SLOW DECAY AND TURNPIKE FOR INFINITE-HORIZON HYPERBOLIC LQ PROBLEMS*

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Abstract. This paper is devoted to analysing the explicit slow decay rate and turnpike in the infinite-horizon linear quadratic optimal control problems for hyperbolic systems. Assume that some weak observability or controllability are satisfied, by which, the lower and upper bounds of the corresponding algebraic Riccati operator are estimated, respectively. Then based on these two bounds, the explicit slow decay rate of the closed-loop system with Riccati-based optimal feedback control is obtained. The averaged turnpike property for this problem is also further discussed. We then apply these results to the LQ optimal control problems constraint to networks of one-dimensional wave equations and also some multi-dimensional ones with local controls which lack of GCC(Geometric Control Condition).

Key words. Optimal control problems, Riccati operator, slow decay rate, weak controllability and observability, turnpike property.

AMS subject classifications. 49J20, 49K20, 93C20, 49N05.

1. Introduction.

The object of this paper is devoted to discussing the large time behaviour and turnpike property in infinite-time linear quadratic(LQ) optimal control problems under weak controllability and observability hypothesises. Specifically, we will discuss the relationship between the bounds of the corresponding algebraic Riccati operator and the weak controllability and observability properties, and based on which we identify the explicit slow decay rate of the closed-loop system with the Riccati-based optimal feedback control. Moreover, under weak controllability and observability hypothesises, we further discuss that how the optimal control and trajectories of the LQ optimal control problems converge to the corresponding stationary optimal control and state, that is the so-called turnpike property.

LQ optimal control problems have been studied extensively in recent thirties years, see [28] for finite dimensional systems, [5], [10], [11], [16] and [22] for the infinite dimensional systems with bounded input or output operator, and [8], [19], [20] for the ones with unbounded input or output operator.

It is known that the solution of LQ optimal control problems can be constructed as a feedback form based on solving its corresponding Riccati equations(see [7], [22]). In other words, the solution to (algebraic) Riccati equation, which is called (algebraic) Riccati operator, is corresponding to the finite-time (or infinite-time) LQ optimal

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feedback control. Especially, the algebraic Riccati operator is usually used to stabilize the system. For instance, Aksikas et. al. in [1] designed the LQ feedback control to stabilize a class of hyperbolic PDE systems exponentially, by solving the matrix Riccati differential equation. Porretta and Zuazua in [18] and [31] proved the exponential decay of the closed-loop system with Riccati-based optimal feedback control under the assumptions of exact observability of (A, C) and exact controllability of (A, B). Indeed, based on the exact observability of B^* and C, they obtained that $\langle \hat{E}x, x\rangle$ is a strict Lyapunov function for the system, where $\hat{E}$ is the algebraic Riccati operator and x is the state of the system in the Hilbert state space H. Meanwhile, it is also proposed in [18] that the exponential-type point-wise turnpike property can be further proved based on the exponential decay rate of the closed-loop system with Riccati-based optimal feedback controls (see [24], [25] and [12]).

Note that in the previous results, both the stabilization of infinite-time LQ optimal control problems and turnpike properties are all discussed under the assumptions of exact controllability and observability. In [18], a kind of slow turpike in average(Logarithmic-type) was concluded for multi-dimensional wave equation without GCC(Geometric Control Condition), which inspired us to give a complete analysis on the turnpike properties for the LQ optimal control problems if lacking of exact controllability or observability. It is obvious that some weaker controllability or observability are still necessary so as to guarantee not only the existence of the solutions to infinite-time LQ optimal control problems but also the feedback stabilization of the system. Thus, to do this, some hypothesises on weak-type observability estimates are chosen as given in the next section.

In this work, we shall first address the following problem:

Q1. To which extent do weak observability or controllability determine the decay rates of the energy for the systems with Riccati-based optimal feedback controls?

We find that if the existence of the solution to the LQ optimal control problems is guaranteed, then the lower bound of $\langle \hat{E}x, x\rangle$ is totally determined by the observability property of output operator C, while the upper bound is totally dependent on the controllability property of the input operator B. Specifically, on one hand, when C is weakly observable, the lower bound can be obtained respect to an weak norm of x, which is estimated completely based on the extent of the weak observability. On the other hand, when B is weakly controllable, then the upper bound of $\langle \hat{E}x, x\rangle$ can be estimated respect to a strong norm of x, totally by the extent of the weak controllability. Based on these properties of the algebraic Riccati operator, we deduce the explicit slow decay rate of the closed-loop system related to the infinite-time LQ optimal control problems.

