\delta}(0) = (\omega+\delta v,\theta,t)\\
&(2) = \mathrm{e}^{\delta^2 Y}(1) = ((\omega+\delta v)\mathrm{e}^{\delta^2}, (\omega+\delta v)(1-\mathrm{e}^{\delta^2})+\theta, t-\delta^2)\\
&(3) = \gamma^{(0)}_{v,-\delta}(2) = ((\omega+\delta v)\mathrm{e}^{\delta^2} - \delta v, (\omega+\delta v)(1-\mathrm{e}^{\delta^2})+\theta, t-\delta^2)\\
&(4) = \mathrm{e}^{-\delta^2 Y}(3) = (\omega+\delta v - \delta v\mathrm{e}^{-\delta^2}, \delta v\mathrm{e}^{-\delta^2} - \delta v + \theta, t)\\
&(5) = \gamma^{(0)}_{v,-\delta^3}(4) = (\omega+\delta v - \delta v\mathrm{e}^{-\delta^2} - \delta^3 v, \delta v\mathrm{e}^{-\delta^2} - \delta v + \theta, t)\\
&(6) = g_{v,\delta}(\omega,\theta,t) = \gamma^{(0)}_{v',\delta'}(5) = (\omega, \theta - \delta v + \delta v\mathrm{e}^{-\delta^2}, t)
\end{aligned}$$

where $\delta' = \delta^5$ and

$$v' = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+2)!}\delta^{2n} v = v\delta^{-4}\mathrm{e}^{-\delta^2} - v\delta^{-4} + v\delta^{-2}$$

Then we can use Lemma 3.2 in [**PP**]:

Lemma 3.1.2 *There exists $\kappa > 0$, only dependent on B, such that: for any $\tilde{\theta} \in \mathbb{R}$ with $|\theta - \tilde{\theta}| \le \kappa$. there exists $v \in \{-1,1\}$ and $\delta \ge 0$ such that*

$$g_{v,\delta}(\omega,\theta,t) = (\omega,\tilde{\theta},t) \quad \textit{and} \quad |\delta| \le c|\theta-\tilde{\theta}|^{\frac{1}{3}}$$

where $c \in \mathbb{R}^+$ only depends on B.

We can take σ small enough such that $\delta \leq \sqrt{t}$, thus implying $t - \delta^2 > 0$, then $\partial_\omega \Gamma_{\Omega,n}$ and $Y\Gamma_{\Omega,n}$ are continuous and bounded through these curves.
For the sake of clarity we omit now the pole in the fundamental solution:

$$\Gamma_{\Omega,n}(\theta,\omega,t;\eta,\xi,0) = \Gamma_{\Omega,n}(\theta,\omega,t)$$

We define as γ every path we used to connect $(1),(2),(3),(4),(5)$ and (6).
We can apply Lagrange mean-value Theorem to the fundamental solution together with (3.21), obtaining the following estimates:

$$|\Gamma_{\Omega,n}(1) - \Gamma(0)_{\Omega,n}| \leq |\delta| \max_\gamma |\partial_\omega \Gamma_{\Omega,n}| \leq |\delta| \frac{\overline{C}}{\sqrt{\mu}} \max_\gamma \Gamma^\varepsilon \leq \frac{C_1}{\sqrt{\mu}} |\theta - \tilde{\theta}|^{\frac{1}{3}} \max_\gamma \Gamma^\varepsilon$$

$$|\Gamma_{\Omega,n}(2) - \Gamma(1)_{\Omega,n}| \leq \left|\delta^2\right| \max_\gamma |Y\Gamma_{\Omega,n}| \leq \left|\delta^2\right| \frac{\overline{C}}{\mu} \max_\gamma \Gamma^\varepsilon \leq |\theta - \tilde{\theta}|^{\frac{1}{3}} \frac{C_1}{\mu} \max_\gamma \Gamma^\varepsilon$$

(*analogous estimates for (3)-(2), (4)-(3) and (5)-(4)*))

$$|\Gamma_{\Omega,n}(6) - \Gamma(5)_{\Omega,n}| \leq \left|\delta(1 - \mathrm{e}^{-\delta^2}) + \delta^3\right| \max_\gamma |\partial_\omega \Gamma_{\Omega,n}| \leq \left(\left|\delta^3\right| + |\delta|\right) \frac{\overline{C}}{\sqrt{\mu}} \max_\gamma \Gamma^\varepsilon$$

$$\leq |\theta - \tilde{\theta}|^{\frac{1}{3}} \frac{C_2}{\sqrt{\mu}} \max_\gamma \Gamma^\varepsilon$$

