OPTIMAL DECAY RATE FOR A DEGENERATE HYPERBOLIC-PARABOLIC COUPLED SYSTEM*

ZHONG-JIE HAN †, KAI YU ‡, AND ENRIQUE ZUAZUA §¶‖

Abstract. We investigate the long-time asymptotic behavior of a one-dimensional degenerate hyperbolic-parabolic coupled PDE system modeling diffusion-wave interactions with degeneracy at the interface. The system consists of a degenerate heat-wave equation on a finite interval, where the degeneracy strengths of the diffusion and wave propagation are characterized respectively by the parameters α and β, with $\alpha, \beta \in [0, 1)$. For smooth initial data, we establish that the system exhibits polynomial stabilization to zero as $t \to \infty$, with an explicit polynomial decay rate of order $\frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}$, determined by the degeneracy exponents α and β. A rigorous spectral analysis of the system operator further confirms the optimality of this decay rate. This constitutes the first proof of the sharp decay rate estimate for degenerate heat-wave coupled systems over the full range of exponents $\alpha, \beta \in [0, 1)$. Our methodology combines frequency-domain techniques with asymptotic properties of Bessel functions and a detailed analysis of the underlying Sturm-Liouville structure. The results reveal the stabilizing role of degenerate diffusion and provide novel insights into the interplay between degeneracy and dissipation in hyperbolic-parabolic coupled systems.

Key words. Degenerate PDE, heat-wave system, optimal polynomial decay rate, resolvent estimates.

1. Introduction.

1.1. Problem formulation.

Coupled heat-wave systems arise in physical and engineering contexts involving fluid-structure interactions, where energy propagates through both conservative and dissipative mechanisms. Such systems appear in applications such as airflow dynamics along aircraft surfaces and heart valve motion (see [27]). In classical settings, the interaction between hyperbolic and parabolic dynamics has been extensively studied under uniform, non-degenerate conditions. However, in many real-world applications, such as fluid flow in narrow channels or heat conduction in inhomogeneous media, the diffusion coefficient or wave speed may vanish at the interface or boundary, leading to degenerate equations (see [1, 14]). These degeneracies significantly alter the dynamics, reduce regularity, and complicate the spectral structure of the associated operators. Despite growing interest in degenerate parabolic and hyperbolic equations separately, little is known about the precise long-time behavior of fully degenerate coupled systems. Our work addresses this gap by analyzing a prototypical one-dimensional model with degeneracy on both components and establishing optimal decay rates that reflect the precise influence of the degeneracy parameters.

In this paper, we focus on the long-time behavior of a coupled degenerate heat-wave system, which is described by the following one-dimensional partial differential

†School of Mathematics and KL-AAGDM, Tianjin University, Tianjin 300350, China. (Email address: zjhan@tju.edu.cn).

‡College of Mathematics and Information Science, Center for Mathematical Sciences, Nanchang Hangkong University, Nanchang 330063, China (Email address: yukyle11757@163.com).

§Chair for Dynamics, Control and Numerics-Alexander von Humboldt-Professorship, Department of Data Science, Friedrich-Alexander-Universität Erlangen-Nürnberg, 91058 Erlangen, Germany. (Email address: enrique.zuazua@fau.de).

¶Chair of Computational Mathematics, Fundacion Deusto, Av. de las Universidades, 24, 48007 Bilbao, Basque Country, Spain.

‖Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.

*Z. J. Han was funded by the Natural Science Foundation of China grant NSFC-62473281, 12571484. E. Zuazua was funded by the Alexander von Humboldt-Professorship program, the European Research Council (ERC) under the European Union's Horizon 2030 research and innovation programme (grant agreement NO. 101096251-CoDeFeL), the ModConFlex Marie Curie Action, HORIZON-MSCA-2021-DN-01, Transregio 154 Project "Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks" of the DFG, AFOSR-24IOE027, grants PID2020-112617GB-C22 and TED2021-131390B-I00 of MICINN (Spain), the SURE-AI Center grant 357482, Research Council of Norway.

equations:

$$\begin{cases} u_{tt}(x,t) - (bu_x)_x(x,t) = 0, & x \in (-1,0),\ t>0,\\ w_t(x,t) - (aw_x)_x(x,t) = 0, & x \in (0,1),\ t>0,\\ u(-1,t) = w(1,t) = 0, & t>0,\\ w(x,0) = w_0(x), & x\in(0,1),\\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), & x \in (-1,0), \end{cases} \tag{1.1}$$

coupled through the following transmission conditions at $x = 0$:

$$u_t(0,t) = w(0,t), \quad (bu_x)(0^-,t) = (aw_x)(0^+,t), \quad \forall t \in \mathbb{R}_+. \tag{1.2}$$

Here u_0, v_0 and w_0 are the initial data. The functions $a(x)$ and $b(x)$ represent the diffusion and wave speed coefficients, respectively. Both degenerate at the interface point $x = 0$, and are given by

$$\begin{cases} a(x) = x^\alpha, & x \in (0,1],\ 0 \le \alpha < 1,\\ b(x) = (-x)^\beta, & x \in [-1,0),\ 0 \le \beta < 1. \end{cases} \tag{1.3}$$

Note that we focus on the range of exponents $0 \le \alpha, \beta < 1$. When $\alpha \ge 1$ or $\beta \ge 1$, the nature of the model changes fundamentally, entering into the strongly degenerate regime as characterized by Cannarsa *et al.* in [12, 13] (see Remark 2.5 below).

The natural energy $E(t)$ of system (1.1)–(1.2) at time $t \ge 0$ is defined as

$$E(t) = \frac{1}{2}\Big[\int_{-1}^{0} \big(b(x)|u_x(x,t)|^2 + |u_t(x,t)|^2\big)\,dx + \int_0^1 |w(x,t)|^2 dx\Big], \quad \forall t \in \mathbb{R}_+.$$

A direct calculation yields that $\frac{d}{dt}E(t) = -\int_0^1 a(x)|w_x(x,t)|^2 dx \le 0, \quad \forall t \in \mathbb{R}_+$. Hence, the energy $E(t)$ is non-increasing over time, with dissipation arising from the heat conduction mechanism in the subregion $(0, 1)$. The objective of this work is to establish a sharp decay rate for the system over the range of exponents $0 \le \alpha, \beta < 1$.

1.2. Main contributions. Recent progress on decay rates for hyperbolic - parabolic coupled systems is summarized below, with the key contributions of this work presented in Table 1.1:

Table 1.1
Decay Rates for Degenerate Coupled Heat-Wave Systems: Past Results and New Contributions

Source(Year)	**Exponent Range**	**Decay Rate**	**Optimality Status**
[38] (2003), [39] (2004), [7] (2016)	$\alpha = \beta = 0$	t^{-2}	Yes
[20] (2020)	$\alpha \in [0,1),\ \beta = 0$	$t^{-\frac{3-\alpha}{2(1-\alpha)}}$	No
[32] (2022)	$\alpha \in [0,1),\ \beta = 0$	Piecewise polynomial	No
[19] (2023)	$\alpha \in [0,1),\ \beta = 0$	$t^{-\frac{2-\alpha}{1-\alpha}}$	Unknown
This work	$\alpha, \beta \in [0,1)$	$t^{-\frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}}$	**Yes**
▷Special case 1	$\alpha \in [0,1),\ \beta = 0$	$t^{-\frac{2-\alpha}{1-\alpha}}$	**Yes**
▷Special case 2	$\beta \in [0,1),\ \alpha = 0$	$t^{-\frac{2(2-\beta)}{2+\beta}}$	**Yes**

- Zhang and Zuazua [38, 39] investigated the constant-coefficient coupled heat-wave system ($\alpha = \beta = 0$ in (1.1)) under two distinct transmission conditions. Through detailed spectral analysis, they established the optimal polynomial decay rate t^{-2} for smooth initial conditions. The same optimal decay rate was also obtained by Batty *et al.* [7] for the system with Neumann boundary condition at the end of the wave component, using frequency domain method.

- Han *et al.* [20] examined the stabilization effect of degenerate parabolic operators in coupled systems, considering constant-coefficient wave equations coupled with degenerate parabolic equations ($\beta = 0$, $\alpha \in [0, 1)$ in (1.1)(1.2)). For smooth initial conditions, they derived the decay rate $t^{-\frac{3-\alpha}{2(1-\alpha)}}$, noting its non-optimality as $\alpha \to 0^+$ due to its inconsistency with the known optimal rate t^{-2} given in [38].
- Tebou [32] subsequently refined the estimates for $\beta = 0$, obtaining piecewise polynomial rates:
  - t^{-2} for $\alpha \in [0, \frac{1}{4}]$;
  - $t^{-\frac{3}{2(1-\alpha)}}$ for $\alpha \in [\frac{1}{4}, \frac{1}{2}]$;
  - $t^{-\frac{9-6\alpha}{4(1-\alpha)}}$ for $\alpha \in [\frac{1}{2}, \frac{3}{4}]$;
  - $t^{-\frac{3-\alpha}{2(1-\alpha)}}$ for $\alpha \in [\frac{3}{4}, 1)$.
- Most recently, Han *et al.* [19] derived the unified decay rate $t^{-\frac{2-\alpha}{1-\alpha}}$ for $\beta = 0$, $\alpha \in [0, 1)$. However, the optimality of this rate remains undetermined.

In contrast, the stability and sharp decay behavior for the jointly degenerate casewhere both the parabolic and hyperbolic components simultaneously degenerate (i.e., $\alpha, \beta \in [0, 1)$)has remained unexplored.

This work makes two fundamental contributions:

- We rigorously confirm the optimality of the decay rate $t^{-\frac{2-\alpha}{1-\alpha}}$ for the heat-degenerate case ($\beta = 0$, $\alpha \in [0, 1)$), originally established in [19].
- More significantly, we establish the optimal polynomial decay rate $t^{-\frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}}$ for the jointly degenerate system, with $\alpha, \beta \in [0, 1)$. This result quantifies the combined effects of degeneracy in both diffusion and wave components, generalizing the partially degenerate case ($\beta = 0$, $\alpha \in [0, 1)$) from [19].

Our findings reveal a dual role of degeneracy: for fixed β, increasing α (enhancing heat degeneracy near the interface) accelerates decay, while for fixed α, increasing β (intensifying wave degeneracy) decelerates decay. Consequently, heat degeneracy enhances stabilization, whereas wave degeneracy tends to impede it. This suggests the following physical interpretation: stronger heat degeneracy (larger α) may facilitate deeper wave penetration from the wave medium into the dissipative heat region, thereby enhancing energy dissipation. By contrast, increased wave degeneracy (larger β) slows wave propagation, delaying energy transfer to the interface and ultimately weakening the stabilizing influence of the parabolic component.

This work presents the first rigorous analysis of optimal energy decay rates for jointly degenerate coupled heat-wave systems. Unlike prior studies on non-degenerate or partially degenerate configurations, the simultaneous degeneracy of both components poses substantial analytical challenges, particularly in the spectral analysis of the associated non-self-adjoint operator. Our main result establishes the sharp polynomial decay rate with explicit dependence on the degeneracy exponents α and β.

The proof combines frequency-domain techniques with refined asymptotic analysis of Bessel functions, and careful analysis of the underlying degenerate Sturm-Liouville problem. More specifically, we exploit properties of Bessel functions to analyze the spectrum of the system operator and thereby derive a lower bound on the growth of the resolvent norm along the imaginary axis. We then establish an explicit resolvent estimate showing that this lower bound is also an upper bound. An application of the frequency-domain criterion for polynomial stability in [5] then yields the optimal decay rate of the system.

These findings advance the understanding of dissipation mechanisms in mixed-type PDEs with interface degeneracies.

1.3. Related work. This subsection reviews closely related literature on controllability and stability for coupled degenerate PDE models.

In [41], null controllability was established for a constant-coefficient heat-wave system with boundary control acting on the wave component. Significant results on decay rates for coupled heat-wave PDEs in networked or multidimensional settings appear in [4, 8, 15, 21, 31, 40] and the references therein. Notably, existing literature primarily focuses on constant-coefficient heat-wave systems, corresponding to $\alpha = \beta = 0$ in (1.1).

The degenerate heat equation

$$w_t - (x^\alpha w_x)_x = 0,\ x \in (0,1), t > 0,$$

was first studied in the context of controllability using Carleman estimates in [12, 14] for $\alpha \in (0,2)$. Subsequently, the coefficient x^α was generalized to functions $a(x) \sim x^\alpha$, with $\alpha \in (0,2)$ in [2]. Further generalizations involving singular potentials were also studied, such as

$$w_t - (x^{\alpha_1} w_x)_x - \frac{\lambda}{x^{\alpha_2}} w = 0,\ x \in (0,1), t > 0,$$

which arise in quantum mechanics and combustion theory (see [6, 10, 11]). The null controllability and stability properties of such systems, as well as the corresponding Carleman estimates, have been studied in [33, 34] for the case $\alpha_1 = 0$, $\alpha_2 = 2$, and in [35] for $\alpha_1 \in [0,2)$ and certain suitable choices of α_2 and λ. Further developments can also be found in the monograph [17].

Compared with the degenerate heat equation, the degenerate hyperbolic equation

$$u_{tt} - (x^\beta u_x)_x = 0,\ x \in (0,1), t > 0,$$

has received considerably less attention. In [1], the authors analyzed the observability of such degenerate wave models for two distinct parameter ranges: $0 < \beta < 1$ and $1 \leq \beta < 2$. Furthermore, they established exponential decay estimates for both linear and nonlinear feedback control frameworks.

1.4. Article's structure. The paper is organized as follows: In Section 2, we present the main results. Section 3 provides the necessary preliminary material for the subsequent analysis. Sections 4 and 5 are devoted to the proofs of the main results. Finally, Section 6 offers concluding remarks and outlines several open problems.

2. Functional setting and main results.

We first introduce the following weighted Sobolev spaces. Fix $\beta \in [0,1)$, and define

$$\begin{cases} H^1_\beta(-1,0) := \left\{ u \in L^2(-1,0) \,\middle|\, \begin{array}{l} u \text{ is absolutely continuous on } [-1,0), \\ |x|^{\beta/2} u' \in L^2(-1,0), \end{array} \right\}; \\ H^1_{\beta,0}(-1,0) := \left\{ u \in H^1_\beta(-1,0) \middle| u(-1) = 0 \right\}; \\ H^2_\beta(-1,0) := \left\{ u \in H^1_\beta(-1,0) \middle| |x|^\beta u' \in H^1(-1,0) \right\}. \end{cases} \tag{2.1}$$

Here $H^1_\beta(-1,0)$ and $H^1_{\beta,0}(-1,0)$ are Hilbert spaces endowed with the following inner product:

$$\langle u, v \rangle_{H^1_\beta(-1,0)} = \int_{-1}^0 \left(|x|^\beta u'(x)\overline{v'(x)} + u(x)\overline{v(x)} \right) dx,\ \forall u, v \in H^1_\beta(-1,0),$$

and

$$\langle u, v \rangle_{H^1_{\beta,0}(-1,0)} = \int_{-1}^0 |x|^\beta u'(x)\overline{v'(x)} dx,\ \forall u, v \in H^1_{\beta,0}(-1,0).$$

By the weighted Hardy's inequality, one easily sees that the norm $\|\cdot\|_{H^1_{\beta,0}}$ is equivalent to the norm $\|\cdot\|_{H^1_\beta}$. Additionally, we note that $H^1_{\beta,0}(-1,0)$ corresponds to the classical

Sobolev space $H^1_\Gamma(-1,0) := \{u \in H^1(-1,0) | u(-1) = 0\}$ as $\beta = 0$. Moreover, we will use the facts that $H^1_{\beta,0}(-1,0)$ is continuously embedded in $C([-1,0])$ and is compactly embedded in $L^2(-1,0)$ ([1, 12, 18, 26]).