The solving of Q1 is one key step to further discuss the turnpike properties of the LQ optimal control problems(see [18], [31]), that is the following problem under consideration:

Q2. To which extent do weak observability or controllability lead to turnpike properties of the LQ optimal control problems?

Based on the weak observability of (A, B^*) and (A, C), we show that the averaged turnpike property of such problems holds under certain conditions on the initial state, the stationary optimal state and its dual. It is worth mentioning that in [18], under the exact controllability and observability, the exponential decay of the closed-loop system with Riccati-based optimal control always holds, based on which the exponential-type point-wise turnpike property for the LQ problems can be proved. Thus, under weak controllability and observability, note that the slow decay rate of the closed-loop system can be estimated in our this work, and then the slow(polynomial-type etc.) point-wise turnpike property should be also reasonable to be expected.

However, the (slow) point-wise turnpike property of the LQ optimal control problems under weak controllability or observability hypothesises is still an open problem and worth further investigating in future. In fact, the non-uniform slow decay rates of the closed-loop system inevitably caused by the weak observability of (A, B^*) and (A, C) are always dependent on the regularity of the initial states. This makes the energy estimates for (slow) point-wise turnpike property become a tough issue to be tackled. This is different from the case under exact controllability and observability, in which the uniform exponential decay of the closed-loop systems always holds for all initial states and the corresponding energy estimates can be easily carried out.

The rest of this paper is given as follows. In Section 2, the preliminary and main results of this paper are presented. The bounds estimates for the algebraic Riccati operator and the explicit slow decay rate are given under weak controllability and observability hypothesises. The averaged turnpike property is also further presented. In Section 3, we prove the main results given in this work. Section 4 is devoted to presenting some examples on the slow decay rates and turnpike for some kinds of hyperbolic systems(networks of wave equations and multi-dimensional ones) without GCC. Finally, in Section 5 a conclusion and future work are given.

2. Preliminary and main results.

This section is devoted to problem formulation and further presenting the decay result and turnpike property of the LQ optimal control problems under the weak assumptions of the controllability and observability.

2.1. Problem formulation.

Similar to the abstract frame setting for control systems in [26], assume that $A : \mathcal{D}(A) \to H$ is a self-adjoint, strictly positive operator with compact resolvent. Thus, A is diagonalizable. Hence, if the eigenvalues of A are given as $(\lambda_n^2)_{n\geq 1}$, its corresponding eigenvectors $(\phi_n)_{n\geq 1}$ forms an orthonormal basis in H. Moreover,

$$D(A) := \{\varphi \in H | \sum_{n\geq 1} \lambda_n^4 |\langle \varphi, \phi_k\rangle|^2 < \infty\}.$$

Define the space X_β as follows.

$$X_\beta := \{\varphi \in H | \sum_{n\geq 1} \lambda_n^{4\beta} |\langle \varphi, \phi_k\rangle|^2 < \infty\}$$

with inner product

$$\langle \varphi, \psi\rangle_\beta = \langle A^\beta \varphi, A^\beta \psi\rangle$$

where

$$A^\beta \varphi := \sum_{n\geq 1} \lambda_n^{2\beta} \langle z, \phi_n\rangle \phi_n.$$

Let us consider the following control system:

$$\begin{cases} w_{tt} + Aw = Bu(t), \\ w(0) = w_0, \quad w_t(0) = w_1. \end{cases} \tag{2.1}$$

In (2.1), assume that $B \in \mathcal{L}(U, H)$ and $C \in \mathcal{L}(X_{1/2}, V)$, where U, V are Hilbert spaces. We know that system (2.1) is well-posed with input space U in Hilbert state space $\mathcal{H} := X_{1/2} \times H$ with norm

$$\|(w,v)\|_{\mathcal{H}}^2 = \langle (w,v),(w,v)\rangle_{\mathcal{H}} = \langle A^{\frac{1}{2}}w, A^{\frac{1}{2}}w\rangle + \langle v, v\rangle.$$

Thus, for $(w_0, w_1) \in \mathcal{H}$ and $u \in L^2([0,\infty), U)$, system (2.1) admits a unique solution $w \in C([0,\infty), X_{1/2}) \cap C^1([0,\infty); H)$. For simplicity, we choose $U = V = H$.