where C_1 and C_2 are positive constant that don't depend on n.
Using triangular inequality these estimates lead to:

$$|\Gamma_{\Omega,n}(\theta,\omega,t) - \Gamma_{\Omega,n}(\tilde{\theta},\omega,t)| = |\Gamma_{\Omega,n}(6) - \Gamma(0)_{\Omega,n}| \leq |\theta - \tilde{\theta}|^{\frac{1}{3}} L_1 \overline{\Gamma^\varepsilon} \tag{3.23}$$

where $L_1 = L_1(\mu)$ is a positive constant that doesn't depend on n, and $\overline{\Gamma^\varepsilon}$ denotes the maximum of Γ^ε among all integral curves we have considered. We observe that these integral curves depend only on matrix B, then they are the same for all $n \in \mathbb{N}$.
This gives us:

$$\begin{aligned}
&\left|\Gamma_{\Omega,n}(\omega,\theta,t) - \Gamma_{\Omega,n}(\tilde{\omega},\tilde{\theta},\tilde{t})\right| \leq \\
\leq &\left|\Gamma_{\Omega,n}(\tilde{\omega},\tilde{\theta},\tilde{t}) - \Gamma_{\Omega,n}(\mathrm{e}^{(t-\tilde{t})Y}(\omega,\theta,t))\right| + \left|\Gamma_{\Omega,n}(\mathrm{e}^{(t-\tilde{t})Y}(\omega,\theta,t)) - \Gamma_{\Omega,n}(\omega,\theta,t)\right| = \\
= &\left|\Gamma_{\Omega,n}(\tilde{\omega},\tilde{\theta},\tilde{t}) - \Gamma_{\Omega,n}(\omega \mathrm{e}^{(t-\tilde{t})}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t}))\right| + \left|\Gamma_{\Omega,n}(\mathrm{e}^{(t-\tilde{t})Y}(\omega,\theta,t)) - \Gamma_{\Omega,n}(\omega,\theta,t)\right| \leq \\
\leq &\left|\Gamma_{\Omega,n}(\tilde{\omega}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t}) - \Gamma_{\Omega,n}(\omega \mathrm{e}^{(t-\tilde{t})}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t}))\right| + \\
+ &\left|\Gamma_{\Omega,n}(\tilde{\omega},\tilde{\theta},\tilde{t}) - \Gamma_{\Omega,n}(\tilde{\omega}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t})\right| + \left|\Gamma_{\Omega,n}(\mathrm{e}^{(t-\tilde{t})Y}(\omega,\theta,t)) - \Gamma_{\Omega,n}(\omega,\theta,t)\right|
\end{aligned}$$

The same argument we used earlier leads to:

$$\left|\Gamma_{\Omega,n}(\tilde{\omega}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t}) - \Gamma_{\Omega,n}(\omega \mathrm{e}^{(t-\tilde{t})}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t}))\right| \leq \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t}))\overline{C} \max_\gamma \Gamma^\varepsilon$$

$$\left|\Gamma_{\Omega,n}(\mathrm{e}^{(t-\tilde{t})Y}(\omega,\theta,t)) - \Gamma_{\Omega,n}(\omega,\theta,t)\right| \leq \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t}))\frac{\overline{C}}{\mu} \max_\gamma \Gamma^\varepsilon$$

The second term can be bounded using (3.23), noticing that we can pick σ small enough such that:

$$|\omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta - \tilde{\theta}| \leq \kappa$$

where $\kappa > 0$ is defined in Lemma 3.1.2. Then we have:

$$|\Gamma_{\Omega,n}(\tilde{\omega},\tilde{\theta},\tilde{t}) - \Gamma_{\Omega,n}(\tilde{\omega}, \omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta, \tilde{t})| \leq |\omega(1 - \mathrm{e}^{(t-\tilde{t})}) + \theta - \tilde{\theta}|^{\frac{1}{3}} L_1 \overline{\Gamma^\varepsilon} \leq \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t})) L_1 \overline{\Gamma^\varepsilon}$$