Notice that $H^1_\alpha(0,1)$, $H^1_{\alpha,0}(0,1)$, and $H^2_\alpha(0,1)$ for any $\alpha \in [0,1)$ can be defined in the same manner. Let us now introduce the energy space as follows:

$$\mathcal{H} = H^1_{\beta,0}(-1,0) \times L^2(-1,0) \times L^2(0,1),$$

with $\|X\|_{\mathcal{H}} := \sqrt{\|f\|^2_{H^1_{\beta,0}(-1,0)} + \|g\|^2_{L^2(-1,0)} + \|h\|^2_{L^2(0,1)}}$, $\forall X = (f,g,h) \in \mathcal{H}$.

Then, for $X = (u,v,w) \in \mathcal{H}$ we define a linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}X = \Big(v, (bu_x)_x, (aw_x)_x\Big), \forall X = (u,v,w) \in D(\mathcal{A}), \tag{2.2}$$

where

$$D(\mathcal{A}) := \Big\{(u,v,w) \in \tilde{\mathcal{H}} : v|_{x=0} = w|_{x=0}, \ \ (bu_x)|_{x=0} = (aw_x)|_{x=0}\Big\}, \tag{2.3}$$

with

$$\tilde{\mathcal{H}} := \left\{(u,v,w) \in \mathcal{H} \ \middle| \ \begin{array}{l} u \in H^2_\beta(-1,0) \cap H^1_{\beta,0}(-1,0), \ v \in H^1_{\beta,0}(-1,0), \\ w \in H^2_\alpha(0,1) \cap H^1_{\alpha,0}(0,1). \end{array} \right\}.$$

Letting $X(t) = (u(\cdot,t), u_t(\cdot,t), w(\cdot,t)) \in \mathcal{H}$, the system (1.1)-(1.2) can be reformulated as the evolution equation:

$$\begin{cases} \dfrac{d}{dt}X(t) = \mathcal{A}X(t), \quad \forall t > 0, \\ X(0) = (u_0, v_0, w_0). \end{cases} \tag{2.4}$$

The first result concerns the well-posedness and strong stability of the evolution equation (2.4), which is achieved by the classical semigroup theory and the use of Bessel functions.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as above. Then:*

(i) $\mathcal{A}$ generates a C_0-contraction semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$;

(ii) $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, and thus, according to [3, 25], the semigroup $e^{\mathcal{A}t}$ is strongly stable.

Now, we give the main results about the long-time asymptotic behavior for system (1.1)-(1.2).

Theorem 2.2. *Assume that the degenerate coefficients $a(x)$ and $b(x)$ are defined by (1.3), with degeneracy orders $\alpha \in [0,1)$ and $\beta \in [0,1)$, and $(u_0,v_0,w_0) \in \mathcal{D}(\mathcal{A})$ is the initial state . Then, the following hold:*

(i) For classical solutions of the system (1.1)–(1.2), there exists a constant $C > 0$ such that

$$\|(u,u_t,w)\|_{\mathcal{H}} \le Ct^{-\frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}} \|(u_0,v_0,w_0)\|_{\mathcal{D}(\mathcal{A})}, \quad t > 1. \tag{2.5}$$

(ii) For any $\alpha \in [0,1)$ and $\beta \in [0,1)$, we have

$$\frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta} = \sup\left\{\gamma > 0 \ \middle| \ \begin{array}{l} \exists C > 0, \ \forall t > 1, \\ \|(u,u_t,w)\|_{\mathcal{H}} \le Ct^{-\gamma}\|(u_0,v_0,w_0)\|_{\mathcal{D}(\mathcal{A})} \end{array} \right\}.$$

Therefore, the decay rate given in (2.5) is optimal.

Remark 2.3. *Theorem 2.2 establishes the optimality of the decay rate $t^{-\frac{2-\alpha}{1-\alpha}}$, which was originally derived for the heat-degenerate case ($\alpha \in [0,1)$, $\beta = 0$) in [19]. As a byproduct of our analysis, we also obtain the optimal decay rate $t^{-\frac{2(2-\beta)}{2+\beta}}$ for the heat-wave system with wave-component degeneracy ($\alpha = 0$, $\beta \in [0,1)$ in* (1.1)-(1.2)*).*

REMARK 2.4. *Theorem 2.2 reveals that for fixed β, decreasing α intensifies heat-induced dissipation yet paradoxically diminishes the decay rate of the system. This phenomenon is similar to the "over-damping effect" in wave equations (see [30]), where excessively strong damping slows down energy decay rather than enhancing it.*

REMARK 2.5. *Throughout this paper, we assume that the degeneracy exponents α and β lie in the interval $[0,1)$. Under this assumption, the differential operators $((-x)^\beta u_x)_x$ and $(x^\alpha w_x)_x$ in equation (1.1) are referred to as weakly degenerate. It is worth noting that when $\alpha \geq 1$ or $\beta \geq 1$, the corresponding operators become strongly degenerate (see, e.g., [1, 12, 13]). In such cases, the homogeneous Neumann boundary conditions $\lim_{x\to 0^-}(-x)^\beta u_x = 0$ and $\lim_{x\to 0^+} x^\alpha w_x = 0$ are automatically satisfied at the degenerate point $x = 0$. As a result, the model described by equations (1.1)(1.2) may no longer be suitable in the strongly degenerate regime. Future research should aim to develop a well-posed alternative model that incorporates appropriate transmission conditions at the interface for such hyperbolicparabolic coupled systems.*

3. Preliminaries.

This section introduces the well-known Bessel functions and outlines some of their fundamental properties. We then examine a specific type of Sturm-Liouville problem that arises in subsequent computations. The structure of solutions to this problem will be determined using the Bessel functions. Throughout this paper, we adopt the convention that the square root of any complex number $\lambda = re^{i\theta}$, with $r \geq 0$ and $\theta \in [-\pi, \pi)$, is defined as $\sqrt{\lambda} := \sqrt{r}\, e^{i\theta/2}$.

3.1. Bessel functions.

3.1.a. Definition of Bessel functions. Let us begin by introducing the definition of Bessel functions. For any fixed real number ν, the Bessel functions of order ν are solutions to the following differential equation:

$$z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0, \forall z \in (0, +\infty). \tag{3.1}$$

This equation is commonly referred to as the Bessel's differential equation of order ν. It is well known that there are four standard solutions to Bessel's differential equation (see [36]). In this work, we will focus exclusively on the first type of Bessel functions, defined as follows: for $\nu \notin \mathbb{N}$,

$$J_{\pm\nu}(z) = \sum_{m=0}^{\infty} c^{\pm}_{\nu,m} z^{2m\pm\nu}, \quad c^{\pm}_{\nu,m} := \frac{(-1)^m}{m!\Gamma(m\pm\nu+1)}\Big(\frac{1}{2}\Big)^{2m\pm\nu}, \ \forall z \geq 0, \tag{3.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

The Wronskian determinant of $J_\nu(z)$ and $J_{-\nu}(z)$ is given by $-\frac{2\sin(\nu\pi)}{\pi z}$ (see [36, Section 3.2]). Consequently, J_ν and $J_{-\nu}$ are linearly independent solutions of (3.1) for any real and non-integer ν. Thus, the pair $(J_\nu, J_{-\nu})$ forms a fundamental system of solutions to equation (3.1), generating a vector space of dimension 2.

3.1.b. Properties of Bessel functions. As shown in [36, Chapter 3], the Bessel function $J_\nu(z)$ is an entire function of z for each fixed $\nu \in \mathbb{C}$. Moreover, from (3.2), it is straightforward to observe that

$$J_\nu(z) \sim c^+_{\nu,0} z^\nu \text{ and } J_{-\nu}(z) \sim c^-_{\nu,0} z^{-\nu} \text{as } z \to 0^+, \tag{3.3}$$

where $c^+_{\nu,0} = \frac{1}{\Gamma(\nu+1)2^\nu}$ and $c^-_{\nu,0} = \frac{2^\nu}{\Gamma(1-\nu)}$. For the asymptotic behavior as $|z|$ sufficiently large, the Bessel functions (3.2) admit the following representation (see Lebedev [23, Section 5.1.6]):

$$\begin{aligned} J_\nu(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \Big[&\cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)\big(1 + O(\frac{1}{|z|^2})\big) \\ &- \frac{(\nu-\frac{1}{2})(\nu+\frac{1}{2})}{2z} \sin(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)\big(1 + O(\frac{1}{|z|})\big)\Big] \end{aligned} \tag{3.4}$$

valid for $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$. Additionally, the Bessel functions satisfy the following recurrence relations:

$$zJ'_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z), \quad zJ'_\nu(z) + \nu J_\nu(z) = zJ_{\nu-1}(z). \tag{3.5}$$

3.1.c. Location of the zeros of the Bessel functions $J_\nu(z)$. The distribution of the zeros of the Bessel functions $J_\nu(z)$ has been thoroughly studied in Watson [36] and Lebedev [23]. Specifically, for any real $\nu \geq -1$, $J_\nu(z)$ has no complex zeros and possesses an infinite number of real zeros, all of which are simple except possibly at $z = 0$. Furthermore, the zeros of $J_\nu(z)$ exhibit several useful properties, which are summarized in the following lemma.

LEMMA 3.1. *[13, 23, 36] For any $\nu \geq -1$, $J_\nu(z)$ has no complex zeros and possesses an infinite number of real zeros. Let $j_{\nu,n}, n \geq 1$, denote the positive zeros. Then:*

(i) The sequence $\{j_{\nu,n}\}_{n\geq 1}$ is infinite and strictly increasing, with $j_{\nu,n} \to +\infty$ as $n \to \infty$.

(ii) The zeros of $J_\nu(z)$ satisfy the asymptotic expansion:

$$j_{\nu,n} = \Big(n + \frac{\nu}{2} - \frac{1}{4}\Big)\pi - \frac{4\nu^2 - 1}{8(n + \frac{1}{2}\nu - \frac{1}{4})\pi} + O\Big(\frac{1}{n^3}\Big) \text{ as } n \to +\infty.$$

(iii) For any $\nu \in [0, \frac{1}{2}]$ and $n \geq 1$, the zeros $j_{\nu,n}$ satisfy

$$\pi(n + \frac{\nu}{2} - \frac{1}{4}) \leq j_{\nu,n} \leq \pi(n + \frac{\nu}{4} - \frac{1}{8}).$$

These results are classical, and complete proofs can be found in Watson's treatise [36].

3.2. The Sturm-Liouville problem.

3.2.a. The homogeneous equation. As an application of Bessel functions, we first consider the following ordinary differential equation:

$$-\big(x^\alpha w'(x)\big)' = \lambda w(x), \forall x \in (0,1),\ \alpha \in [0,1),\ \lambda \in \mathbb{C}. \tag{3.6}$$

The general solution structure for this equation has been established in [13, 18, 22], as summarized in the following proposition:

PROPOSITION 3.2. *For $\nu_\alpha := \frac{1-\alpha}{2-\alpha}$ with $\alpha \in [0,1)$, there exist constants C_+ and C_- such that (3.6) admits the general solution:*

$$w(x) = C_+\Phi_\alpha^+(\lambda, x) + C_-\Phi_\alpha^-(\lambda, x), \quad \forall x \in (0,1), \tag{3.7}$$

where $\Phi_\alpha^+(\lambda, x) := x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big), \quad \Phi_\alpha^-(\lambda, x) := x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big).$

Proof. Following the methodology in [13, 18], we employ the ansatz

$$w(x) := x^{\frac{1-\alpha}{2}} y\big(z(x)\big),\ z(x) := \frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}},$$

where $y(z)$ is assumed to be a smooth function. Direct substitution reveals that $y(z)$ satisfies the classical Bessel equation:

$$z^2 y''(z) + zy'(z) + \Big(z^2 - (\frac{1-\alpha}{2-\alpha})^2\Big)y(z) = 0, \forall z \in (0, \frac{2\sqrt{\lambda}}{2-\alpha}). \tag{3.8}$$

Since $\nu_\alpha = \frac{1-\alpha}{2-\alpha} \in \big(0, \frac{1}{2}\big]$ is non-integer, the fundamental solutions to (3.8) are $J_{\nu_\alpha}(z)$ and $J_{-\nu_\alpha}(z)$, yielding the general solution (3.7). □

3.2.b. The non-homogeneous equation. We now consider the associated non-homogeneous equation:

$$\big(x^{\alpha} w'(x)\big)' + \lambda w(x) = \xi(x), \forall x \in (0,1),\ \alpha \in [0,1),\ \lambda \in \mathbb{C}, \tag{3.9}$$

for $\xi(x) \in L^2(0,1)$. The general solution is derived via the variation of parameters formula, using the fact that the Wronskian determinant of $J_\nu(z)$ and $J_{-\nu}(z)$ is given by $-\frac{2\sin(\nu\pi)}{\pi z}$ for $\nu \notin \mathbb{Z}$; see [36, Section 3.2].

PROPOSITION 3.3. *For any $\alpha \in [0,1)$, let ν_α, $\Phi_\alpha^+(\lambda,x)$, and $\Phi_\alpha^-(\lambda,x)$ be defined as in Proposition 3.2. Then for any $\xi(x) \in L^2(0,1)$, there exist two constants C_+ and C_- such that equation (3.9) has the following general solution: $\forall x \in (0,1)$,*

$$\begin{aligned} w(x) &= C_+\Phi_\alpha^+(\lambda,x) + C_-\Phi_\alpha^-(\lambda,x) \\ &+ \frac{\pi}{(2-\alpha)\sin(\nu_\alpha\pi)} \int_0^x \xi(\tau)\Big(\Phi_\alpha^+(\lambda,x)\Phi_\alpha^-(\lambda,\tau) - \Phi_\alpha^+(\lambda,\tau)\Phi_\alpha^-(\lambda,x)\Big)d\tau. \end{aligned} \tag{3.10}$$

The proof follows the standard variation of parameters method and is detailed in [37].

3.3. Asymptotics and technical estimates. We now establish fundamental properties of the solutions $\Phi_\alpha^\pm(\lambda,x)$. Let

$$\tilde{c}^{+}_{\nu_\alpha,m}(\lambda) := c^{+}_{\nu_\alpha,m}\Big(\frac{2\sqrt{\lambda}}{2-\alpha}\Big)^{2m+\nu_\alpha}, \quad \tilde{c}^{-}_{\nu_\alpha,m}(\lambda) := c^{-}_{\nu_\alpha,m}\Big(\frac{2\sqrt{\lambda}}{2-\alpha}\Big)^{2m-\nu_\alpha}.$$

Using the series expansion (3.2), we derive

$$\Phi_\alpha^+(\lambda,x) = \sum_{m=0}^{\infty} \tilde{c}^{+}_{\nu_\alpha,m}(\lambda) x^{m(2-\alpha)+(1-\alpha)}, \quad \Phi_\alpha^-(\lambda,x) = \sum_{m=0}^{\infty} \tilde{c}^{-}_{\nu_\alpha,m}(\lambda) x^{m(2-\alpha)}, \tag{3.11}$$

from which we obtain the asymptotic behavior:

$$\Phi_\alpha^+(\lambda,x) \sim \tilde{c}^{+}_{\nu_\alpha,0}(\lambda)x^{1-\alpha}, \quad \Phi_\alpha^-(\lambda,x) \sim \tilde{c}^{-}_{\nu_\alpha,0}(\lambda) \ \text{ as } x \to 0^+. \tag{3.12}$$

Applying the recurrence relation (3.5), we derive the derivative relationships:

$$x^{\frac{\alpha}{2}}(\Phi_\alpha^+)_x(\lambda,x) = (1-\alpha)x^{-\frac{1}{2}} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big) - \sqrt{\lambda}x^{\frac{1-\alpha}{2}} J_{\nu_\alpha+1}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big), \tag{3.13}$$

and

$$x^{\frac{\alpha}{2}}(\Phi_\alpha^-)_x(\lambda,x) = -\sqrt{\lambda}x^{\frac{1-\alpha}{2}} J_{1-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big). \tag{3.14}$$

Then using (3.2) again, we obtain, as $x \to 0^+$,

$$x^{\frac{\alpha}{2}}(\Phi_\alpha^+)_x(\lambda,x) \sim (1-\alpha)\tilde{c}^{+}_{\nu_\alpha,0}(\lambda)x^{-\frac{\alpha}{2}},\ x^{\frac{\alpha}{2}}(\Phi_\alpha^-)_x(\lambda,x) \sim -\sqrt{\lambda}\tilde{c}^{-}_{1-\nu_\alpha,0}(\lambda)x^{1-\frac{\alpha}{2}}. \tag{3.15}$$

Thus, we observe from (3.12) and (3.15) that $\Phi_\alpha^\pm(\lambda,\cdot) \in H^1_\alpha(0,1)$.