Set $\mathcal{A} := \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}$ with $\mathcal{D}(\mathcal{A}) = X_1 \times X_{1/2}$, and $\mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}$. Then system (2.1) also can be written as follows:

$$\begin{cases} \frac{dW}{dt} = \mathcal{A}W + \mathcal{B}(0, u(t))^T, \\ W(0) = (w_0, w_1)^T, \end{cases} \tag{2.2}$$

where $W = (w, w_t)^T$.

It is well-known that $\mathcal{A}$ generates a C_0 contraction semigroup on $\mathcal{H}$. Define the interpolation space $\mathcal{D}(\mathcal{A}^s) = X_{(s+1)/2} \times X_{s/2}$ and so its dual one $\mathcal{D}(\mathcal{A}^s)' = X_{-s/2} \times X_{-(s+1)/2}$. Thus, we get

$$\|(w_0,w_1)\|_{\mathcal{D}(\mathcal{A}^s)}^2 = \sum_{j\geq 1} \lambda_n^{2s}(\lambda_n^2 a_n^2 + b_n^2), \tag{2.3}$$

and

$$\|(w_0,w_1)\|_{\mathcal{D}(\mathcal{A}^s)'}^2 = \sum_{j\geq 1} \lambda_n^{-2(s+1)}(\lambda_n^2 a_n^2 + b_n^2), \tag{2.4}$$

where $a_n,\ b_n$ are the Fourier's coefficients and

$$w_0 = \sum_{n\geq 1} a_n \phi_n(x), \quad w_1 = \sum_{n\geq 1} b_n \phi_n(x). \tag{2.5}$$

Assume that the operator B and C satisfy the following two estimates.

(H1). (A, B^*) is weakly observable, that is, for the system

$$\begin{cases} w_{tt} + Aw = 0, \\ w(0) = w_0, \quad w_t(0) = w_1, \end{cases} \tag{2.6}$$

there exist positive constants T_0 and c such that

$$c\int_0^{T_0} \|B^* w_t\|_{U'}^2 dt \geq \|(w_0,w_1)\|_{X_{1/2-1/(2\varrho)}\times X_{-1/(2\varrho)}}^2 = \sum_{j\geq 1} \lambda_n^{-\frac{2}{\varrho}}(\lambda_n^2 a_n^2 + b_n^2), \tag{2.7}$$

where $\varrho > 0$ is some constant, $\lambda_n,\ a_n,\ b_n,\ n = 1, 2, ...$ are given as in (2.5).

(H2). (A, C) is weakly observable, that is, there exist positive constants T_1 and c such that

$$c\int_0^{T_1} \|Cw\|_V^2 dt \geq \|(w_0,w_1)\|_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}}^2 = \sum_{j\geq 1} \lambda_n^{-\frac{2}{\eta}}(\lambda_n^2 a_n^2 + b_n^2), \tag{2.8}$$

where $\eta > 0$ is some constant, $\lambda_n,\ a_n,\ b_n,\ n = 1, 2, ...$ are also given as in (2.5).

Based on (H1), together with the semigroup theory, we can get the following slow decay rate for the system with the collocated feedback controls (see [2]).

LEMMA 2.1. *Assume that (H1) is fulfilled. Under the feedback control law*

$$u(t) = -B^* w_t, \tag{2.9}$$

it holds that for all $t > 0$*, the solution to the closed-loop system (2.1) decays polynomially for any* $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^k)$*,* $k > 0$*, that is*

$$\|(w, w_t)\|_{\mathcal{H}}^2 \leq c_1 (t+1)^{-k\varrho} \|(w_0, w_1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \ k > 0, \tag{2.10}$$

where $\varrho > 0$ *is the same as in (H1) and* c_1 *is a constant independent of initial data.*

Thus, by multiplying (2.1) with w_t, we have

$$\begin{aligned}\int_0^T ||B^* w_t||_H^2 dt &= -\frac{1}{2}\|(w(T), w_t(T))\|_{\mathcal{H}}^2 + \frac{1}{2}\|(w_0, w_1)\|_{\mathcal{H}}^2 \\ &\leq \frac{1}{2}\|(w_0, w_1)\|_{\mathcal{H}}^2 + \frac{1}{2} c_1 (T+1)^{-k\varrho} \|(w_0, w_1)\|_{\mathcal{D}(\mathcal{A}^k)}^2. \end{aligned} \tag{2.11}$$

If using the transformation $\phi = \int_0^t w dt + \phi(0)$ in (2.6), it holds that

$$\begin{cases} \phi_{tt} + A\phi = 0, \\ \phi(0) = A^{-1} w_1, \quad \phi_t(0) = w_0. \end{cases}$$

By (H2), we get

$$c \int_0^{T_1} \|C\phi_t\|_V^2 dt \geq \|(\phi(0), \phi_t(0))\|_{X_{1-1/(2\eta)} \times X_{1/2 - 1/(2\eta)}}^2. \tag{2.12}$$

Then, similar to Lemma 2.1, we obtain the following result.