Then, denoting as $\underline{\Gamma^\varepsilon}$ the maximum among all integral curves we considered, it follows:

$$|\Gamma_{\Omega,n}(\omega,\theta,t)-\Gamma_{\Omega,n}(\tilde{\omega},\tilde{\theta},\tilde{t})|\leq \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t}))L_2\,\underline{\Gamma^\varepsilon}$$

where $L_2=L_2(\mu)=\max\{C,L_1\}$ doesn't depend on n.
We denote by (a,b,c) the maximum point:

$$\underline{\Gamma^\varepsilon}=\Gamma^\varepsilon(a,b,c;\eta,\xi,0)$$

Representation formula for $\{\rho_n\}_n$ gives:

$$|\rho_n(\theta,\omega,\Omega,t)-\rho_n(\tilde{\theta},\tilde{\omega},\Omega,\tilde{t})|\leq\int_{\mathbb{R}}\int_{\mathbb{R}}|\Gamma_{\Omega,n}(\theta,\omega,t;\eta,\xi,0)-\Gamma_{\Omega,n}(\tilde{\theta},\tilde{\omega},\tilde{t};\eta,\xi,0)|\rho_0(\eta,\xi,\Omega)d\eta d\xi\leq$$
$$\leq \mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t}))L_2C_0\int_{\mathbb{R}}\int_{\mathbb{R}}\Gamma^\varepsilon(a,b,c;\eta,\xi,0)d\eta d\xi=L\,\mathrm{d}((\theta,\omega,t),(\tilde{\theta},\tilde{\omega},\tilde{t}))$$

where $L=L(\mu)$ is a positive constant that doesn't depend on n.

We show now that $\{\rho_n\}_{n\in\mathbb{N}}$ converges uniformly to a continuous function in $\overline{S_T^*}=\mathbb{R}^2\times[0,T-\nu]$ for every $\nu>0$, that means:

$$\lim_{n\to+\infty}\|\rho_n-\rho\|_{C(\overline{S_T^*})}=0$$

Let fix a parameter $t_0>0$ such that:

$$q:=2\pi^{\frac{3}{2}}\frac{AC}{M}\sqrt{t_0}<1$$

where C,M are the constant that give exponential decay, and A is defined as in (3.18).
Consider the finite set of numbers $T_k=kt_0$ for $k=0,...,N-1$, with $T_N=T$ $(Nt_0\geq T)$.
Then the function $u(\theta,\omega,\Omega,t):=\rho_{n+1}(\theta,\omega,\Omega,t)-\rho_n(\theta,\omega,\Omega,t)$ solves:

$$\partial^2_{\omega\omega}u+\omega\partial_\omega u-\omega\partial_\theta u+(-\Omega-K_{\rho_{n+1}})\partial_\omega u+u-\partial_t u=(K_{\rho_{n+1}}-K_{\rho_n})\partial_\omega\rho_n,\qquad \text{in } S_T^*\cap(T_k,T_{k+1}) \tag{3.24}$$

with the following initial condition:

$$u(\theta,\omega,\Omega,T_k)=\rho_{n+1}(\theta,\omega,\Omega,T_k)-\rho_n(\theta,\omega,\Omega,T_k)$$

Now we have a nonnull term in the rightside of the equation:

$$f(\theta,\omega,\Omega,t)=(K_{\rho_{n+1}}(\theta,t)-K_{\rho_n}(\theta,t))\partial_\omega\rho_n(\theta,\omega,\Omega,t)$$

but we can still apply all the results stated in Theorem 2.2.1, indeed:

- $|f(\theta,\omega,\Omega,t)|\leq\frac{C_1\mathrm{e}^{C_1(\omega^2+\theta^2)}}{\sqrt{t}}\qquad\forall(\theta,\omega,\Omega,t)\in S_T.$

This is a direct consequence of the exponential decay of $\partial_\omega\rho_n$ and the boundedness of K_{ρ_n}.

- $|f(\theta_1,\omega_1,\Omega,t)-f(\theta_2,\omega_2,\Omega,t)|\leq\frac{C}{t^{1-\gamma}}\,\mathrm{d}((\theta_1,\omega_1,\Omega,t),(\theta_2,\omega_2,\Omega,t))^\beta$

for every $(\theta_1,\omega_1,\Omega)(\theta_2,\omega_2,\Omega)\in M\subset\subset\mathbb{R}^2, t\in(0,T),\gamma,\beta<1$.
Indeed we have shown in (3.16) that $K_{\rho_{n+1}}(\theta,t)$ and $K_{\rho_n}(\theta,t)$ satisfy the previous relation and are bounded; $\partial_\omega\rho_n$ satisfies the previous relation, because of the representation formula and the following relation for the fundamental solution, stated in Theorem 3.2 of [**DPa**]:

$$|\partial_\omega\Gamma_{n,\Omega}(\theta,\omega,t;\xi,\eta,0)-\partial_\omega\Gamma_{n,\Omega}(\tilde{\theta},\tilde{\omega},t;\xi,\eta,0)|\leq c\frac{\mathrm{d}((\theta,\omega),(\tilde{\theta},\tilde{\omega}))^{\frac{1}{2}}}{t^{\frac{3}{4}}}\Gamma^\varepsilon(\theta,\omega,t;\xi,\eta,0)$$