4. Well-posedness and spectral analysis.

4.1. Well-posedness (proof of Theorem 2.1–(i)). In this subsection, we establish that the operator $\mathcal{A}$ associated with the degenerate coupled system (1.1)-(1.2) is the infinitesimal generator of a C_0 contraction semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$. This result proves the well-posedness of the evolution equation (2.4).

We now prove Theorem 2.1–(i).

Proof of Theorem 2.1–(i). We employ classical semigroup theory. A direct calculation shows that for any $X=(u,v,w)\in D(\mathcal{A})$,

$$\mathcal{R}e\langle \mathcal{A}X, X\rangle_{\mathcal{H}} = -\int_0^1 a|w_x|^2 dx \le 0,$$

which implies that the operator $\mathcal{A}$ is dissipative.

To proceed, it suffices to verify that $0\in\rho(\mathcal{A})$. If this holds, then since $\rho(\mathcal{A})$ is an open set, there exists a sufficiently small $\lambda>0$ such that $R(\lambda I-\mathcal{A})=\mathcal{H}$. Combining this with the dissipativeness of $\mathcal{A}$, it follows from [28, Theorem 1.4.6] that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Consequently, by the Lumer-Phillips theorem (see [28]), the operator $\mathcal{A}$ generates a C_0 contraction semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$.

To show that $0\in\rho(\mathcal{A})$, we first prove the existence of the inverse operator $\mathcal{A}^{-1}$. For any given $(f,g,h)\in\mathcal{H}$, consider $(u,v,w)\in D(\mathcal{A})$ satisfying

$$\mathcal{A}(u,v,w)=(f,g,h). \tag{4.1}$$

Combining this with (1.3), we obtain the system

$$\begin{cases} v=f & \text{in } H^1_{\beta,0}(-1,0),\\ ((-x)^\beta u_x)_x = g & \text{in } L^2(-1,0),\\ (x^\alpha (w)_x)_x = h & \text{in } L^2(0,1),\\ u(-1)=w(1)=0,\ v(0)=w(0), & \\ (-x)^\beta u_x(0^-) = x^\alpha w_x(0^+). & \end{cases} \tag{4.2}$$

Following the classical ODE methods, we solve for u and w explicitly, obtaining constants C_1,C_2,C_3,C_4 such that

$$\begin{cases} u(x) = C_1(-x)^{1-\beta} + C_2 + \displaystyle\int_x^0 \Big(\frac{(-x)^{1-\beta}-(-s)^{1-\beta}}{1-\beta}\Big) g(s)ds,\\ w(x) = C_3 x^{1-\alpha} + C_4 + \displaystyle\int_0^x \Big(\frac{x^{1-\alpha}-s^{1-\alpha}}{1-\alpha}\Big) h(s)ds. \end{cases} \tag{4.3}$$

Let $\mathbf{C_1} := (C_1,C_2,C_3,C_4)$. Using the boundary and transmission conditions in (4.2), we derive the matrix equation: $\mathbf{M_1}\mathbf{C_1}^T = \mathbf{F_1}$, where

$$\mathbf{M_1} := \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ (1-\beta) & 0 & (1-\alpha) & 0 \end{pmatrix}, \quad \mathbf{F_1} := \begin{pmatrix} -\int_{-1}^0 \big(\frac{1-(-s)^{1-\beta}}{1-\beta}\big) g(s)ds\\ -\int_0^1 \big(\frac{1-s^{1-\alpha}}{1-\alpha}\big) h(s)ds\\ f(0)\\ 0 \end{pmatrix}.$$

Since $\det \mathbf{M_1} = -(1-\beta)\neq 0$ for any $\beta\in[0,1)$, there exists a unique solution (u,v,w) to (4.2), given by $v(x)=f(x),\ x\in(-1,0)$, and

$$\begin{cases} u(x) = \big(\frac{1-\alpha}{1-\beta}\big)\Big[\displaystyle\int_0^1 \Big(\frac{1-s^{1-\alpha}}{1-\alpha}\Big)h(s)ds + f(0)\Big]\Big((-x)^{1-\beta}-1\Big)\\ \quad + \displaystyle\int_x^0 \Big(\frac{(-x)^{1-\beta}-(-s)^{1-\beta}}{1-\beta}\Big)g(s)ds - \int_{-1}^0 \Big(\frac{1-(-s)^{1-\beta}}{1-\beta}\Big)g(s)ds,\ x\in(-1,0);\\ w(x) = -\Big[\displaystyle\int_0^1 \Big(\frac{1-s^{1-\alpha}}{1-\alpha}\Big)h(s)ds + f(0)\Big]x^{1-\alpha} + \int_0^x \Big(\frac{x^{1-\alpha}-s^{1-\alpha}}{1-\alpha}\Big)h(s)ds\\ \qquad + f(0),\ x\in(0,1). \end{cases}$$

Thus, $\mathcal{A}^{-1}$ exists. To complete the proof, it remains to verify that $\mathcal{A}^{-1}$ is bounded.

A direct calculation yields

$$\begin{aligned}\|(-x)^{\frac{\beta}{2}}u_x\|_{L^2(-1,0)} \leq &(1-\alpha)\Big(\Big|\int_0^1 (\frac{1-s^{1-\alpha}}{1-\alpha})h(s)ds\Big| + |f(0)|\Big)\|(-x)^{-\frac{\beta}{2}}\|_{L^2(-1,0)} \\ &+ \Big\|(-x)^{-\frac{\beta}{2}}\int_{-1}^0 g(s)ds\Big\|_{L^2(-1,0)} \\ \leq &C\Big(\|f\|_{H^1_{\beta,0}(-1,0)} + \|g\|_{L^2(-1,0)} + \|h\|_{L^2(0,1)}\Big)\end{aligned}$$

and

$$\begin{aligned}\|w\|_{L^2(0,1)} \leq &\Big|\int_0^1 (\frac{1-s^{1-\alpha}}{1-\alpha})h(s)ds\Big|\|x^{1-\alpha}\|_{L^2(0,1)} + 2|f(0)| \\ &+ \Big\|\int_0^x (\frac{x^{1-\alpha}-s^{1-\alpha}}{1-\alpha})h(s)ds\Big\|_{L^2(0,1)} \\ \leq &C\Big(\|f\|_{H^1_{\beta,0}(-1,0)} + \|h\|_{L^2(0,1)}\Big).\end{aligned}$$

Here we have used the estimate $|f(0)| = |\int_{-1}^0 f'(s)ds| \leq \frac{1}{1-\beta}\|(-x)^{\frac{\beta}{2}}f'\|_{L^2(-1,0)}$. Consequently, we obtain $\|(u,v,w)\|_{\mathcal{H}} \leq C\|(f,g,h)\|_{\mathcal{H}}$, which implies $\mathcal{A}^{-1} \in \mathcal{B}(\mathcal{H})$. Therefore, $0 \in \rho(\mathcal{A})$. This completes the proof of Theorem 2.1–(i), establishing the well-posedness of system (1.1)-(1.2). □

4.2. Spectral analysis. This subsection is devoted to analyzing the distribution of the spectrum of the operator $\mathcal{A}$. We begin by showing that $\mathcal{A}$ has no spectrum on the imaginary axis. Then, it follows from the spectral criterion for the strong stability of bounded C_0-semigroups established by Arendt and Batty [3] and Lyubich and Vũ [25] that the semigroup $e^{t\mathcal{A}}$ is strongly stable on $\mathcal{H}$.

4.2.1. $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ (Proof of Theorem 2.1–(ii)). We first present the following lemma, whose proof is given in Appendix A.

LEMMA 4.1. *Let $Y := H^2_\beta(-1,0) \cap H^1_{\beta,0}(-1,0)$ for any $\beta \in [0,1)$. Then space Y is compactly embedded in $H^1_{\beta,0}(-1,0)$.*

Proof of Theorem 2.1–(ii). We now establish that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Note that the spaces $H^1_{\beta,0}(-1,0)$ and $H^1_{\alpha,0}(0,1)$ are compactly embedded in $L^2(-1,0)$ and $L^2(0,1)$, respectively. Combined with Lemma 4.1, this implies that the space $\widetilde{\mathcal{H}}$ is compactly embedded in $\mathcal{H}$. From Theorem 2.1–(i), we know that $0 \in \rho(\mathcal{A})$, and that $\mathcal{A}^{-1}$ is an isomorphism from $\mathcal{H}$ onto $D(\mathcal{A})$ endowed with the graph norm. Furthermore, $D(\mathcal{A})$ is continuously embedded in $\widetilde{\mathcal{H}}$. Consequently, $\mathcal{A}^{-1}$ is a compact operator on $\mathcal{H}$. This implies that the spectrum of $\mathcal{A}^{-1}$ consists only of eigenvalues with finite multiplicity, and the only possible accumulation point is at the origin. By the spectral mapping theorem for the point spectrum, it follows that $\sigma(\mathcal{A})$ consists only of eigenvalues of finite multiplicity, with the only possible accumulation point at infinity.

To complete the proof of Theorem 2.1–(ii), we establish that $\mathcal{A}$ has no eigenvalue on the imaginary axis. This is equivalent to showing that $is - \mathcal{A}$ is injective for any $s \neq 0$. Assume that $X = (u,v,w) \in D(\mathcal{A})$ satisfies:

$$(is - \mathcal{A})(u,v,w) = 0, \tag{4.4}$$

that is,

$$\begin{cases} isu - v = 0 & \text{in } H^1_{\beta,0}(-1,0), \\ ((-x)^\beta u_x)_x + s^2 u = 0 & \text{in } L^2(-1,0), \\ (x^\alpha w_x)_x - isw = 0 & \text{in } L^2(0,1), \\ u(-1) = w(1) = 0, \\ v(0) = w(0), \quad (-x)^\beta u_x(0^-) = x^\alpha w_x(0^+). \end{cases} \tag{4.5}$$

In Proposition 3.2, the general solution structure of $(4.5)_2$ and $(4.5)_3$ has been clarified. It follows that u satisfies

$$u(x) = C_1(s)U^+(x) + C_2(s)U^-(x), \quad \forall x \in (-1, 0), \tag{4.6}$$

where

$$\begin{cases} U^+(x) := \Phi_\beta^+(s^2, -x) = (-x)^{\frac{1-\beta}{2}} J_{\nu_\beta}\Big(\frac{2}{2-\beta} s(-x)^{\frac{2-\beta}{2}}\Big), \\ U^-(x) := \Phi_\beta^-(s^2, -x) = (-x)^{\frac{1-\beta}{2}} J_{-\nu_\beta}\Big(\frac{2}{2-\beta} s(-x)^{\frac{2-\beta}{2}}\Big), \end{cases} \tag{4.7}$$

with $\nu_\beta = \frac{1-\beta}{2-\beta}$, and $C_1,\ C_2$ are some complex constants depending on s. Similarly, the general solution of w is given as follows:

$$w(x) = C_3(s)W^+(x) + C_4(s)W^-(x), \quad \forall x \in (0, 1), \tag{4.8}$$

where

$$\begin{cases} W^+(x) := \Phi_\alpha^+(-is, x) = x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha} \sqrt{-is}x^{\frac{2-\alpha}{2}}\Big), \\ W^-(x) := \Phi_\alpha^-(-is, x) = x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha} \sqrt{-is}x^{\frac{2-\alpha}{2}}\Big), \end{cases} \tag{4.9}$$

with $\nu_\alpha = \frac{1-\alpha}{2-\alpha}$, and $C_3,\ C_4$ are some complex constants depending on s.

In what follows, we determine the relationships among the above four constants $C_j,\ j = 1, 2, 3, 4$. Firstly, it follows from the boundary conditions $u(-1) = w(1) = 0$ that

$$\begin{aligned} J_{\nu_\beta}\Big(\frac{2s}{2-\beta}\Big)C_1(s) + J_{-\nu_\beta}\Big(\frac{2s}{2-\beta}\Big)C_2(s) = 0, \\ J_{\nu_\alpha}\Big(\frac{2\sqrt{-is}}{2-\alpha}\Big)C_3(s) + J_{-\nu_\alpha}\Big(\frac{2\sqrt{-is}}{2-\alpha}\Big)C_4(s) = 0. \end{aligned} \tag{4.10}$$

Moreover, we consider the first coupling condition $v(0) = w(0)$, which, in view of $(4.5)_1$, also reads as $isu(0) = w(0)$. Using the asymptotic behavior (3.12) of $\Phi_\alpha^\pm(\lambda, x)$ near $x = 0$, we have

$$\begin{aligned} &U^+(x) \sim \tilde{c}^+_{\nu_\beta,0}(\lambda)\Big|_{\lambda=s^2} \cdot (-x)^{1-\beta} = c^+_{\nu_\beta,0}\Big(\frac{2s}{2-\beta}\Big)^{\nu_\beta}(-x)^{1-\beta}, \ \text{ as } x \to 0^-, \\ &U^-(x) \sim \tilde{c}^-_{\nu_\beta,0}(\lambda)\Big|_{\lambda=s^2} = c^-_{\nu_\beta,0}\Big(\frac{2s}{2-\beta}\Big)^{-\nu_\beta}, \ \text{ as } x \to 0^-. \end{aligned}$$

Thus, we have

$$\begin{aligned} u(0) &= \lim_{x\to 0^-}\Big(C_1(s)U^+ + C_2(s)U^-\Big) \\ &= C_2(s) \cdot \lim_{x\to 0^-} U^-(x) = C_2(s) \cdot c^-_{\nu_\beta,0}\Big(\frac{2s}{2-\beta}\Big)^{-\nu_\beta}, \end{aligned} \tag{4.11}$$

where $c^-_{\nu_\beta,0} = \frac{2^{\nu_\beta}}{\Gamma(-\nu_\beta+1)} \neq 0$ is independent of s. In the same manner, we have

$$\begin{aligned} &W^+(x) \sim \tilde{c}^+_{\nu_\alpha,0}(\lambda)\Big|_{\lambda=\sqrt{-is}} \cdot x^{1-\alpha} = c^+_{\nu_\alpha,0}\Big(\frac{2\sqrt{-is}}{2-\alpha}\Big)^{\nu_\alpha} x^{1-\alpha}, \ \text{ as } x \to 0^+, \\ &W^-(x) \sim \tilde{c}^-_{\nu_\alpha,0}(\lambda)\Big|_{\lambda=\sqrt{-is}} = c^-_{\nu_\alpha,0}\Big(\frac{2\sqrt{-is}}{2-\alpha}\Big)^{-\nu_\alpha}, \ \text{ as } x \to 0^+. \end{aligned}$$

Hence,

$$w(0) = \lim_{x\to 0^+}\Big(C_3(s)W^+(x) + C_4(s)W^-(x)\Big) = C_4(s) \cdot c^-_{\nu_\alpha,0}\Big(\frac{2\sqrt{-is}}{2-\alpha}\Big)^{-\nu_\alpha}, \tag{4.12}$$

where $c^-_{\nu_\alpha,0} = \frac{2^{\nu_\alpha}}{\Gamma(-\nu_\alpha+1)} \neq 0$ is independent of s. Consequently, combining (4.11) and (4.12) yields

$$c^-_{\nu_\beta,0}\Big(\frac{2s}{2-\beta}\Big)^{-\nu_\beta} is \cdot C_2(s) - c^-_{\nu_\alpha,0}\Big(\frac{2\sqrt{-is}}{2-\alpha}\Big)^{-\nu_\alpha} \cdot C_4(s) = 0. \tag{4.13}$$