LEMMA 2.2. *Assume that (H2) is fulfilled. Then for all* $0 < t < T$ *and* $(\phi_0^T, \phi_1^T) \in \mathcal{D}(\mathcal{A}^k)$*,* $k \geq 1$*, the solution to the following backward closed-loop system*

$$\begin{cases} \phi_{tt} + A\phi = C^* C \phi_t, \\ \phi(T) = \phi_0^T \in X_1, \quad \phi_t(T) = \phi_1^T \in X_{1/2}, \end{cases} \tag{2.13}$$

satisfies

$$\|(\phi, \phi_t)\|_{\mathcal{H}}^2 \leq c_1 (T - t + 1)^{-k\eta} \|(\phi_0^T, \phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \ k \geq 1, \tag{2.14}$$

where $\eta > 0$ *is the same as in (H2) and* c_1 *is a constant independent of terminal data.*

Thus, by multiplying (2.13) with ϕ_t, we have

$$\begin{aligned}\int_0^T ||C\phi_t||_H^2 dt &= \frac{1}{2}\|(\phi_0^T, \phi_1^T)\|_{\mathcal{H}}^2 - \frac{1}{2}\|(\phi(0), \phi_t(0))\|_{\mathcal{H}}^2 \\ &\leq \frac{1}{2}\|(\phi_0^T, \phi_1^T)\|_{\mathcal{H}}^2 + \frac{1}{2} c_1 (T+1)^{-k\eta} \|(\phi_0^T, \phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \ k \geq 1. \end{aligned} \tag{2.15}$$

We consider the following quadratic performance index associated with the control system (2.1).

$$\min J_t^T(u) = \frac{1}{2} \int_t^T [\|u(t)\|_U^2 + \|Cw(t)\|_V^2] dt, \ u \in L^2(t, T; U). \tag{2.16}$$

The corresponding OS (Optimality System) is given as follows:

$$\begin{cases} w_{tt}^T + Aw^T = Bu^T, \quad t \le s \le T, \\ w(t) = w_0, \quad w_t(t) = w_1, \\ u^T = -B^* p^T, \\ p_{tt}^T + Ap^T = C^* C w^T, \quad t \le s \le T, \\ p^T(T) = p_t^T(T) = 0. \end{cases} \tag{2.17}$$

Lemma 2.3. *Assume that (H2) is fulfilled. Then there exists a unique linear operator $\mathcal{E}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, where $\mathcal{H}'$ is the dual one for $\mathcal{H}$ and $\mathcal{H}' = X_{-1/2} \times H$, such that $\mathcal{E}(t)$ is strictly positive and monotone increasing in $\mathcal{H}$, and*

$$(-p_t^T, p^T) = \mathcal{E}(T-t)(w^T, w_t^T),$$

where (w^T, w_t^T) is the optimal state for system (2.1) in the sense of (2.16), and $\mathcal{E}(\cdot)$ is the solution to the Riccati equation with initial condition 0, that is,

$$\begin{cases} \mathcal{E}_t = \mathcal{C}^*\mathcal{C} + (\mathcal{E}\mathcal{A} + \mathcal{A}^*\mathcal{E}) - \mathcal{E}\mathcal{B}\mathcal{B}^*\mathcal{E}, \quad \text{in } (0, +\infty), \\ \mathcal{E}(0) = 0, \end{cases} \tag{2.18}$$

in which $\mathcal{C} = [C, 0]$, $\mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}$.

Proof. Since $B \in \mathcal{L}(U, H)$ and $C \in \mathcal{L}(X_{1/2}, V)$ and $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{H}$, by Theorem 2.1 (p. 393) in [5], we obtain the unique existence of the Riccati operator $\mathcal{E}(t)$. In fact, it is a consequence of the fact that the optimality system (2.17) has a unique solution, and the adjoint state $(-p_t, p)$ at time t is a linear function of the state (w, w_t) at time t.