where c is a positive constant. Moreover we know that:

$$|\partial_\omega \rho_n| \leq \frac{C}{\sqrt{t}}$$

Therefore we can apply the parametrix method to this Cauchy problem.
Calling Γ_Ω the fundamental solution of the operator in the left side, the following representation formula holds:

$$\begin{aligned} u(\theta,\omega,\Omega,t) = &\int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\xi,\eta,T_k)(\rho_{n+1}(\xi,\eta,\Omega,T_k) - \rho_n(\xi,\eta,\Omega,T_k))\, d\xi d\eta - \\ - &\int_{T_k}^t \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\xi,\eta,\tau)\left(K_{\rho_{n+1}}(\xi,\tau) - K_{\rho_n}(\xi,\tau)\right)\partial_\omega \rho_n(\xi,\eta,\Omega,\tau) d\xi d\eta d\tau \end{aligned} \tag{3.25}$$

Let us define:

$$\begin{aligned} \|\rho_{n+1}-\rho_n\|_k &:= \sup_{S_T^* \cap (T_{k-1},T_k]} |\rho_{n+1}(\theta,\omega,\Omega,t) - \rho_n(\theta,\omega,\Omega,t)| \mathrm{e}^{M\omega^2}, \qquad k = 0,...,N-1 \\ \|\rho_{n+1}-\rho_n\|_N &:= \sup_{S_T^* \cap (T_{N-1},T)} |\rho_{n+1}(\theta,\omega,\Omega,t) - \rho_n(\theta,\omega,\Omega,t)| \mathrm{e}^{M\omega^2} \end{aligned}$$

which are finite for all n thanks to property **(E)**.
We will now prove the following estimates:

$$\begin{aligned} &\|\rho_{n+1}-\rho_n\|_1 \leq q\|\rho_n - \rho_{n-1}\|_1 \\ &\|\rho_{n+1}-\rho_n\|_{k+1} \leq q\|\rho_n - \rho_{n-1}\|_{k+1} + \|\rho_{n+1}-\rho_n\|_k \end{aligned} \tag{3.26}$$

that hold for all $n = 2, 3, ..$ and $k = 1, 2, ..., N-1$.
Let's split the representation formula (3.25) in the sum of two integrals I_1 and I_2. For $k = 0$ we have $I_1 \equiv 0$, indeed we have:

$$u(\theta,\omega,\Omega,0) = \rho_{n+1}(\theta,\omega,\Omega,0) - \rho_n(\theta,\omega,\Omega,0) \equiv 0$$

because all ρ_n take the initial datum ρ_0. If $k = 1, 2, ..., N-1$ we have:

$$\begin{aligned} |I_1| &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\xi,\eta,T_k)\,|(\rho_{n+1}(\xi,\eta,\Omega,T_k) - \rho_n(\xi,\eta,\Omega,T_k))|\, d\xi d\eta \leq \\ &\leq \|\rho_{n+1}-\rho_n\|_k \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\xi,\eta,T_k)\mathrm{e}^{-M\eta^2} d\xi d\eta \leq \\ &\leq \|\rho_{n+1}-\rho_n\|_k \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\xi,\eta,T_k) d\xi d\eta \leq \|\rho_{n+1}-\rho_n\|_k \end{aligned}$$

where we used property (2.55) of the fundamental solution ($c \equiv 1$).
To estimate I_2 we need first the following:

$$|K_{\rho_{n+1}}(\theta,t) - K_{\rho_n}(\theta,t)| \leq \int_{\mathbb{R}} g(\Omega) \int_{\mathbb{R}}\int_0^{2\pi} |\sin(\theta-\theta')(\rho_n(\theta',\omega,\Omega,t) - \rho_{n-1}(\theta',\omega,\Omega,t))| d\theta' d\omega d\Omega \leq$$

$$\leq A\|\rho_n - \rho_{n-1}\|_k \int_{\mathbb{R}}\int_0^{2\pi} \mathrm{e}^{-M\omega^2} d\theta d\omega = 2\pi^{\frac{3}{2}} \frac{A}{M} \|\rho_n - \rho_{n-1}\|_k$$