Besides, for the second coupling condition $(-x)^\beta u_x(0^-) = x^\alpha w_x(0^+)$, we utilize the formulas (3.13) and (3.14) to obtain that

$$\begin{cases} (-x)^\beta U_x^+(x) = -(1-\beta)(-x)^{-\frac{1-\beta}{2}} J_{\nu_\beta}\left(\frac{2}{2-\beta}s(-x)^{\frac{2-\beta}{2}}\right) + s(-x)^{\frac{1}{2}} J_{\nu_\beta+1}\left(\frac{2}{2-\beta}s(-x)^{\frac{2-\beta}{2}}\right), \\ (-x)^\beta U_x^-(x) = s(-x)^{\frac{1}{2}} J_{1-\nu_\beta}\left(\frac{2}{2-\beta}s(-x)^{\frac{2-\beta}{2}}\right), \end{cases}$$

and

$$\begin{cases} x^\alpha W_x^+(x) = (1-\alpha)x^{-\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{-is}x^{\frac{2-\alpha}{2}}\right) - \sqrt{-is}x^{\frac{1}{2}} J_{\nu_\alpha+1}\left(\frac{2}{2-\alpha}\sqrt{-is}x^{\frac{2-\alpha}{2}}\right), \\ x^\alpha W_x^-(x) = -\sqrt{-is}x^{\frac{1}{2}} J_{1-\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{-is}x^{\frac{2-\alpha}{2}}\right). \end{cases}$$

Combining these with the series expansions of (3.2) yield, as $x \to 0^-$,

$$(-x)^\beta U_x^+(x) \sim -(1-\beta)c_{\nu_\beta,0}^+\left(\frac{2s}{2-\beta}\right)^{\nu_\beta},\ (-x)^\beta U_x^-(x) \sim sc_{1-\nu_\beta,0}^+\left(\frac{2s}{2-\beta}\right)^{1-\nu_\beta} \cdot (-x),$$

and, as $x \to 0^+$,

$$x^\alpha W_x^+(x) \sim (1-\alpha)c_{\nu_\alpha,0}^+\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{\nu_\alpha},\ x^\alpha W_x^-(x) \sim -\sqrt{-is}c_{1-\nu_\alpha,0}^+\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{1-\nu_\alpha} \cdot x.$$

Thus, we have

$$\begin{aligned} (-x)^\beta u_x(0^-) &= \lim_{x\to 0^-}\left(C_1(s)(-x)^\beta U_x^+ + C_2(s)(-x)^\beta U_x^-\right) \\ &= C_1(s)\lim_{x\to 0^-}(-x)^\beta U_x^+ = -C_1(s)(1-\beta)c_{\nu_\beta,0}^+\left(\frac{2s}{2-\beta}\right)^{\nu_\beta}, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} x^\alpha w_x(0^+) &= \lim_{x\to 0^+}\left(C_3(s)x^\alpha W_x^+ + C_4(s)x^\alpha W_x^-\right) \\ &= C_3(s)\lim_{x\to 0^+}x^\alpha W_x^+ = C_3(s)(1-\alpha)c_{\nu_\alpha,0}^+\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{\nu_\alpha}, \end{aligned} \tag{4.15}$$

where $c_{\nu_\beta,0}^+ = \frac{1}{\Gamma(\nu_\beta+1)2^{\nu_\beta}}$ and $c_{\nu_\alpha,0}^+ = \frac{1}{\Gamma(\nu_\alpha+1)2^{\nu_\alpha}}$ do not depend on s. Then, it follows that

$$(1-\beta)c_{\nu_\beta,0}^+\left(\frac{2s}{2-\beta}\right)^{\nu_\beta} \cdot C_1(s) + (1-\alpha)c_{\nu_\alpha,0}^+\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{\nu_\alpha} \cdot C_3(s) = 0. \tag{4.16}$$

Let $\mathbf{C_2}(s) := (C_1(s), C_2(s), C_3(s), C_4(s))$. By collecting the above four equalities in (4.10), (4.13), and (4.16), we derive that $\mathbf{C_2}(s)$ satisfies

$$\mathbf{M_2}(s)\mathbf{C_2}^T(s) = \mathbf{0}, \tag{4.17}$$

where the matrix $\mathbf{M_2}(s)$ is defined by

$$\begin{pmatrix} J_{\nu_\beta}\left(\frac{2s}{2-\beta}\right) & J_{-\nu_\beta}\left(\frac{2s}{2-\beta}\right) & 0 & 0 \\ 0 & 0 & J_{\nu_\alpha}\left(\frac{2\sqrt{-is}}{2-\alpha}\right) & J_{-\nu_\alpha}\left(\frac{2\sqrt{-is}}{2-\alpha}\right) \\ 0 & c_{\nu_\beta,0}^-\left(\frac{2s}{2-\beta}\right)^{-\nu_\beta} is & 0 & -c_{\nu_\alpha,0}^-\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{-\nu_\alpha} \\ (1-\beta)c_{\nu_\beta,0}^+\left(\frac{2s}{2-\beta}\right)^{\nu_\beta} & 0 & (1-\alpha)c_{\nu_\alpha,0}^+\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{\nu_\alpha} & 0 \end{pmatrix}.$$

By direct computation, we get that

$$\begin{aligned} &\det \mathbf{M_2}(s) \\ := &\underbrace{(1-\alpha)c_{\nu_\alpha,0}^+\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{\nu_\alpha} c_{\nu_\beta,0}^-\left(\frac{2s}{2-\beta}\right)^{-\nu_\beta} \cdot is \cdot J_{-\nu_\alpha}\left(\frac{2\sqrt{-is}}{2-\alpha}\right)}_{=:A(s)} \cdot J_{\nu_\beta}\left(\frac{2s}{2-\beta}\right) \\ &+ \underbrace{(1-\beta)c_{\nu_\beta,0}^+\left(\frac{2s}{2-\beta}\right)^{\nu_\beta} c_{\nu_\alpha,0}^-\left(\frac{2\sqrt{-is}}{2-\alpha}\right)^{-\nu_\alpha} \cdot J_{\nu_\alpha}\left(\frac{2\sqrt{-is}}{2-\alpha}\right)}_{=:B(s)} \cdot J_{-\nu_\beta}\left(\frac{2s}{2-\beta}\right). \end{aligned}$$

We claim that $\det \mathbf{M_2}(s) \neq 0$ for any $s \in \mathbb{R}$ and $s \neq 0$. In fact, by the theory of Bessel functions, we know that $J_{\nu_\beta}(z)$ and $J_{-\nu_\beta}(z)$ are linearly independent, which implies that $\det \mathbf{M_2}(s) = 0$ if and only if $A(s) = B(s) = 0$. However, this is impossible. According to Lemma 3.1, $J_{\nu_\alpha}, J_{-\nu_\alpha}$ have no complex zeros. Furthermore, we observe that $\frac{2}{2-\alpha}\sqrt{-is}$ is not a real number for any $s \in \mathbb{R}$ and $s \neq 0$. Thus, for all nonzero real numbers s, $A(s), B(s) \neq 0$, which reduces to $\det \mathbf{M_2}(s) \neq 0$.

Hence, we establish the uniqueness of the solution to equation (4.4) and conclude that $\mathrm{Ker}(is - \mathcal{A}) = \{0\}$ for any $s \in \mathbb{R}$ $(s \neq 0)$. Therefore, the operator $(is - \mathcal{A})$ is injective, and consequently, $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Consequently, by the spectral criterion for the strong stability of bounded C_0-semigroups [3, 25], the C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable on $\mathcal{H}$. This completes the proof of Theorem 2.1–(ii). □

4.2.2. Spectral distribution of $\mathcal{A}$. We next analyze the asymptotic behavior of the point spectrum of the operator $\mathcal{A}$ near the imaginary axis. This facilitates the subsequent stability analysis in the next section. We first establish the following result.

PROPOSITION 4.2. *Fix $\alpha, \beta \in [0,1)$ and set $\nu_\alpha := \frac{1-\alpha}{2-\alpha}$, $\nu_\beta := \frac{1-\beta}{2-\beta}$. Let $\mathcal{A}$ be defined as in (2.2) and assume that J_ν is the classical Bessel function of the first kind. Then the point spectrum of $\mathcal{A}$ can be described by the following expression:*

$$\sigma_p(\mathcal{A}) = \left\{ \lambda \in \mathbb{C}_- \;\middle|\; F(\lambda) = 0 \;\right\} \tag{4.18}$$

where

$$\begin{aligned} F(\lambda) :=& (1-\alpha)c^+_{\nu_\alpha,0}\lambda\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)^{\nu_\alpha} \cdot c^-_{\nu_\beta,0}\Big(\frac{2}{2-\beta}i\lambda\Big)^{-\nu_\beta} \cdot J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big) J_{\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big) \\ &+ (1-\beta)c^-_{\nu_\alpha,0}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)^{-\nu_\alpha} \cdot c^+_{\nu_\beta,0}\Big(\frac{2}{2-\beta}i\lambda\Big)^{\nu_\beta} \cdot J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big) J_{-\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big), \end{aligned}$$

as well as $c^{\pm}_{\nu_\alpha,0} = \frac{1}{\Gamma(\pm\nu_\alpha+1)2^{\pm\nu_\alpha}}$ and $c^{\pm}_{\nu_\beta,0} = \frac{1}{\Gamma(\pm\nu_\beta+1)2^{\pm\nu_\beta}}$ are the same constants defined in (3.2).

Proof. Observe from Theorem 2.1 that $i\mathbb{R} \subset \rho(\mathcal{A})$, which combined with the semigroup theory implies that $\sigma_p(\mathcal{A}) \subseteq \mathbb{C}_-$. Now, we take $\lambda \in \mathbb{C}_-$ and set $X := (u, v, w) \in \mathrm{Ker}(\lambda I - \mathcal{A})$. By the definition of $\mathcal{A}$, we have

$$\begin{cases} \lambda u - v = 0 & \text{in } H^1_{\beta,0}(-1,0), \\ \lambda^2 u - ((-x)^\beta u')' = 0 & \text{in } L^2(-1,0), \\ \lambda w - (x^\alpha (w)')' = 0 & \text{in } L^2(0,1), \end{cases} \tag{4.19}$$

with boundary conditions $u(-1) = w(1) = 0$ and the transmission conditions:

$$v(0) = w(0), \quad (-x)^\beta (u)'(0^-) = x^\alpha (w)'(0^+). \tag{4.20}$$

Note that in order to get the point spectrum of $\mathcal{A}$, it suffices to seek the non-zero solutions to system (4.19)-(4.20). Utilizing Proposition 3.2, we solve system (4.19) directly along with the above boundary and coupling conditions and obtain that

$$\begin{aligned} u(x) =& C_1(\lambda) \cdot (-x)^{\frac{1-\beta}{2}} J_{\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda(-x)^{\frac{2-\beta}{2}}\Big) \\ &+ C_2(\lambda) \cdot (-x)^{\frac{1-\beta}{2}} J_{-\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda(-x)^{\frac{2-\beta}{2}}\Big), \ \forall x \in (-1,0), \\ w(x) =& C_3(\lambda) \cdot x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big) \\ &+ C_4(\lambda) \cdot x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big), \ \forall x \in (0,1), \end{aligned} \tag{4.21}$$

where $\nu_\alpha = \frac{1-\alpha}{2-\alpha}$, $\nu_\beta := \frac{1-\beta}{2-\beta}$, and C_1, C_2, C_3, and C_4 are some complex constants depending on λ, which satisfy the following equalities:

$$\begin{cases} J_{\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big)\cdot C_1(\lambda) + J_{-\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big)\cdot C_2(\lambda) = 0, \\ J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)\cdot C_3(\lambda) + J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)\cdot C_4(\lambda) = 0, \\ c^-_{\nu_\beta,0}(\frac{2}{2-\beta}i\lambda)^{-\nu_\beta}\cdot\lambda\cdot C_2(\lambda) - c^-_{\nu_\alpha,0}(\frac{2}{2-\alpha}i\sqrt{\lambda})^{-\nu_\alpha}\cdot C_4(\lambda) = 0, \\ (1-\beta)c^+_{\nu_\beta,0}(\frac{2}{2-\beta}i\lambda)^{\nu_\beta}\cdot C_1(\lambda) + (1-\alpha)c^+_{\nu_\alpha,0}(\frac{2}{2-\alpha}i\sqrt{\lambda})^{\nu_\alpha}\cdot C_3(\lambda) = 0. \end{cases}$$

Set $\mathbf{C_3}(\lambda) := (C_1(\lambda), C_2(\lambda), C_3(\lambda), C_4(\lambda))$. Then, the equation satisfied by $\mathbf{C_3}(\lambda)$ can be formulated as

$$\mathbf{M_3}(\lambda)\mathbf{C_3}^T(\lambda) = \mathbf{0}, \tag{4.22}$$

where the matrix $\mathbf{M_3}(\lambda)$ is defined by

$$\begin{pmatrix} J_{\nu_\beta}\Big(\frac{2i\lambda}{2-\beta}\Big) & J_{-\nu_\beta}\Big(\frac{2i\lambda}{2-\beta}\Big) & 0 & 0 \\ 0 & 0 & J_{\nu_\alpha}\Big(\frac{2i\sqrt{\lambda}}{2-\alpha}\Big) & J_{-\nu_\alpha}\Big(\frac{2i\sqrt{\lambda}}{2-\alpha}\Big) \\ 0 & c^-_{\nu_\beta,0}(\frac{2i\lambda}{2-\beta})^{-\nu_\beta}\cdot\lambda & 0 & -c^-_{\nu_\alpha,0}(\frac{2i\sqrt{\lambda}}{2-\alpha})^{-\nu_\alpha} \\ (1-\beta)c^+_{\nu_\beta,0}(\frac{2i\lambda}{2-\beta})^{\nu_\beta} & 0 & (1-\alpha)c^+_{\nu_\alpha,0}(\frac{2i\sqrt{\lambda}}{2-\alpha})^{\nu_\alpha} & 0 \end{pmatrix}.$$

Note that equation (4.19) admits nonzero solutions if and only if there exist non-trivial solutions to the matrix equation (4.22), which is equivalent to $F(\lambda) := \det \mathbf{M_3}(\lambda) = 0$. Therefore, the point spectrum of $\mathcal{A}$ is given by (4.18). The desired result follows. □

By using the structure of $\sigma_p(\mathcal{A})$ as given in Proposition 4.2, we have the following estimate on the asymptotic distribution of the spectrum of $\mathcal{A}$ near the imaginary axis.