Note that it can be checked directly that

$$\langle \mathcal{E}(T)(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}', \mathcal{H}} = \min_u J_0^T. \tag{2.19}$$

Thus, due to the (weak) observability of (A, C) (see (H2)), together with (2.19), we get that

$$\langle \mathcal{E}(T)(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}', \mathcal{H}} \ge \frac{1}{c} \|(w_0, w_1)\|^2_{X_{1/2-1/(2\eta)} \times X_{-1/(2\eta)}},$$

and hence $\mathcal{E}(\cdot)$ is strictly positive in $\mathcal{H}$. Moreover, $\mathcal{E}(t)$ is monotone increasing. Indeed, for $t_1 \le t_2$, let $u_i,\ \varphi_i,\ i = 1, 2$ be the optimal control and trajectory in $[0, t_i]$. Then, we have

$$\begin{aligned} \langle \mathcal{E}(t_2)(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}', \mathcal{H}} &= \min_u J_0^{t_2} = J_0^{t_2}(u_2) \ge J_0^{t_1}(u_2) \\ &\ge \min_u J_0^{t_1} = J_0^{t_1}(u_1) = \langle \mathcal{E}(t_1)(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}', \mathcal{H}}. \end{aligned}$$

Thus, $\mathcal{E}(t_2) \ge \mathcal{E}(t_1)$ for $t_2 > t_1 > 0$. The proof is completed. □

Consider the corresponding infinite-horizon quadratic cost functional associated with the control system (2.1):

$$\min J^\infty(u) = \frac{1}{2} \int_0^\infty [\|u(t)\|_U^2 + \|Cw(t)\|_V^2] dt,\ u \in L^2(0, T; U). \tag{2.20}$$

We have the following result on the well-posedness of the above infinite-horizon optimal control problem.

LEMMA 2.4. *Assume that (H1) is fulfilled. Then the set of admissible controls for problem (2.20) is non-empty if the initial state* $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^k) = X_{(k+1)/2} \times X_{k/2}$ *is sufficiently smooth satisfying* $k \geq \frac{1}{\varrho}$.

Proof. Due to the weak observability of (A, B^*) as given in (H1), by the HUM method and Proposition 3.25 in [9], p.43, we obtain that for any given $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$, there always exists a control $u(t) = \omega_{T_0}(t) \in L^2((0, T_0], U)$, such that the solution to (2.1) satisfying $(w(T_0), w_t(T_0)) = 0$ and

$$\int_0^{T_0} \|\omega_{T_0}(t)\|_U^2 dt \leq c\|(w_0, w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}. \tag{2.21}$$

Consider the control $\widetilde{u}$ on $[0, \infty)$ that is equal to ω_{T_0} for $t \in (0, T_0]$ and is identically zero for $t > T_0$. Thus, by (2.21), along with the well-posedness of control system (2.1) and $C \in \mathcal{L}(X_{1/2}, H)$, we have

$$\begin{aligned}\min_u J^\infty(u) \leq J^\infty(\widetilde{u}) &= \frac{1}{2}\int_0^{T_0} [\|\omega_{T_0}\|_U^2 + \|Cw(t)\|_V^2] dt \\ &\leq \widetilde{c}\|(w_0, w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}.\end{aligned} \tag{2.22}$$

Hence, by [5], we obtain the set of admissible controls for (2.20) is non-empty for $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$. □

Let $(\hat{w}, \hat{w}_t)$ and $\hat{u}$ be the optimal state and control for system (2.1) respect to (2.20), and by [15], we see that $\hat{w}$ and $\hat{u}$ satisfies

$$\begin{cases} \hat{w}_{tt} + A\hat{w} = B\hat{u}, \\ w(0) = w_0, \quad w_t(0) = w_1, \\ \hat{u} = -B^*\hat{p}, \\ \hat{p}_{tt} + A\hat{p} = C^*C\hat{w}, \\ \hat{p}(t) \to 0, \ \hat{p}_t(t) \to 0, \quad t \to \infty. \end{cases} \tag{2.23}$$

PROPOSITION 2.5. *Assume that (H1) holds true. Then there exists a unique minimal solution*

$$\hat{E} \in \mathcal{L}(\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}}), \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})')$$

of the algebraic Riccati equation

$$\mathcal{C}^*\mathcal{C} + (\hat{E}\mathcal{A} + \mathcal{A}^*\hat{E}) - \hat{E}\mathcal{B}\mathcal{B}^*\hat{E} = 0, \tag{2.24}$$

such that $(-\hat{p}_t, \hat{p}) = \hat{E}(\hat{w}, \hat{w}_t)$.