Thus we have, using again (2.55),

$$\begin{aligned}
|I_2| &\leq \int_{T_k}^{t}\int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma_\Omega(\theta,\omega,t;\xi,\eta,\tau)\left|K_{\rho_{n+1}}(\xi,\tau)-K_{\rho_n}(\xi,\tau)\right|\left|\partial_\omega\rho_n(\xi,\eta,\Omega,\tau)\right|d\xi d\eta d\tau \leq \\
&\leq 2\pi^{\frac{3}{2}}\frac{AC}{M}\|\rho_n-\rho_{n-1}\|_k \int_{T_k}^{t}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\Gamma_\Omega(\theta,\omega,t;\xi,\eta,\tau)\mathrm{e}^{-M\eta^2}}{\sqrt{\tau}} d\xi d\eta d\tau \leq \\
&\leq 2\pi^{\frac{3}{2}}\frac{AC}{M}\|\rho_n-\rho_{n-1}\|_k(\sqrt{t}-\sqrt{T_k}) \leq 2\pi^{\frac{3}{2}}\frac{AC}{M}\|\rho_n-\rho_{n-1}\|_k(\sqrt{T_{k+1}}-\sqrt{T_k}) = \\
&= 2\pi^{\frac{3}{2}}\frac{AC}{M}\|\rho_n-\rho_{n-1}\|_k\sqrt{t_0} = q\|\rho_n-\rho_{n-1}\|_k
\end{aligned}$$

thus proving (3.26).
We show now that, for all $n = 3, 4, ...$ and $k = 1, ..., N$, the following inequality holds:

$$\|\rho_{n+1}-\rho_n\|_k \leq q^{n-2}\sum_{i=0}^{k-1}(n-2)^i\|\rho_3-\rho_2\|_{k-i} \tag{3.27}$$

The proof runs by induction. Let's show that for $k = 1$ the inequality holds. Using (3.26) we get:

$$\|\rho_{n+1}-\rho_n\|_1 \leq q\|\rho_n-\rho_{n-1}\|_1 \leq ... \leq q^{n-2}\|\rho_3-\rho_2\|_1$$

Suppose now that the relation holds for k, we show this implies it is true also for $k+1$. Using (3.26) we get:

$$\begin{aligned}
&\|\rho_{n+1}-\rho_n\|_{k+1} \leq q\|\rho_n-\rho_{n-1}\|_{k+1} + q^{n-2}\sum_{i=0}^{k-1}(n-2)^i\|\rho_3-\rho_2\|_{k-i} \leq \\
&\leq q\left(q\|\rho_{n-1}-\rho_{n-2}\|_{k+1} + q^{n-3}\sum_{i=0}^{k-1}(n-3)^i\|\rho_3-\rho_2\|_{k-i}\right) + q^{n-2}\sum_{i=0}^{k-1}(n-2)^i\|\rho_3-\rho_2\|_{k-i} \leq \\
&\leq q^2\|\rho_{n-1}-\rho_{n-2}\|_{k+1} + 2q^{n-2}\sum_{i=0}^{k-1}(n-2)^i\|\rho_3-\rho_2\|_{k-i} \leq ... \leq \\
&\leq q^{n-2}\|\rho_3-\rho_2\|_{k+1} + (n-2)q^{n-2}\sum_{i=0}^{k-1}(n-2)^i\|\rho_3-\rho_2\|_{k-i} = \\
&= q^{n-2}\sum_{i=-1}^{k-1}(n-2)^{i+1}\|\rho_3-\rho_2\|_{k-i} = q^{n-2}\sum_{i=0}^{k}(n-2)^i\|\rho_3-\rho_2\|_{k+1-i}
\end{aligned}$$

To show the uniform convergence to ρ we first note that:

$$\|\rho_{n+1}-\rho_n\|_{C(\overline{S_T^*})} \leq \sum_{k=1}^{N}\|\rho_{n+1}-\rho_n\|_k$$

Then, relation (3.27) implies

$$\|\rho_{n+1}-\rho_n\|_{C(\overline{S_T^*})} \leq \sum_{k=1}^{N} q^{n-2}\sum_{i=0}^{k-1}(n-2)^i\|\rho_3-\rho_2\|_{k-i}$$

Using the following relation:

$$\sum_{k=1}^{N}\sum_{i=0}^{k-1}\|\rho_3-\rho_2\|_{k-i} = \sum_{k=1}^{N}\sum_{j=1}^{k}\|\rho_3-\rho_2\|_j \leq \sum_{k=1}^{N}\sum_{j=1}^{N}\|\rho_3-\rho_2\|_j = N\sum_{j=1}^{N}\|\rho_3-\rho_2\|_j$$

we have

$$\|\rho_{n+1}-\rho_n\|_{C(\overline{S_T^*})} \leq q^{n-2}(n-2)^N N \sum_{j=1}^{N} \|\rho_3-\rho_2\|_j$$