THEOREM 4.3. *Define the function*

$$\gamma(\alpha,\beta) := \frac{2(1-\alpha)+\beta}{(2-\alpha)(2-\beta)},\ \alpha\in[0,1),\ \beta\in[0,1).$$

There exists a sequence $\{\lambda_n\}\subset\sigma_p(\mathcal{A})$ *and a positive constant* $\sigma_1\in\mathbb{R}_+$ *such that*

$$\begin{cases} |\lambda_n| \sim |\mathcal{I}m\lambda_n| \sim \frac{2-\beta}{2}n\pi, & \textit{as } n\to\infty; \\ 0 > \mathcal{R}e\lambda_n \geq -\sigma_1 n^{-\gamma(\alpha,\beta)}, & \forall \textit{ large } n\in\mathbb{N}. \end{cases} \tag{4.23}$$

Proof. To establish (4.23), we analyze the distribution of the roots of $F(\lambda)$ defined in (4.18). Proposition 4.2 states that $\lambda\in\mathbb{C}_-$ belongs to $\sigma_p(\mathcal{A})$ if and only if λ is the root of

$$F(\lambda) = \underbrace{(1-\alpha)c^+_{\nu_\alpha,0}(\frac{2i}{2-\alpha})^{\nu_\alpha}\cdot c^-_{\nu_\beta,0}(\frac{2i}{2-\beta})^{-\nu_\beta}}_{=:c_1}\cdot\lambda^{1-\nu_\beta+\frac{\nu_\alpha}{2}}\cdot J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)J_{\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big)$$
$$+\underbrace{(1-\beta)c^-_{\nu_\alpha,0}(\frac{2i}{2-\alpha})^{-\nu_\alpha}\cdot c^+_{\nu_\beta,0}(\frac{2i}{2-\beta})^{\nu_\beta}}_{=:c_2}\cdot\lambda^{\nu_\beta-\frac{\nu_\alpha}{2}}\cdot J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)J_{-\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big).$$

We can rewrite $F(\lambda)$ as

$$F(\lambda) = c_1\cdot\lambda^{1-\nu_\beta+\frac{\nu_\alpha}{2}}\cdot J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}i\sqrt{\lambda}\Big)\cdot\tilde{F}(\lambda),$$

where

$$\tilde{F}(\lambda) = \underbrace{J_{\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big)}_{=:\tilde{F}_0(\lambda)} + \underbrace{\frac{c_2}{c_1}\cdot\frac{J_{\nu_\alpha}(\frac{2}{2-\alpha}i\sqrt{\lambda})}{J_{-\nu_\alpha}(\frac{2}{2-\alpha}i\sqrt{\lambda})}J_{-\nu_\beta}\Big(\frac{2}{2-\beta}i\lambda\Big)\lambda^{-1+2\nu_\beta-\nu_\alpha}}_{=:\tilde{F}_1(\lambda)}.$$

Thus, let us analyze the zeros of $\tilde{F}(\lambda)$, which obviously are the zeros of $F(\lambda)$.

Let $\{j_{\nu_\beta,n}\}_{n=1}^{\infty}$ denote the positive real zeros of Bessel function $J_{\nu_\beta}(z)$. From Lemma 3.1, $j_{\nu_\beta,n}$ satisfies

$$\begin{cases} \text{(i) } \pi(n+\frac{\nu_\beta}{2}-\frac{1}{4}) \le j_{\nu_\beta,n} \le \pi(n+\frac{\nu_\beta}{4}-\frac{1}{8}),\ \forall n\ge 1; \\ \text{(ii) } |j_{\nu_\beta,n+1}-j_{\nu_\beta,n}| \ge\ \pi(\frac{\nu_\beta}{4}+\frac{7}{8})>0,\ \forall n\ge 1; \\ \text{(iii) } j_{\nu_\beta,n} = \big(n+\frac{\nu_\beta}{2}-\frac{1}{4}\big)\pi - \frac{4\nu_\beta^2-1}{8(n+\frac{1}{2}\nu_\beta-\frac{1}{4})\pi} + O\big(\frac{1}{n^3}\big),\ \text{as } n\to+\infty. \end{cases} \tag{4.24}$$

Define $\mu_n := -i\frac{(2-\beta)}{2}j_{\nu_\beta,n}$. It is clear that each μ_n is purely imaginary and satisfies $\tilde{F}_0(\mu_n)=0$ for all $n\ge 1$. Additionally, set

$$B_n := \big\{\lambda\in\mathbb{C}\big||\lambda-\mu_n| < \sigma_1 n^{-\gamma(\alpha,\beta)},\ \forall \alpha,\beta\in[0,1)\big\},\ \forall n\in\mathbb{N},$$

where σ_1 is a positive constant to be specified later. Inspired by [8, 7], we apply Rouché's theorem to analyze the spectrum of $\mathcal{A}$. Specifically, we aim to prove that

$$|\tilde{F}_0(\lambda)| > |\tilde{F}_1(\lambda)|,\ \forall\lambda\in\partial B_n, \tag{4.25}$$

for n sufficiently large. If this holds, then, as both $\tilde{F}_0$ and $\tilde{F}_1$ are analytic in $\overline{B_n}$ and $\tilde{F}=\tilde{F}_0+\tilde{F}_1$, Rouché's theorem ensures that $\tilde{F}(\lambda)$ and $\tilde{F}_0(\lambda)$ have the same number of zeros inside B_n for large n. Since each μ_n is a simple zero of $\tilde{F}_0$, it follows that $\tilde{F}(\lambda)$ possesses a unique zero in each B_n. This establishes the existence of an eigenvalue sequence that satisfies (4.23).

To establish (4.25), we first estimate $|\tilde{F}_0(\lambda)|$ on ∂B_n. For any $\lambda\in\partial B_n$, λ can be expressed as

$$\lambda = \mu_n + \sigma_1 n^{-\gamma(\alpha,\beta)}e^{i\theta} =: a_n + ib_n = (a_n^2+b_n^2)^{\frac{1}{2}}\cdot e^{i\varphi_n},\ \forall\theta\in[-\pi,\pi),$$

where $a_n := \sigma_1 n^{-\gamma(\alpha,\beta)}\cos\theta$, $b_n := -\frac{(2-\beta)}{2}j_{\nu_\beta,n} + \sigma_1 n^{-\gamma(\alpha,\beta)}\sin\theta$, as well as $\varphi_n := \arctan(b_n/a_n)$. From (4.24), it holds that $\forall\lambda\in\overline{B}_n$, as $n\to+\infty$,

$$\begin{cases} \text{①}\ |\lambda| \sim |\mu_n| = \frac{(2-\beta)}{2}j_{\nu_\beta,n} \sim \frac{(2-\beta)}{2}n\pi; \\ \text{②}\ \lambda\sim\mu_n = -i\frac{(2-\beta)}{2}j_{\nu_\beta,n} \sim -i\frac{(2-\beta)}{2}n\pi; \\ \text{③}\ \sqrt{\lambda} = (a_n^2+b_n^2)^{\frac{1}{4}}\cdot e^{i\frac{\varphi_n}{2}} \sim \sqrt{\frac{(2-\beta)}{2}n\pi}\cdot e^{-\frac{\pi}{4}i}. \end{cases} \tag{4.26}$$

Let $z(\lambda) := \frac{2}{2-\beta}i\lambda$ and define the closed contour $\mathcal{C}_n := \{\xi\in\mathbb{C}\mid|\xi-\mu_n| = r_0\}$ for each $n\in\mathbb{N}$, where $r_0>0$ is some fixed small constant. It follows from Taylor's theorem that, for all $\lambda\in\partial B_n$ and n sufficiently large,

$$\begin{aligned}\tilde{F}_0(\lambda) =& \tilde{F}_0(\lambda) - \tilde{F}_0(\mu_n) \\ =& J_{\nu_\beta}(z(\lambda)) - J_{\nu_\beta}(z(\mu_n)) \\ =& \underbrace{\frac{2i}{2-\beta}\frac{d}{dz}J_{\nu_\beta}(z(\mu_n))\cdot(\lambda-\mu_n)}_{=:\Xi_n^1} + \underbrace{\frac{1}{2\pi i}\int_{\mathcal{C}_n}\frac{J_{\nu_\beta}(z(\xi))}{(\xi-\mu_n)^2(\xi-\lambda)}d\xi\cdot\big(\lambda-\mu_n\big)^2}_{=:\Xi_n^2}.\end{aligned} \tag{4.27}$$

We now proceed to estimate the two terms above.

• *The estimate for* Ξ_n^1. From the first formula in (3.5), we have

$$\frac{d}{dz}J_{\nu_\beta}(z)\Big|_{z=z(\mu_n)} = \Big(\frac{\nu_\beta}{z}J_{\nu_\beta}(z) - J_{\nu_\beta+1}(z)\Big)\Big|_{z=z(\mu_n)} = -J_{\nu_\beta+1}(z(\mu_n)). \tag{4.28}$$

We also note that J_ν satisfies the following asymptotic formula (see [36]), for $z\in\mathbb{R}_+$,

$$J_\nu^2(z) + J_{\nu+1}^2(z) = \frac{2}{\pi z}\big(1+o(1)\big),\ \text{as } z\to+\infty. \tag{4.29}$$

Since $z(\mu_n) = j_{\nu_\beta,n} \sim n\pi$ as $n \to +\infty$, combining this with (4.28) and (4.29) yields that

$$\left(\frac{d}{dz} J_{\nu_\beta}(z(\mu_n))\right)^2 = \left(J_{\nu_\beta+1}(z(\mu_n))\right)^2 \sim \frac{2}{\pi^2 n}, \text{ as } n \to \infty.$$

Thus, we obtain that, $\forall \lambda \in \partial B_n$,

$$\left|\Xi_n^1\right| = \frac{2\sqrt{2}\sigma_1}{(2-\beta)\pi} \cdot n^{-1/2-\gamma(\alpha,\beta)} \left(1 + o(1)\right), \text{ as } n \to \infty.$$

• *The estimate for* Ξ_n^2*.* Based on the choice of the contour $\mathcal{C}_n$, we observe that for any $\xi \in \mathcal{C}_n$, one has $z(\xi) \sim z(\mu_n) \sim n\pi$ as $n \to +\infty$, and

$$|\xi - \lambda| \geq |\xi - \mu_n| - |\lambda - \mu_n| \geq \frac{1}{2} r_0, \ \forall \lambda \in \partial B_n,$$

for sufficiently large n. Applying the asymptotic formula (3.4), we then obtain

$$\left|\frac{1}{2\pi i}\int_{\mathcal{C}_n} \frac{J_{\nu_\beta}(z(\xi))}{(\xi-\mu_n)^2(\xi-\lambda)} d\xi\right| \leq \frac{1}{2\pi}\int_{\mathcal{C}_n} \left|C\Big(\frac{2}{\pi^2 n}\Big)^{\frac{1}{2}} \cdot \frac{2}{r_0} \cdot \frac{1}{r_0^2}\right| |d\xi| = C\frac{2}{r_0^2}\Big(\frac{2}{\pi^2 n}\Big)^{\frac{1}{2}}, \ \forall \lambda \in \partial B_n,$$

for some constant $C > 0$ independent of n. Therefore, for the second term of (4.27), we have

$$\left|\Xi_n^2\right| = O\left(n^{-\frac{1}{2}-2\gamma(\alpha,\beta)}\right), \text{ as } n \to \infty.$$

Hence, combining the estimates of Ξ_n^1 and Ξ_n^2 into (4.27), we obtain that for any $\lambda \in \partial B_n$, as $n \to \infty$,

$$|\tilde{F}_0(\lambda)| = \frac{2\sqrt{2}\sigma_1}{(2-\beta)\pi} \cdot n^{-\frac{1}{2}-\gamma(\alpha,\beta)} \left(1 + O\left(\frac{1}{n^{\gamma(\alpha,\beta)}}\right)\right).$$

Therefore, it follows that

$$|\tilde{F}_0(\lambda)| \geq \frac{\sqrt{2}\sigma_1}{(2-\beta)\pi} \cdot n^{-\frac{1}{2}-\gamma(\alpha,\beta)}, \quad \forall\, \lambda \in \partial B_n, \tag{4.30}$$

for all sufficiently large n.

Next, we estimate $|\tilde{F}_1(\lambda)|$ on ∂B_n. From (3.4) and the second estimate in (4.26), for any $\lambda \in \partial B_n$,

$$\left|J_{-\nu_\beta}\Big(\frac{2}{2-\beta} i\lambda\Big)\right| \leq \frac{2\sqrt{2}}{\pi} \cdot n^{-\frac{1}{2}}, \quad \text{for } n \text{ sufficiently large.} \tag{4.31}$$

Moreover, we claim that

$$\left|\frac{J_{\nu_\alpha}\big(\frac{2}{2-\alpha} i\sqrt{\lambda}\big)}{J_{-\nu_\alpha}\big(\frac{2}{2-\alpha} i\sqrt{\lambda}\big)}\right| \longrightarrow 1, \ \forall \lambda \in \partial B_n, \text{ as } n \to \infty. \tag{4.32}$$

Indeed, using (3.4) and the third estimate in (4.26) along with $\nu_\alpha = \frac{1-\alpha}{2-\alpha}$, we obtain that for any $\lambda \in \partial B_n$,

$$\begin{aligned}
\frac{J_{\nu_\alpha}\big(\frac{2}{2-\alpha} i\sqrt{\lambda}\big)}{J_{-\nu_\alpha}\big(\frac{2}{2-\alpha} i\sqrt{\lambda}\big)} &\sim \frac{\cos\left(\frac{2}{2-\alpha} i\sqrt{\lambda} - \frac{\nu_\alpha}{2}\pi - \frac{1}{4}\pi\right)}{\cos\left(\frac{2}{2-\alpha} i\sqrt{\lambda} + \frac{\nu_\alpha}{2}\pi - \frac{1}{4}\pi\right)} \\
&\sim \frac{\cos\left(\frac{\sqrt{(2-\beta)\pi}}{2-\alpha} n^{\frac{1}{2}} - \frac{(4-3\alpha)}{4(2-\alpha)}\pi + \frac{\sqrt{(2-\beta)\pi}}{2-\alpha} n^{\frac{1}{2}} i\right)}{\cos\left(\frac{\sqrt{(2-\beta)\pi}}{2-\alpha} n^{\frac{1}{2}} - \frac{\alpha}{4(2-\alpha)}\pi + \frac{\sqrt{(2-\beta)\pi}}{2-\alpha} n^{\frac{1}{2}} i\right)} \\
&=: f(n)
\end{aligned}$$

as $n \to +\infty$. Moreover, considering $f(n)$ and using the identity $\cos z = (e^{iz} + e^{-iz})/2,\ \forall z \in \mathbb{C}$, we have $\lim_{n\to+\infty} f(n) = e^{\frac{1-\alpha}{2-\alpha}\pi i}$. This establishes (4.32).

Observe that

$$-1 + 2\nu_\beta - \nu_\alpha = -1 + \frac{2(1-\beta)}{2-\beta} - \frac{1-\alpha}{2-\alpha} = -\frac{2(1-\alpha)+\beta}{(2-\alpha)(2-\beta)} = -\gamma(\alpha,\beta).$$

Thus, summarizing the estimates $(4.26)_1$, (4.31) and (4.32), we infer that

$$|\tilde{F}_1(\lambda)| \le |\frac{c_2}{c_1}| \cdot \frac{4\sqrt{2}}{\pi}\Big(\frac{(2-\beta)\pi}{2}\Big)^{-\gamma(\alpha,\beta)} \cdot n^{-\frac{1}{2}-\gamma(\alpha,\beta)},\ \forall \lambda \in \partial B_n, \tag{4.33}$$

for n sufficiently large.

Hence, by choosing $\sigma_1 = 2 \cdot |\frac{c_2}{c_1}|\frac{4\cdot 2^{\gamma(\alpha,\beta)}}{\pi}((2-\beta)\pi)^{1-\gamma(\alpha,\beta)}$, from (4.30) and (4.33), we deduce that $|\tilde{F}_0(\lambda)| > |\tilde{F}_1(\lambda)|$ holds on ∂B_n for sufficiently large $n \in \mathbb{N}$. This proves (4.25), and the desired eigenvalue sequence of $\mathcal{A}$ follows. The proof is complete. □

5. Stability analysis. In this section, we mainly study the rate of energy decay for classical solutions of the abstract Cauchy problem (2.4) associated with the degenerate heat-wave coupled system (1.1)–(1.2).

5.1. Decay rate. The goal of this subsection is to present the proof of the decay rate in Theorem 2.2–(i).

Firstly, we introduce the following frequency-domain characterization of bounded semigroups, as established by Borichev and Tomilov in [5] (see [24] for a related weaker result and [9, 29] for general cases).

LEMMA 5.1. *Let $\{T(t)\}_{t\ge 0}$ be a bounded C_0−semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then for fixed $\gamma > 0$ the following are equivalent:*

(i) $\|R(is;\mathcal{A})\| = \mathcal{O}(|s|^\gamma),\ s \in \mathbb{R},\ |s| \to \infty$*;*

(ii) There exists a constant $C > 0$ such that $\|T(t)x\|_{\mathcal{H}} \le Ct^{-\frac{1}{\gamma}}\|x\|_{\mathcal{D}(\mathcal{A})},\ \forall x \in \mathcal{D}(\mathcal{A}),\ t \ge 1$, where $\|\cdot\|_{\mathcal{D}(\mathcal{A})}$ is the graph norm corresponding to $\mathcal{A}$.