If we set

$$M = N\sum_{j=1}^{N} \|\rho_3-\rho_2\|_j$$

the following relation holds:

$$\sum_{n=3}^{+\infty} \|\rho_{n+1}-\rho_n\|_{C(\overline{S_T^*})} \leq \sum_{n=3}^{+\infty} Mq^{n-2}(n-2)^N = M\sum_{n=1}^{+\infty} q^n n^N < \infty$$

as we have

$$\lim_{n\to+\infty} \sqrt[n]{q^n n^N} = q < 1$$

This means that $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\overline{S_T^*})$, thus there exists a function $\rho(\theta,\omega,\Omega,t)$ such that:

$$\lim_{n\to+\infty} \|\rho_n-\rho\|_{C(\overline{S_T^*})} = 0$$

We conclude the proof of the existence of a solution leaning on Ascoli-Arzelà theorem. As we saw, for every fixed $\delta > 0$ the sequences $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\partial\rho_n\}_{n\in\mathbb{N}}$, where ∂ stands for $\partial_\omega, \partial^2_{\omega\omega}$ or Y, are uniformly bounded and uniformly continuous in $S_T^*\cap\{t\geq\mu\}$, moreover $\{\rho_n\}_{n\in\mathbb{N}}$ is uniformly bounded and uniformly continuous in $\overline{S_T^*}$. This means that, using the periodicity in θ, there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} \|\rho_{n_k}-\rho\|_{C(\overline{S_T^*}\cap\omega\in[-W,W])} = 0$$
$$\lim_{k\to+\infty} \|\partial\rho_{n_k}-\partial\rho\|_{C(\overline{S_T^*}\cap\omega\in[-W,W]\cap\{t\geq\mu\})} = 0$$

for any $W > 0$.
Then we have uniform convergence in $\overline{S_T^*}$ for the exponential decay in ω.
The uniform bound in Lipschitz norm implies ρ is Lipschitz in $S_T^*\cap\{t\geq\mu\}$. Indeed from (3.22) follows:

$$|\rho(\theta,\omega,\Omega,t)-\rho(\tilde\theta,\tilde\omega,\Omega,\tilde t)| \leq \limsup_k |\rho_{n_k}(\theta,\omega,\Omega,t)-\rho_{n_k}(\tilde\theta,\tilde\omega,\Omega,\tilde t)| \leq L\,\mathrm{d}((\theta,\omega,t),(\tilde\theta,\tilde\omega,\tilde t))$$

Let consider the subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ in (3.20): passing to the limit as $k\to+\infty$ and using Lebesgue theorem we get:

$$\begin{cases}\partial^2_{\omega\omega}\rho + \partial_\omega[(\omega-\Omega-K_\rho(\theta,t))\rho] - \omega\partial_\theta\rho - \partial_t\rho = 0\\ \rho(\theta,\omega,\Omega,0) = \rho_0(\theta,\omega,\Omega)\end{cases}$$

where:

$$K_\rho(\theta,t) = K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi} g(\Omega')\sin(\theta'-\theta)\rho(\theta',\omega',\Omega',t)d\theta' d\omega' d\Omega'$$

Thus ρ is a classical solution of our Cauchy problem.

We show now the continuity with respect to Ω.
For every $\Omega_1,\Omega_2\in\mathbb{R}$, let's consider $\rho(\theta,\omega,\Omega_1,t)$ and $\rho(\theta,\omega,\Omega_2,t)$. The continuity of ρ_0 with respect to Ω implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$|\rho_0(\theta,\omega,\Omega_1)-\rho_0(\theta,\omega,\Omega_2)| < \varepsilon$$

for every $\Omega_1, \Omega_2 \in \mathbb{R}$ such that $|\Omega_2 - \Omega_1| < \delta$. We suppose without loss of generality that $\delta < \varepsilon$.
Let define $f(\theta, \omega, t, \Omega_1, \Omega_2) := \rho(\theta, \omega, \Omega_1, t) - \rho(\theta, \omega, \Omega_2, t)$.
This function satisfies the following Cauchy problem in S_T^*:

$$\begin{cases} \partial^2_{\omega\omega} f + \omega\partial_\omega f - \omega\partial_\theta f - \Omega_1\partial_\omega f - K_\rho(\theta,t)\partial_\omega f + f - \partial_t f = (\Omega_1 - \Omega_2)\partial_\omega\rho(\theta,\omega,\Omega_2,t) \\ f(\theta,\omega,0,\Omega_1,\Omega_2) = \rho_0(\theta,\omega,\Omega_1,0) - \rho_0(\theta,\omega,\Omega_2,0) \end{cases} \tag{3.28}$$

The operator associated to the equation in (3.28) is in the form (2.52), and we have a nonnull right term:

$$h(\theta,\omega,t,\Omega_1,\Omega_2) = (\Omega_1 - \Omega_2)\partial_\omega\rho(\theta,\omega,\Omega_2,t)$$