Next, we recall the following weighted Hardy inequality (see [19, 20]), which will play a key role in the proof.

LEMMA 5.2. *Given two constants $\gamma > -1$ and $\alpha < 1$, there exists a constant $C = C(\alpha,\gamma) > 0$ such that*

$$\int_0^1 x^\gamma |f(x)|^2 dx \le C\int_0^1 x^\alpha |f'(x)|^2 dx, \tag{5.1}$$

for any $f(x) \in H^1_{\alpha,0}(0,1)$.

Proof of Theorem 2.2–(i).

Recall that $\gamma(\alpha,\beta) = \frac{2(1-\alpha)+\beta}{(2-\alpha)(2-\beta)}$ for any $\alpha \in [0,1)$ and $\beta \in [0,1)$. According to the necessary and sufficient conditions in Lemma 5.1, and with $i\mathbb{R} \subset \rho(\mathcal{A})$ already established in Theorem 2.1, we need to verify

$$\limsup_{|s|\to+\infty} \frac{\|R(is;\mathcal{A})\|}{|s|^{\gamma(\alpha,\beta)}} < +\infty. \tag{5.2}$$

If (5.2) does not hold, then by the Banach–Steinhaus theorem, there exist sequences $\{s_n\} \subset \mathbb{R}$ and $\{X^n\} \subset \mathcal{D}(\mathcal{A})$ where $X^n = (u^n, v^n, w^n)$ and $\|X^n\|_{\mathcal{H}} = 1$, such that $\lim\limits_{n\to\infty} |s_n| = +\infty$ and

$$|s_n|^{\gamma(\alpha,\beta)}\|(is_n - \mathcal{A})X^n\|_{\mathcal{H}} = o(1), \quad n \to \infty. \tag{5.3}$$

Let $(is_n - \mathcal{A})X^n =: (f^n, g^n, h^n)$. Then, by the definition of $\mathcal{A}$, (5.3) can be expressed as

$$\begin{cases} is_n u^n - v^n = f^n = |s_n|^{-\gamma(\alpha,\beta)}o(1), & \text{in} \quad H^1_{\beta,0}(-1,0), \\ is_n v^n - ((-x)^\beta u^n_x)_x = g^n = |s_n|^{-\gamma(\alpha,\beta)}o(1), & \text{in} \quad L^2(-1,0), \\ is_n w^n - (x^\alpha w^n_x)_x = h^n = |s_n|^{-\gamma(\alpha,\beta)}o(1), & \text{in} \quad L^2(0,1), \end{cases} \tag{5.4}$$

equipped with the boundary conditions $u^n(-1) = w^n(1) = 0$ and the transmission conditions:

$$\begin{cases} (-x)^\beta u_x^n(0^-) = x^\alpha w_x^n(0^+), \\ w^n(0) = v^n(0) = is_n u^n(0) - f^n(0). \end{cases} \tag{5.5}$$

In what follows, we shall prove that $\|X^n\|_{\mathcal{H}} = o(1)$, which contradicts the fact that $\|X^n\|_{\mathcal{H}} = 1$. Consequently, (5.2) must hold. Then by Lemma 5.1, the desired polynomial stability for system (1.1)–(1.2) follows.

In the proof, it is important to note that due to the degeneracy of the system operator $\mathcal{A}$ near the interface, the traditional frequency multiplier method used in [19, 20] is not fully applicable here. To address the effects of this degeneracy, we will employ direct estimates for Bessel functions provided in [37], combined with local analysis, to complete the proof. The proof is divided into three main steps.

Step 1. We show

$$\int_0^1 x^{2-\alpha}|w^n|^2 dx = |s_n|^{-2-\gamma(\alpha,\beta)} o(1) \quad \text{and} \quad \int_0^1 |w^n|^2 dx = |s_n|^{-1-\gamma(\alpha,\beta)} o(1). \tag{5.6}$$

We begin with some preliminary observations.

Observation I. $\|x^{\frac{\alpha}{2}} w_x^n\|_{L^2(0,1)} = |s_n|^{-\frac{\gamma(\alpha,\beta)}{2}} o(1)$. Indeed, by the dissipativeness of $\mathcal{A}$, and the fact that $\|X^n\|_{\mathcal{H}} = 1$, we have

$$\mathcal{R}e\langle (is_n - \mathcal{A})X^n, X^n\rangle_{\mathcal{H}} = -\mathcal{R}e\langle \mathcal{A}X^n, X^n\rangle_{\mathcal{H}} = \left\|x^{\frac{\alpha}{2}} w_x^n\right\|^2_{L^2(0,1)} = |s_n|^{-\gamma(\alpha,\beta)} o(1).$$

Hence, $\|x^{\frac{\alpha}{2}} w_x^n\|_{L^2(0,1)} = |s_n|^{-\frac{\gamma(\alpha,\beta)}{2}} o(1)$.

Observation II. $|w^n(0)| = |s_n|^{-\frac{\gamma(\alpha,\beta)}{2}} o(1)$. In fact, since $w^n(1) = 0$ and by *Observation I*, we have

$$|w^n(0)| = \Big|\int_0^1 x^{-\frac{\alpha}{2}} x^{\frac{\alpha}{2}} w_x^n dx\Big| \leq \sqrt{\frac{1}{1-\alpha}} \|x^{\frac{\alpha}{2}} w_x^n\|_{L^2(0,1)} = |s_n|^{-\frac{\gamma(\alpha,\beta)}{2}} o(1).$$

Thus, *Observation II* holds.

Observation III. $\lim\limits_{x\to 0^+}(xw_x^n)(x) = 0$ *for any* $n \geq 1$. Let $\epsilon_n := \lim\limits_{x\to 0^+}(xw_x^n)(x)$. If there exists some $n_0 \geq 1$ such that $\epsilon_{n_0} \neq 0$, then we can find some $\delta > 0$ such that $\left|x^{\frac{\alpha}{2}} w_x^{n_0}(x)\right| \geq \frac{1}{2}|\epsilon_{n_0}| \cdot x^{-1+\frac{\alpha}{2}}$, $\forall x \in (0,\delta)$, which implies

$$\infty = \left\|x^{\frac{\alpha}{2}} w_x^{n_0}\right\|^2_{L^2(0,1)} = |\mathcal{R}e\langle \mathcal{A}X^{n_0}, X^{n_0}\rangle_{\mathcal{H}}| < \infty.$$

This is a contradiction. Therefore, *Observation III* holds.

Now, taking the inner product in $L^2(0,1)$ of the third equation in (5.4) with $x^{2-\alpha}\overline{w^n}$ and integrating by parts, we obtain

$$is_n\int_0^1 x^{2-\alpha}|w^n|^2 dx = \left[x^2 w_x^n \overline{w^n}\right]_0^1 - (2-\alpha)\int_0^1 x w_x^n \overline{w^n} dx - \int_0^1 x^2 |w_x^n|^2 dx + \int_0^1 x^{2-\alpha} h^n \overline{w^n} dx. \tag{5.7}$$

By *Observation II, III*, the boundary term vanishes. Then taking the imaginary part of (5.7) yields

$$\int_0^1 s_n x^{2-\alpha}|w^n|^2 dx = -(2-\alpha)\mathcal{I}m\int_0^1 x w_x^n \overline{w^n} dx + \mathcal{I}m\int_0^1 x^{2-\alpha} h^n \overline{w^n} dx.$$

Thus,

$$\begin{aligned}\int_0^1 s_n x^{2-\alpha}|w^n|^2dx \leq&(2-\alpha)\Big|\int_0^1 xw_x^n\overline{w^n}dx\Big|+\Big|\int_0^1 x^{2-\alpha}h^n\overline{w^n}dx\Big|\\ \leq&(2-\alpha)\Big[\int_0^1 s_n^{-1}x^\alpha|w_x^n|^2dx\Big]^{\frac12}\Big[\int_0^1 s_nx^{2-\alpha}|w^n|^2dx\Big]^{\frac12}\\ &+\Big[\int_0^1 s_n^{-1}|h^n|^2dx\Big]^{\frac12}\Big[\int_0^1 s_nx^{2-\alpha}|w^n|^2dx\Big]^{\frac12}\\ \leq&\frac{(3-\alpha)}{4}\int_0^1 s_nx^{2-\alpha}|w^n|^2dx+(2-\alpha)\int_0^1 s_n^{-1}x^\alpha|w_x^n|^2dx+\int_0^1 s_n^{-1}|h^n|^2dx.\end{aligned}$$

By *Observation I* and the third equation in (5.4), we conclude that

$$\int_0^1 x^{2-\alpha}|w^n|^2dx=|s_n|^{-2-\gamma(\alpha,\beta)}o(1). \tag{5.8}$$

Next, we prove the second estimate in (5.6). By direct calculation, we get

$$\int_0^1 |w^n|^2dx=-2\mathcal{R}e\int_0^1 xw_x^n\overline{w^n}dx. \tag{5.9}$$

Thus, using (5.8), (5.9) and *Observation I*, we obtain

$$\int_0^1 |w^n|^2dx\leq 2\Big[\int_0^1 x^\alpha|w_x^n|^2dx\Big]^{\frac12}\Big[\int_0^1 x^{2-\alpha}|w^n|^2dx\Big]^{\frac12}=|s_n|^{-1-\gamma(\alpha,\beta)}o(1).$$

This completes the proof of (5.6).
Step 2. We show

$$|(x^\alpha w_x^n)(0)|=|s_n|^{-\frac{\beta}{2(2-\beta)}}o(1). \tag{5.10}$$

Let us choose the interval $I_n:=[\frac12|s_n|^{-\frac{1}{2-\alpha}},|s_n|^{-\frac{1}{2-\alpha}}]$. In this interval, we compute that

$$\begin{aligned}\min_{x\in I_n}|(x^\alpha w_x^n)(x)|\leq&\frac{1}{|I_n|}\int_{I_n}|x^\alpha w_x^n|dx\\ \leq&\frac{1}{\frac12|s_n|^{-\frac{1}{2-\alpha}}}[\frac12|s_n|^{-\frac{1}{2-\alpha}}]^{\frac12}[\int_{\frac12 s_n^{-\frac{1}{2-\alpha}}}^{s_n^{-\frac{1}{2-\alpha}}}|x^\alpha w_x^n|^2dx]^{\frac12}\\ \leq&\sqrt2|s_n|^{\frac{1}{2(2-\alpha)}}(\max_{x\in I_n}|x^{\frac\alpha2}|)\|x^{\frac\alpha2}w_x^n\|_{L^2(0,1)}\\ \leq&C|s_n|^{\frac{1-\alpha}{2(2-\alpha)}}\|x^{\frac\alpha2}w_x^n\|_{L^2(0,1)}=|s_n|^{-g(\alpha,\beta)}o(1),\end{aligned}$$

where $g(\alpha,\beta):=-(\frac{1-\alpha}{2(2-\alpha)}-\frac12\gamma(\alpha,\beta))=-\frac{1-\alpha}{2(2-\alpha)}+\frac{2(1-\alpha)+\beta}{2(2-\alpha)(2-\beta)}=\frac{\beta}{2(2-\beta)}>0$. Thus, we can always find some sequence $\{\theta_n\}_{n=1}^\infty\subset I_n$ such that

$$|(x^\alpha w_x^n)(\theta_n)|=|s_n|^{-g(\alpha,\beta)}o(1). \tag{5.11}$$

Now, considering $|(x^\alpha w_x^n)(0)|$, we integrate the third equation in (5.4) from 0 to θ_n and consequently obtain

$$(x^\alpha w_x^n)(0)=(x^\alpha w_x^n)(\theta_n)-is_n\int_0^{\theta_n}w^ndx+\int_0^{\theta_n}h^ndx. \tag{5.12}$$

Observe that by the choice of θ_n, along with the second estimate in (5.6) and Hölder inequality, we get

$$|is_n \int_0^{\theta_n} w^n dx| \le |s_n||\theta_n|^{\frac{1}{2}} \|w^n\|_{L^2(0,1)} = |s_n|^{-g(\alpha,\beta)} o(1), \tag{5.13}$$

and

$$|\int_0^{\theta_n} h^n dx| \le |\theta_n|^{\frac{1}{2}} \|h^n\|_{L^2(0,1)} = |s_n|^{-g(\alpha,\beta)-\frac{1}{2}-\frac{1}{2}\gamma(\alpha,\beta)} o(1). \tag{5.14}$$

Then by recalling $g(\alpha,\beta) = \frac{\beta}{2(2-\beta)}$ and substituting (5.11), (5.13) and (5.14) into (5.12), we finally get $|(x^\alpha w_x^n)(0)| = |s_n|^{-\frac{\beta}{2(2-\beta)}} o(1)$, which gives (5.10).
Step 3. We show that

$$\|u^n\|_{H^1_{\beta,0}(-1,0)} = o(1), \ \|v^n\|_{L^2(-1,0)} = o(1). \tag{5.15}$$

From (5.4) and (5.5), u^n satisfies

$$\begin{cases} \left((-x)^\beta u_x^n\right)_x + s_n^2 u^n = -(is_n f^n + g^n) \ \text{ in } L^2(-1,0), \\ is_n u^n(0) = w^n(0) + f^n(0), \ (-x)^\beta u_x^n(0^-) = x^\alpha w_x^n(0^+). \end{cases} \tag{5.16}$$

By (4.6) and (4.7) in the proof of Theorem 2.1–(ii), the general solution for u is given by

$$u^n(x) = C_{1n} U_n^+(x) + C_{2n} U_n^-(x) + F_n(x), \quad \forall x \in (-1,0), \tag{5.17}$$

where

$$\begin{cases} U_n^+(x) := \Phi_\beta^+(s_n^2, -x) = (-x)^{\frac{1-\beta}{2}} J_{\nu_\beta}\Big(\frac{2}{2-\beta} s_n (-x)^{\frac{2-\beta}{2}}\Big), \\ U_n^-(x) := \Phi_\beta^-(s_n^2, -x) = (-x)^{\frac{1-\beta}{2}} J_{-\nu_\beta}\Big(\frac{2}{2-\beta} s_n (-x)^{\frac{2-\beta}{2}}\Big), \\ F_n(x) := \frac{\pi}{(2-\beta)\sin(\nu_\beta \pi)} \int_x^0 (-is_n f^n(\tau) - g^n(\tau))\Big(U_n^+(x) U_n^-(\tau) - U_n^-(x) U_n^+(\tau)\Big) d\tau, \end{cases} \tag{5.18}$$

with $\nu_\beta = \frac{1-\beta}{2-\beta}$, and C_{1n}, C_{2n} are constants depending on n.