Arguing as we did when we considered (3.24), we can consider the fundamental solution Γ of the operator in the left side. Then the following representation formula holds:

$$f(\theta,\omega,t,\Omega_1,\Omega_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma(\theta,\omega,t;\xi,\eta,0)\left(\rho_0(\xi,\eta,\Omega_1,0) - \rho_0(\xi,\eta,\Omega_2,0)\right) d\xi d\eta -$$
$$- \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \Gamma(\theta,\omega,t;\xi,\eta,0)\left((\Omega_1 - \Omega_2)\partial_\omega\rho(\xi,\eta,\Omega_2,t)\right)) d\xi d\eta d\tau$$

Then property (2.55), the continuity of ρ_0 and property **(E)** for $\partial_\omega\rho(\theta,\omega,\Omega_2,t)$ give us:

$$|f(\theta,\omega,t,\Omega_1,\Omega_2)| = |\rho(\theta,\omega,\Omega_1,t) - \rho(\theta,\omega,\Omega_2,t)| \leq \varepsilon + K\varepsilon$$

where K is a positive constant that does not depend on Ω_1 and Ω_2.
This means that ρ is continuous also with respect to Ω.

We conclude with the proof of the uniqueness of the solution. We can't use directly the maximum principle, because when we consider two solutions the term (3.19) change, but we can still apply a maximum like method to prove our thesis.
Let's suppose that ρ_1 and ρ_2 are two solution of our Cauchy problem. For every $\lambda > 0$ we define:

$$\overline{\rho} := (\rho_1 - \rho_2)\mathrm{e}^{-\lambda t}$$

From property **(E)** we have that $\overline{\rho}$ is a Lipschitz continuous bounded function. A direct computation shows that $\overline{\rho}$ is solution of:

$$\partial^2_{\omega\omega}\overline{\rho} + \omega\partial_\omega\overline{\rho} - \omega\partial_\theta\overline{\rho} + (-\Omega - K_{\overline{\rho}})\partial_\omega\overline{\rho} + \overline{\rho} - \partial_t\overline{\rho} = \lambda\overline{\rho} + \mathrm{e}^{-\lambda t}\left[K_{\rho_1}\partial_\omega\rho_1 - K_{\rho_2}\partial_\omega\rho_2 - K_{\overline{\rho}}\partial_\omega(\rho_1 - \rho_2)\right]$$

We first note that the rightside term which is multiplied by $\mathrm{e}^{-\lambda t}$ is finite for every $t > 0$ because of the continuity, the periodicity and the exponential decay of ρ_1, ρ_2, $\partial_\omega\rho_1$ and $\partial_\omega\rho_2$.
Let define:

$$L := \max_{\overline{S_T}} |\overline{\rho}| < +\infty$$

which is finite, for the reason we explained above. Let's call $M \in \overline{S_T}$ the point such that $\overline{\rho}(M) = L$, there are two cases:

1. M is a nonnegative maximum point;

2. M is a nonpositive minimum point;

Let's start with the first case, being M a maximum point we have:

$$\partial_\omega \overline{\rho}(M) = 0; \quad \partial^2_{\omega\omega}\overline{\rho}(M) \leq 0; \quad Y\overline{\rho}(M) \leq 0$$

where the last relation comes from the fact that $M \in (0,T]$, because in $t = 0$ we have $\overline{\rho} = \rho_0 - \rho_0 = 0$.
This implies:

$$\overline{\rho}(M) \leq -\frac{\mathrm{e}^{-\lambda t_M}\left[K_{\rho_1}(M)\partial_\omega\rho_1(M) - K_{\rho_2}(M)\partial_\omega\rho_2(M) - K_{\overline{\rho}}(M)\partial_\omega(\rho_1-\rho_2)(M)\right]}{\lambda - 1}$$

This relation holds for every $\lambda > 0$, this means that we can choose λ such that:

$$\overline{\rho}(M) \leq 0$$

But M is a nonnegative point, so $\overline{\rho}(M) = 0$.
For the second case analogous considerations follows, being M a minimum point we have:

$$\partial_\omega \overline{\rho}(M) = 0; \quad \partial^2_{\omega\omega}\overline{\rho}(M) \geq 0; \quad Y\overline{\rho}(M) \geq 0$$

where the last relation comes from the fact that $M \in (0,T]$, because in $t = 0$ we have $\overline{\rho} = \rho_0 - \rho_0 = 0$.
This implies:

$$\overline{\rho}(M) \geq -\frac{\mathrm{e}^{-\lambda t_M}\left[K_{\rho_1}(M)\partial_\omega\rho_1(M) - K_{\rho_2}(M)\partial_\omega\rho_2(M) - K_{\overline{\rho}}(M)\partial_\omega(\rho_1-\rho_2)(M)\right]}{\lambda - 1}$$

This relation holds for every $\lambda > 0$, this means that we can choose λ such that:

$$\overline{\rho}(M) \geq 0$$

But M is a nonpositive minimum point, so $\overline{\rho}(M) = 0$.
Then we have $\overline{\rho} \equiv 0$, and therefore $\rho_1 \equiv \rho_2$. □

Bibliography

[AP] F.Anceschi, S.Polidoro, *A survey on the classical theory for Kolmogorov equation*, https://arxiv.org/abs/1907.05155, (2019).

[AS] J.A. Acebroan, R. Spigler, *Adaptive frequency model for phase-frequency synchronization in large populations of globally coupled nonlinear oscillators*, Phys. Rev. Lett. 81, 14 September 1998, 229-2232.

[BLU] A.Bonfiglioli, E.Lanconelli, F.Uguzzoni, *Stratified Lie Groups and Potential Theory for their Sub-Laplacians*, Springer, (2007).

[DP] M. Di Francesco and A. Pascucci, *On a class of degenerate parabolic equations of Kolmogorov type*, AMRX Appl. Math. Res. Express, (2005), pp. 77–116.

[DPa] M. Di Francesco and A. Pascucci,*A continuous dependence result for ultra-parabolic equations in option pricing*, J. Math. Anal. Appl. (2007), doi:10.1016/j.jmaa.2007.03.031

[DPo] M. Di Francesco and S. Polidoro, *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form*, Adv. Differential Equations, 11 (2006), pp. 1261–1320.

[E] B. Ermentrout, *An adaptive model for synchrony in the firefly Pteroptyx malaccae*, J. Math. Biol. 29, no. 6 (1991), 571-585.

[H] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), pp. 147–171.

[K] A. Kolmogorov, *Zufllige bewegungen. (zur theorie der brownschen bewegung.)*., Ann. of Math., II. Ser., 35 (1934), pp. 116–117.

[Ku] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer, Berlin, 1984.

[1] PP S.Pagliarani, M.Pignotti *Intrinsic Taylor formula for non-homogeneous Kolmogorov-type Lie groups*, http://arxiv.org/abs/1707.01422v2

[LP] E. Lanconelli, S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Pol. Torino 51,4 (1993), 137-171.

[M] M. Manfredini, *The Dirichlet problem for a class of ultraparabolic equations*, Adv. Differential Equations, 2 (1997), pp. 831–866.

[P] S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Matematiche (Catania), 49 (1994), 53-105.

[K1] L.P. Kuptsov, *The mean value property and the maximum principle for parabolic equation of second order*, Dokl. Akad. Nauk SSSR. 242 (1978), 529-532. English transl. Soviet Math. Dokl. 19 (1978), 1140-1144.

[S] S.Strogatz, *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators*, Physica D: Nonlinear Phenomena, vol. 143, pagg. 1-20, 2000

[Sa] H. Sakaguchi, *Cooperative phenomena in coupled oscillator systems under external fields* Progr. Theoret. Phys. 79 (1988) 39

[SM] S.H. Strogatz, R.E. Mirollo, *Stability of incoherence in a population of coupled oscillators* J. Statist. Phys. 63 (1991) 613.

[Sp] R.Spigler, *The mathematics of Kuramoto models which describe synchronization phenomena*, Matematica, Cultura e Società. Rivista dell'Unione Matematica Italiana, Serie 1, Vol. 1 (2016), n.2, p. 123–132

[Sp1] M.M.Lavrentiev Jr.,R. Spigler, and A. Tani, *Existence, uniqueness, and regularity for the Kuramoto-Sakaguchi equation with unboundedly supported frequency distribution*, Differential Integral Equations,27, No. 9-10 (2014), 879-892.

[Sp2] D.R.Akhmetov, M.M.Lavrentiev Jr., and R.Spigler, *Existence and uniqueness of classical solutions to certain nonlinear integrodifferential Fokker-Planck-type equations*, Electron. J. Differential Equations, Vol. 2002 (2002), No. 24, pp. 1-17.

[TLO] H.Tanaka, A.Lichtenberg, S.Oishi, *Self-synchronization of coupled oscillators with hysteretic responses*, Physica D 100(1997),279-300

[K2] L.P.Kuptsov, *Fundamental solution of certain degenerate second order parabolic equations*, Mat. Zametki 31 (1982), 559-570. English transl. Math. Notes 31 (1982), 283-289.