From the analysis in (4.11) and (4.14), we derive

$$\begin{aligned} u^n(0) &= \lim_{x\to 0^-} \Big(C_{1n} U_n^+ + C_{2n} U_n^- + F_n\Big) \\ &= C_{2n} \cdot \lim_{x \to 0^-} U_n^-(x) = C_{2n} \cdot c^-_{\nu_\beta,0} \Big(\frac{2s_n}{2-\beta}\Big)^{-\nu_\beta}, \end{aligned}$$

and

$$\begin{aligned} (-x)^\beta u_x^n(0^-) &= \lim_{x\to 0^-} \Big(C_{1n} (-x)^\beta U_{nx}^+ + C_{2n} (-x)^\beta U_{nx}^- + (-x)^\beta F_{nx}\Big) \\ &= C_{1n} \cdot \lim_{x\to 0^-} (-x)^\beta U_{nx}^+ = -C_{1n} \cdot (1-\beta) c^+_{\nu_\beta,0} \Big(\frac{2s_n}{2-\beta}\Big)^{\nu_\beta}, \end{aligned}$$

where $c^\mp_{\pm\nu_\beta,0} = \frac{2^{\pm\nu_\beta}}{\Gamma(\pm\nu_\beta+1)} \neq 0$. Thus, recalling $\nu_\beta = \frac{1-\beta}{2-\beta}$ and using the transmission conditions in (5.16), we obtain

$$\begin{cases} C_{1n} = \frac{-1}{(1-\beta) c^+_{\nu_\beta,0}} \Big(\frac{2}{2-\beta}\Big)^{-\frac{1-\beta}{2-\beta}} \cdot s_n^{-\frac{1-\beta}{2-\beta}} \cdot (x^\alpha w_x^n)(0), \\ C_{2n} = -\frac{i}{c^-_{\nu_\beta,0}} \Big(\frac{2}{2-\beta}\Big)^{\frac{1-\beta}{2-\beta}} \cdot s_n^{-\frac{1}{2-\beta}} \cdot \big(w^n(0) + f^n(0)\big). \end{cases}$$

From (5.10), *Observation II*, and the estimate $|f^n(0)| \le \|(-x)^{\frac{\beta}{2}} f^n_x\|_{L^2(-1,0)}$, we deduce

$$\left|C_{1n}\right| = |s_n|^{-\frac{1}{2}} o(1), \quad \left|C_{2n}\right| = |s_n|^{-\frac{1}{2-\beta}-\frac{1}{2}\gamma(\alpha,\beta)} o(1). \tag{5.19}$$

Following Zerkouk *et al.* in [37, Lemma 5.3], we have the following estimates for $U_n^+(x), U_n^-(x)$ and $F_n(x)$:

Proposition 5.3. *Let $U_n^\pm(x), F_n(x)$ be defined in (5.18). For $f^n \in H^1_{\beta,0}(-1,0)$ and $g^n \in L^2(-1,0)$, the following estimates hold:*

(i) $|s_n|^{\frac{1}{2}} \|U_n^+\|_{L^2(-1,0)} = O(1),\ |s_n|^{\frac{1}{2}} \|U_n^-\|_{L^2(-1,0)} = O(1)$;

(ii) $|s_n|^{-\frac{1}{2}} \|(-x)^{\frac{\beta}{2}} U^+_{nx}\|_{L^2(-1,0)} = O(1),\ |s_n|^{-\frac{1}{2}} \|(-x)^{\frac{\beta}{2}} U^-_{nx}\|_{L^2(-1,0)} = O(1)$;

(iii) $\|(-x)^{\frac{\beta}{2}} F_{nx}\|_{L^2(-1,0)} \le C(\|f^n\|_{H^1_{\beta,0}(-1,0)} + \|g^n\|_{L^2(-1,0)})$,

for some constant $C > 0$ independent of n.

For the sake of completeness, we provide the sketch of the proof of this proposition in Appendix B, and we refer to [37] for more details.

From (5.17), we have

$$(-x)^{\frac{\beta}{2}} u^n_x(x) = C_{1n}(-x)^{\frac{\beta}{2}} U^+_{nx}(x) + C_{2n}(-x)^{\frac{\beta}{2}} U^-_{nx}(x) + (-x)^{\frac{\beta}{2}} F_{nx}(x),\ x \in (-1,0).$$

Using Proposition 5.3-(iii), we obtain

$$\begin{aligned}\|(-x)^{\frac{\beta}{2}} u^n_x\|_{L^2(-1,0)} \le & |C_{1n}| \cdot \|(-x)^{\frac{\beta}{2}} U^+_{nx}\|_{L^2(-1,0)} + |C_{2n}| \cdot \|(-x)^{\frac{\beta}{2}} U^-_{nx}(x)\|_{L^2(-1,0)} \\ & + C(\|f^n\|_{H^1_{\beta,0}(-1,0)} + \|g^n\|_{L^2(-1,0)}).\end{aligned}$$

Therefore, using the estimate (5.19) and Proposition 5.3-(ii), together with (5.4), we obtain

$$\|u^n\|_{H^1_{\beta,0}(-1,0)} \sim \|(-x)^{\frac{\beta}{2}} u^n_x\|_{L^2(-1,0)} = o(1). \tag{5.20}$$

To estimate $\|v^n\|_{L^2(-1,0)}$, we take the inner product in $L^2(-1,0)$ of the second equation in (5.4) with v^n and in $L^2(0,1)$ of the third equation in (5.4) with w^n, respectively. Then, adding them and using integration by parts, together with the coupling conditions in (5.5), we get

$$\begin{aligned}& is_n\|v^n\|^2_{L^2(-1,0)} + \langle(-x)^{\frac{\beta}{2}} u^n_x, (-x)^{\frac{\beta}{2}} v^n_x\rangle + is_n\|w^n\|^2_{L^2(0,1)} + \|x^{\frac{\alpha}{2}} w^n_x\|^2_{L^2(0,1)} \\ =& \langle g^n, v^n\rangle + \langle h^n, w^n\rangle.\end{aligned} \tag{5.21}$$

Notice from the second equation in (5.4) that $(-x)^{\frac{\beta}{2}} v^n_x = is_n(-x)^{\frac{\beta}{2}} u^n_x - (-x)^{\frac{\beta}{2}} f^n_x$ holds in $L^2(-1,0)$. Inserting this into (5.21) and then taking the imaginary part, we derive

$$\begin{aligned}& s_n\|v^n\|^2_{L^2(-1,0)} - s_n\|(-x)^{\frac{\beta}{2}} u^n_x\|^2_{L^2(-1,0)} + s_n\|w^n\|^2_{L^2(0,1)} - \mathcal{I}m\langle(-x)^{\frac{\beta}{2}} u^n_x, (-x)^{\frac{\beta}{2}} f^n_x\rangle \\ =& \mathcal{I}m\langle g^n, v^n\rangle + \mathcal{I}m\langle h^n, w^n\rangle.\end{aligned}$$

So,

$$\begin{aligned}\|v^n\|^2_{L^2(-1,0)} \le & \|(-x)^{\frac{\beta}{2}} u^n_x\|^2_{L^2(-1,0)} + \|w^n\|^2_{L^2(0,1)} + |s_n|^{-1}\left|\langle(-x)^{\frac{\beta}{2}} u^n_x, (-x)^{\frac{\beta}{2}} f^n_x\rangle\right| \\ & + |s_n|^{-1}\left|\langle g^n, v^n\rangle\right| + |s_n|^{-1}\left|\langle h^n, w^n\rangle\right| \\ \le & \frac{1}{4}\|v^n\|^2_{L^2(-1,0)} + \|(-x)^{\frac{\beta}{2}} u^n_x\|^2_{L^2(-1,0)} + |s_n|^{-2}\|g^n\|^2_{L^2(-1,0)} \\ & + \|w^n\|^2_{L^2(0,1)} + |s_n|^{-1}\|(-x)^{\frac{\beta}{2}} u^n_x\|_{L^2(-1,0)}\|(-x)^{\frac{\beta}{2}} f^n_x\|_{L^2(-1,0)} \\ & + |s_n|^{-1}\|h^n\|_{L^2(0,1)}\|w^n\|_{L^2(0,1)}.\end{aligned}$$

Finally, from (5.4), (5.6) and (5.20), we conclude that

$$\|v^n\|_{L^2(-1,0)} = o(1). \tag{5.22}$$

Combining (5.6), (5.20), and (5.22), we obtain the contradiction $\|X^n\|_{\mathcal{H}} = o(1)$, completing the proof of Theorem 2.2–(i). □

5.2. Optimality of the decay rate. This section is devoted to proving the optimality of the decay rate established in Theorem 2.2–(i) based on the distribution of the spectrum given in Theorem 4.3 and the frequency characteristics of semigroups.
Proof of Theorem 2.2–(ii). First, we define

$$\gamma^* := \sup\left\{\gamma > 0 \,\middle|\, \exists C > 0,\ \forall t > 1, \|(u, u_t, w)\|_{\mathcal{H}} \leq C t^{-\gamma} \|(u_0, v_0, w_0)\|_{\mathcal{D}(\mathcal{A})}\right\}.$$

It follows from Theorem 2.2–(i) that $\gamma^* \geq \frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}$. From Lemma 5.1 and the proof of Theorem 2.2–(i), we know that the resolvent norm of $\mathcal{A}$ satisfies

$$\|R(is;\mathcal{A})\| = \mathcal{O}(|s|^{\gamma(\alpha,\beta)}), \text{as } |s| \to +\infty,\ s \in \mathbb{R}, \tag{5.23}$$

where $\gamma(\alpha, \beta)$ is given as in Theorem 4.3.

We prove that there exists a constant $\underline{C} > 0$ independent of s such that

$$\liminf_{|s|\to+\infty} \frac{\|R(is;\mathcal{A})\|}{|s|^{\gamma(\alpha,\beta)}} \geq \underline{C}. \tag{5.24}$$

In fact, by Theorem 4.3, we may find a sequence $\{\lambda_n\} \subset \sigma_p(\mathcal{A})$ and a constant $\sigma_1 \in \mathbb{R}_+$ such that

$$\begin{cases} |\lambda_n| \sim |\mathcal{I}m\lambda_n| \sim \frac{2-\beta}{2} n\pi, & n \to \infty; \\ 0 > \mathcal{R}e\lambda_n \geq -\sigma_1 n^{-\gamma(\alpha,\beta)}, & \forall \text{ large } n \in \mathbb{N}. \end{cases}$$

Let $\tilde{X}_n$ be an eigenfunction corresponding to λ_n. Set $s_n := \mathcal{I}m\lambda_n \in \mathbb{R}_+$ and define $X_n := (is_n - \mathcal{A})\tilde{X}_n$. Thus, for sufficiently large n, we compute

$$\begin{aligned} \left\|R(is_n;\mathcal{A})\right\| &= \sup_{X\in\mathcal{H}, |X|\neq 0} \frac{\left\|(is_n - \mathcal{A})^{-1}X\right\|}{\|X\|} \\ &\geq \frac{\left\|(is_n - \mathcal{A})^{-1}X_n\right\|}{\|X_n\|} = \frac{\|\tilde{X}_n\|}{\|(is_n - \mathcal{A})\tilde{X}_n\|} = \frac{1}{|\mathcal{R}e\lambda_n|} \geq \frac{|n|^{\gamma(\alpha,\beta)}}{\sigma_1} \geq \underline{C} \cdot |s_n|^{\gamma(\alpha,\beta)}, \end{aligned}$$

for some constant $\underline{C} > 0$ independent of n. This establishes (5.24).

Finally, combining the upper and lower bounds in (5.23)-(5.24), we deduce that

$$\gamma(\alpha, \beta) = \inf\left\{\theta > 0 \,\middle|\, \|R(is;\mathcal{A})\| = \mathcal{O}(|s|^\theta),\ \text{as } |s| \to +\infty,\ s \in \mathbb{R}\right\}.$$

Then, using Lemma 5.1 again yields $\gamma^* \leq \frac{1}{\gamma(\alpha,\beta)} = \frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}$. Therefore, we conclude that $\gamma^* = \frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}$, which confirms the optimality of the decay rate of system (1.1)–(1.2) established in Theorem 2.2–(i). The proof of Theorem 2.2-(ii) is complete. □

6. Conclusions and open problems. In this work, we investigated the polynomial stability of a degenerate coupled heat-wave system defined on two connected intervals, where both components simultaneously degenerate at the interface. The interplay between degeneracy exponents α (associated with the parabolic equation) and β (associated with the hyperbolic equation) critically determines the system's long-time dynamics. Through detailed frequency-domain analysis and asymptotic properties of Bessel functions, we established the polynomial decay rate $t^{-\frac{(2-\alpha)(2-\beta)}{2(1-\alpha)+\beta}}$ for the jointly degenerate system (1.1)–(1.2) with smooth initial data. A refined spectral analysis of the system operator confirms the sharpness of this decay rate, providing the first rigorous proof of optimal energy decay for such coupled PDE systems.

This result underscores the decisive influence of interface degeneracy structures on energy dissipation mechanisms. The analytical framework developed herein — integrating frequency-domain techniques, spectral estimates, and Bessel function properties — offers a versatile approach for analyzing coupled systems with mixed dynamics and degeneracy.

Despite advances in characterizing optimal decay rates, several open problems remain:

- **Strong Degeneracy:** The case $\alpha \geq 1$ or $\beta \geq 1$ (Remark 2.5) requires careful study. Modified interface conditions must be developed to ensure well-posedness and dissipativity.
- **Singular Degeneracy:** When $\alpha < 0$ or $\beta < 0$ in system (1.1), diffusion or elasticity coefficients diverge at $x = 0$. This singularity fundamentally alters energy propagation and dissipation, presenting significant mathematical challenges.
- **Multidimensional Extensions:** Generalizing the analysis to higher- dimensional settings and complex geometries could deepen theoretical understanding of degenerate systems' dynamics. Notably, investigating this problem through geometric optics offers a promising research direction.

Appendix A. Proof of Lemma 4.1. Observe that Y is a Banach space equipped with the following norm:

$$\|u\|_Y := \left(\|u\|^2_{L^2(-1,0)} + \left\|(-x)^{\frac{\beta}{2}}u_x\right\|^2_{L^2(-1,0)} + \left\|((-x)^{\beta}u_x)_x\right\|^2_{L^2(-1,0)}\right)^{\frac{1}{2}},\ u \in Y. \tag{A.1}$$

It is evident that $\|u\|_{H^1_{\beta,0}} \leq \|u\|_Y$, which implies that the space Y is continuously embedded in $H^1_{\beta,0}(-1,0)$.

In what follows, we will demonstrate that this embedding is not only continuous but also compact. To establish this, it suffices to show that if $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in Y, there exists a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ which converges in $H^1_{\beta,0}(-1,0)$.

Assume that $\{u_n\}_{n=1}^{\infty}$ is a sequence in Y satisfying $\sup_n \|u_n\|_Y < \infty$. We now define $y_n(x) := (-x)^{\beta}u_{nx}(x)$, $n \geq 1$. From the definition of Y, we conclude that $\{y_n\}_{n=1}^{\infty} \subset H^1(-1,0)$ and $\sup_n \|y_n\|_{H^1(-1,0)} < \infty$. By the Sobolev embedding theorem, we see that $\{y_n\}_{n=1}^{\infty} \subset C([-1,0])$ and $\sup_n \|y_n\|_{C([-1,0])} \leq \sup_n \|y_n\|_{H^1(-1,0)} < \infty$. This implies that the sequence $\{y_n(x)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[-1,0]$. Hence, by the Arzelà-Ascoli theorem, there exists a subsequence $\{y_{n_j}\}_{j=1}^{\infty}$ that converges to a function $y \in C([-1,0])$, i.e.,

$$y_{n_j} \to y \quad \text{in} \quad C([-1,0]) \quad \text{as} \quad j \to \infty. \tag{A.2}$$

Next, we set $\psi(x) := \int_{-1}^{x}(-s)^{-\beta}y(s)ds$. It is easy to verify that $\psi(x) \in H^1_{\beta,0}(-1,0)$. We then prove that

$$u_{n_j} \to \psi \quad \text{in} \quad H^1_{\beta,0}(-1,0) \quad \text{as} \quad j \to \infty. \tag{A.3}$$

Indeed, by direct calculation, we estimate

$$\begin{aligned}\|u_{n_j} - \psi\|_{H^1_{\beta,0}(-1,0)} &= \left\|(-x)^{\frac{\beta}{2}}u_{n_jx} - (-x)^{\frac{\beta}{2}}\psi_x\right\|_{L^2(-1,0)}\\ &= \left\|(-x)^{-\frac{\beta}{2}}(y_{n_j} - y)\right\|_{L^2(-1,0)} \leq \frac{1}{\sqrt{1-\beta}}\left\|y_{n_j} - y\right\|_{C([-1,0])}.\end{aligned} \tag{A.4}$$

Thus, assertion (A.3) follows from (A.2) and (A.4). Therefore, Y is compactly embedded in $H^1_{\beta,0}(-1,0)$ and the proof is complete. □

Appendix B. Proof of Proposition 5.3. We mainly prove Proposition 5.3 through a series of lemmas, which provide estimates for $U_n^+(x)$, $U_n^-(x)$ and $F_n(x)$, as detailed below.

$$\begin{cases} U_n^+(x) = (-x)^{\frac{1-\beta}{2}} J_{\nu_\beta}\Big(\frac{2}{2-\beta} s_n (-x)^{\frac{2-\beta}{2}}\Big), \quad U_n^-(x) = (-x)^{\frac{1-\beta}{2}} J_{-\nu_\beta}\Big(\frac{2}{2-\beta} s_n (-x)^{\frac{2-\beta}{2}}\Big), \\ F_n(x) := \frac{\pi}{(2-\beta) sin(\nu_\beta \pi)} \int_x^0 (-is_n f^n(\tau) - g^n(\tau))\Big(U_n^+(x)U_n^-(\tau) - U_n^-(x)U_n^+(\tau)\Big) d\tau. \end{cases}$$

Moreover, let $G_n^\pm(x) := \frac{\pi}{(2-\beta)sin(\nu_\beta\pi)} \int_x^0 (-is_n f^n(\tau) - g^n(\tau)) U_n^\pm(\tau) d\tau$, and hence $F_n(x) = U_n^+(x)G_n^-(x) - U_n^-(x)G_n^+(x)$.

We primarily provide the proof sketches for the following lemmas. For more detailed computational steps, we refer readers to Appendix B in [37].

Lemma B.1. *Set $\theta_n := \frac{2}{2-\beta} s_n$ and $z_n(x) := \theta_n(-x)^{\frac{2-\beta}{2}}$. The following relationships hold:*

(i) $\|U_n^+\|_{L^2(-1,0)} = \Big\|(-x)^{\frac{1-\beta}{2}} J_{\nu_\beta}(z_n(x))\Big\|_{L^2(-1,0)} = (\dfrac{2-\beta}{2})^{\frac12} s_n^{-1} \Big\| z^{\frac12} J_{\nu_\beta}(z)\Big\|_{L^2(0,\theta_n)}$,
$\|U_n^-\|_{L^2(-1,0)} = \Big\|(-x)^{\frac{1-\beta}{2}} J_{-\nu_\beta}(z_n(x))\Big\|_{L^2(-1,0)} = (\dfrac{2-\beta}{2})^{\frac12} s_n^{-1} \Big\| z^{\frac12} J_{-\nu_\beta}(z)\Big\|_{L^2(0,\theta_n)}$;

(ii) $s_n^{-\frac12}\Big\| z^{\frac12} J_{\nu_\beta}(z)\Big\|_{L^2(0,\theta_n)}$, $s_n^{-\frac12}\Big\| z^{\frac12} J_{-\nu_\beta}(z)\Big\|_{L^2(0,\theta_n)} = O(1)$;

(iii) $s_n^{\frac12}\|U_n^+\|_{L^2(-1,0)}$, $s_n^{\frac12}\|U_n^-\|_{L^2(-1,0)} = O(1)$.

Sketch of proof. Firstly, it is easy to check that the equalities in (i) follow from direct computations. And for (ii), on the one hand, we have the recurrence formula $zJ_\nu'(z) + \nu J_\nu(z) = zJ_{\nu-1}(z)$; on the other hand, assuming that $\alpha \neq \beta$ are complex numbers and $\mathcal{R}e\ \nu > -1$, $J_\nu(z)$ satisfies the integral formulas (see [23]):

$$\begin{aligned} &(\alpha^2-\beta^2)\int_0^z xJ_\nu(\alpha x)J_\nu(\beta x)dx = z\Big[J_\nu(\alpha z)\frac{d}{dz}\{J_\nu(\beta z)\} - J_\nu(\beta z)\frac{d}{dz}\{J_\nu(\alpha z)\}\Big], \\ &2\alpha^2\int_0^z x|J_\nu(\alpha x)|^2 dx = \Big(z\frac{d}{dz}\{J_\nu(\alpha z)\}\Big)^2 + (\alpha^2 z^2 - \nu^2)\Big(J_\nu(\alpha z)\Big)^2. \end{aligned} \tag{B.1}$$

Thus, by recurrence formula and $(B.1)_2$, we compute that

$$\Big\|z^{\frac12} J_{\nu_\beta}(z)\Big\|^2_{L^2(0,\theta_n)} = \frac12\Big[\theta_n^2 J^2_{\nu_\beta}(\theta_n) - 2\theta_n\nu_\beta J_{\nu_\beta}(\theta_n)J_{\nu_\beta+1}(\theta_n) + \theta_n^2 J^2_{\nu_\beta+1}(\theta_n)\Big],$$

which combined with the asymptotic representations (3.4) of J_ν yields (ii). And then (iii) immediately follows from (i) and (ii). □

Lemma B.2. *For such $\theta_n = \frac{2}{2-\beta}s_n$ and $z_n(x) = \theta_n|x|^{\frac{2-\beta}{2}}$, we have*

(i) $\|(-x)^{\frac{\beta}{2}}U_{nx}^+\|^2_{L^2(-1,0)} \le (1-\beta)\Big\|(-x)^{-\frac12}J_{\nu_\beta}(z_n(x))\Big\|^2_{L^2(-1,0)} + s_n^2\Big\|(-x)^{\frac{1-\beta}{2}}J_{\nu_\beta+1}(z_n(x))\Big\|^2_{L^2(-1,0)}$,
$\|(-x)^{\frac{\beta}{2}}U_{nx}^-\|^2_{L^2(-1,0)} = s_n^2\Big\|(-x)^{\frac{1-\beta}{2}}J_{1-\nu_\beta}(z_n(x))\Big\|^2_{L^2(-1,0)}$;

(ii) $s_n^{-\frac12}\Big\|(-x)^{-\frac12}J_{\nu_\beta}(z_n(x))\Big\|_{L^2(-1,0)} = (\dfrac{1}{2-\beta})^{\frac12} s_n^{-\frac12}\Big\|z^{-\frac12}J_{\nu_\beta}(z)\Big\|_{L^2(0,\theta_n)} = O(1)$;

(iii) $s_n^{-\frac12}\ \|(-x)^{\frac{\beta}{2}}U_{nx}^+\|_{L^2(-1,0)}$, $s_n^{-\frac12}\|(-x)^{\frac{\beta}{2}}U_{nx}^-\|_{L^2(-1,0)} = O(1)$.

Sketch of proof. Recall from (3.13) and (3.14) that (i) holds. Then using integration by parts and the recurrence formula $zJ_\nu'(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z)$, together with the Cauchy-Schwarz inequality, gives

$$\Big\|(-x)^{-\frac12}J_{\nu_\beta}(z_n(x))\Big\|^2_{L^2(-1,0)} = (\frac{1}{2-\beta})\Big\|z^{-\frac12}J_{\nu_\beta}(z)\Big\|^2_{L^2(0,\theta_n)} \le C\Big(J^2_{\nu_\beta}(\theta_n) + \Big\|z^{\frac12}J_{\nu_\beta+1}(z)\Big\|^2_{L^2(0,\theta_n)}\Big),$$

which along with Lemma B.1-(ii) and (3.4) leads to (ii). It then follows from (i) and (ii) that (iii) holds. □

Lemma B.3. *For any $f^n \in H^1_{\beta,0}(-1,0)$, $g^n \in L^2(-1,0)$, we have*

(i) $\Big\|\displaystyle\int_x^0 is_n f^n U_n^+ dt\Big\|_{L^\infty(-1,0)} \le C|s_n|^{-\frac12}\|f^n\|_{H^1_{\beta,0}(-1,0)}$,
$\Big\|\displaystyle\int_x^0 is_n f^n U_n^- dt\Big\|_{L^\infty(-1,0)} \le C|s_n|^{-\frac12}\|f^n\|_{H^1_{\beta,0}(-1,0)}$;

(ii) $\|G_n^+\|_{L^\infty(-1,0)}$, $\|G_n^-\|_{L^\infty(-1,0)} \le C|s_n|^{-\frac12}\Big(\|f^n\|_{H^1_{\beta,0}(-1,0)} + \|g^n\|_{L^2(-1,0)}\Big)$;

(iii) $\|(-x)^{\frac{\beta}{2}}F_{nx}\|_{L^2(-1,0)} \le C\Big(\|f^n\|_{H^1_{\beta,0}(-1,0)} + \|g^n\|_{L^2(-1,0)}\Big)$.

Sketch of proof. The estimate (i) can be established through integration by parts, in conjunction with the Cauchy-Schwarz inequality and the differential recurrence formula: $-z^{1-\nu}J_\nu(z) = \frac{d}{dz}(z^{1-\nu}J_{\nu-1}(z))$. Subsequently, (ii) and (iii) can be directly derived from (i), Lemma B.1-(iii), and Lemma B.2-(iii). □

Finally, observing from Lemma B.1-(iii), Lemma B.2-(iii), and Lemma B.3-(iii), we have completed the proof of Proposition 5.3.

Acknowledgment. The authors are grateful to the editors and anonymous referees for their insightful comments and constructive suggestions.

REFERENCES

[1] F. Alabau-Boussouira, P. Cannarsa, and G. Leugering, Control and stabilization of degenerate wave equations, *SIAM J. Control Optim.*, 2017, 55(3), pp. 2052–2087.
[2] F. Alabau-Boussouira, P. Cannarsa, and G. Leugering, Carleman estimates for degenerate parabolic operators with applications to null controllability, *J. Evol. Equ.*, 2006, 6, pp. 161–204.
[3] W. Arendt and C. J. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.*, 1988, 306, pp. 837–852.
[4] G. Avalos, I. Lasiecka, and R. Triggiani, Heat-wave interaction in 2-3 dimensions: optimal rational decay rate, *J. Math. Anal. Appl.*, 2016, 437, pp. 782–815.
[5] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.*, 2010, 347(2), pp. 455–478.
[6] P. Baras and J. Goldstein, Remarks on the inverse square potential in quantum mechanics, *Differ. Equ. North-Holland Math. Stud.*, 1984, 92, pp. 31–35.
[7] C. J. K. Batty, L. Paunonen, and D. Seifert, Optimal energy decay in a one-dimensional coupled wave-heat system, *J. Evol. Equ.*, 2016, 16, pp. 649–664.
[8] C. J. K. Batty, L. Paunonen, and D. Seifert, Optimal energy decay for the wave-heat system on a rectangular domain, *SIAM J. Math. Anal.*, 2018, 51(2), pp. 808–819.
[9] C. J. K. Batty, R. Chill, and Y. Tomilov, Fine scales of decay of operator semigroups, *J. Eur. Math. Soc.*, 2016, 18(4), pp. 853–929.
[10] J. Bebernes and D. Eberly, Mathematical Problems from Combustion Theory, Applied Mathematical Sciences, vol. 83. Springer, New York, 1989.
[11] H. Brezis and J. L. Vazquez, Blow-up solutions of some nonlinear elliptic equations, *Rev. Mat. Complut.*, 1997, 10, pp. 443–469.
[12] P. Cannarsa, P. Martinez, and J. Vancostenoble, Null controllability of degenerate heat equations, *Adv. Differential Equations*, 2005, 10, pp. 153–190.
[13] P. Cannarsa, P. Martinez, and J. Vancostenoble, The cost of controlling weakly degenerate parabolic equations by boundary controls, *Math. Control Relat. Fields*, 2017, 7, pp. 171–211.
[14] P. Cannarsa, P. Martinez, and J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, *SIAM J. Control Optim.*, 2008, 47, pp. 1–19.
[15] T. Duyckaerts, Optimal decay rates of the energy of a hyperbolic-parabolic system coupled by an interface, *Asymptot. Anal.*, 2007, 51, pp. 17–45.
[16] M. Fotouhi and L. Salimi: Controllability results for a class of one dimensional degenerate/singular parabolic equations, *Commun. Pure Appl. Anal.*, 2013, 12, pp. 1415–1430.
[17] G. Fragnelli and D. Mugnai, Control of Degenerate and Singular Parabolic Equations: Carleman Estimates and Observability, Cham, Switzerland: Springer, 2021.
[18] M. Gueye, Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations, *SIAM J. Control Optim.*, 2014, 52, pp. 2037–2054.
[19] Z. J. Han, H. Q. Song, and K. Yu, Sharp decay rates of degenerate hyperbolic-parabolic coupled system: Rectangular domain vs one-dimensional domain, *J. Differential Equations*, 2023, 349, pp. 53–82.
[20] Z. J. Han, G. Wang, and J. Wang, Explicit decay rate for a degenerate hyperbolic-parabolic coupled system, *ESAIM Control Optim. Calc. Var.*, 2020, 26, 116.
[21] Z. J. Han and E. Zuazua, Decay rates for 1-d heat-wave planar networks, *Netw. Heterog. Media*, 2016, 11, pp. 655–692.
[22] E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen*, 3rd ed., Chelsea Publishing, New York, 1948.
[23] N. N. Lebedev, *Special Functions and their Applications*, Dover Publications, New York, 1972.
[24] Z. Liu and B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, *Z. Angew. Math. Phys.*, 2005, 56, pp. 630–644.
[25] Y. Lyubich and P. Vũ, Asymptotic stability of linear differential equations in Banach spaces, *Studia Mathematica*, 1988, 88(1), pp. 37–42.
[26] R. D. Meyer, Degenerate elliptic differential systems, *J. Math. Anal. Appl.*, 1970, 29, pp. 436–442.

[27] H. Morand and R. Ohayon, *Fluid Structure Interaction: Applied Numerical Methods*, Wiley, New York, 1995.

[28] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[29] J. Rozendaal, D. Seifert, and R. Stahn, Optimal rates of decay for operator semigroups on Hilbert spaces, *Adv. Math.*, 2019, 346, pp. 359–388.

[30] J. Rauch, Qualitative behavior of dissipative wave equations on bounded domains, *Arch. Rational Mech. Anal.*, 1976, 62, pp. 77–85.

[31] J. Rauch, X. Zhang and E. Zuazua, *Polynomial decay for a hyperbolic-parabolic coupled system*, *J. Math. Pures Appl.*, 2005, 84, pp. 407–470.

[32] L. Tebou, Sharp decay estimates for semigroups associated with some one-dimensional fluid-structure interactions involving degeneracy, *SIAM J. Control Optim.*, 2022, 60, pp. 2787–2810.

[33] J. Vancostenoble and E. Zuazua, Null controllability for the heat equation with singular inverse-square potentials, *J. Funct. Anal.*, 2008, 254, pp. 1864–1902.

[34] J. Vancostenoble, Lipschitz stability in inverse source problems for singular parabolic equations, *Commun. Partial Differ. Equ.*, 2011, 36, pp. 1287–1317.

[35] J. Vancostenoble, Improved Hardy-Poincaré inequalities and sharp Carleman estimates for degenerate/singular parabolic problems, *Discrete Contin. Dyn. Syst. Ser. S*, 2011, 4(3), pp. 761–790.

[36] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1958.

[37] H. Zerkouk, C. Aichi, and A. Benaissa, On the stability of a degenerate wave Equation under fractional feedbacks acting on the degenerate boundary, *J. Dyn. Control Syst.*, 2022, 28, pp. 601–633.

[38] X. Zhang and E. Zuazua, Control, observation and polynomial decay for a coupled heat-wave system, *C. R. Acad. Sci. Paris, Ser. I.*, 2003, 336, pp. 823–828.

[39] X. Zhang and E. Zuazua, Polynomial decay and control of a 1-d hyperbolic-parabolic coupled system, *J. Differential Equations*, 2004, 204, pp. 380–438.

[40] X. Zhang and E. Zuazua, Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction, *Arch. Ration. Mech. Anal.*, 2007, 184, pp. 49–120.

[41] E. Zuazua, Null control of a 1-d model of mixed hyperbolic-parabolic type, in Optimal Control and Partial Differential Equations, J. L. Menaldi et al., eds., IOS Press, 2001, pp. 198–210.