Thus, we can get the Riccati-based optimal feedback control law for the infinite horizon problem.

$$(0, \hat{u}(t)) = -\mathcal{B}^*\hat{E}(\hat{w}, \hat{w}_t) = (0, -B^*\hat{p}(t)). \tag{2.25}$$

and similarly we get that for $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$,

$$\langle \hat{E}(w_0, w_1), (w_0, w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} = \min_u J^\infty(u).$$

2.2. Main results. Under the assumptions (H1) and (H2), in order to discuss the large time behaviour of system (2.1) with Riccati-based optimal feedback control, we estimate the upper and lower bounds of $\langle \hat{E}(w_0, w_1), (w_0, w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}$, and based on which, the explicit slow decay rate can be derived.

THEOREM 2.6. *Suppose that (H1) and (H2) are satisfied. Then*

(1). There exists constants $c_j > 0,\ j = 1, 2$ *such that*

$$c_1\|(w_0, w_1)\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}} \leq (\hat{E}(w_0, w_1), (w_0, w_1))_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} \leq c_2\|(w_0, w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}, \tag{2.26}$$

where ϱ *and* η *is given as in (H1) and (H2), respectively.*

(2). There exists a constant $M > 0$ *such that*

$$\|(\hat{w}(t), \hat{w}_t(t))\|^2_{\mathcal{H}} \leq M(t + 1)^{-\frac{s\eta\varrho}{\varrho+\eta}}\|(w_0, w_1)\|^2_{X_{(1/\varrho+1/\eta+s+1)/2}\times X_{(1/\varrho+1/\eta+s)/2}},\ s > 0, \tag{2.27}$$

where $(\hat{w}(t), \hat{w}_t(t))$ *is the solution to system (2.1) with the Riccati-based optimal feedback control (2.25),* $\varrho > 0$ *and* $\eta > 0$ *is given as in (H1) and (H2), respectively.*

REMARK 2.7. *By the proof for Theorem 2.6, we find that the lower bound in (2.26) is for* $(w_0, w_1) \in X_{1/2-1/(2\eta)} \times X_{-1/(2\eta)}$ *due to the weak observability of* (A, C) *given as in (H2), while the upper bound is for* $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$ *mainly caused by the weak observability of* (A, B^*) *(see (H1)). Both of these two weak observability hypothesises determine the slow decay rate of the closed-loop system with the Riccati-based optimal feedback control.*

REMARK 2.8. *if assuming that the exact observability of* (A, B^*) *and* (A, C) *are fulfilled, i.e., there exist positive constants* T_0, T_1 *and* c *such that*

$$c\int_0^{T_0} \|B^*w_t\|^2_{U'}dt \geq \|(w_0, w_1)\|^2_{\mathcal{H}}, \quad c\int_0^{T_1} \|Cw\|^2_{V}dt \geq \|(w_0, w_1)\|^2_{\mathcal{H}}, \tag{2.28}$$

we can see from (2.27) that the system can be stabilized exponentially under Riccati-based optimal feedback control for this case. However, it should be noted that these strong assumptions only can be satisfied under suitable geometric conditions on the medium in which waves propagate. For instance, for waves on planar networks one needs a tree-like graph with control or observation in all the free extreme edges except for one. Besides, for the multi-dimensional wave equation, the GCC has to be satisfied for control areas.

If either of the exact observation of (A, B^*) *and* (A, C) *is fulfilled, then the exponential decay no longer holds, but the polynomial decay can still be achieved. Specifically, the decay rates can be also derived from (2.27) and given as* $(t + 1)^{-s\eta}$ *for* $(w_0, w_1) \in X_{(1/\eta+s+1)/2} \times X_{(1/\eta+s)/2}$ *and* $(t + 1)^{-s\varrho}$ *for* $X_{(1/\varrho+s+1)/2} \times X_{(1/\varrho+s)/2}$*, respectively.*

REMARK 2.9. *If some other kinds of weak observability hypothesises are fulfilled, the corresponding slow decay rates can also be deduced similarly. For instance, If (H1) and (H2) are replaced by the following much weaker ones:*

$$c\int_0^{T} \|B^*w_t\|^2_{U'}dt \geq \|(w_0, w_1)\|^2_{\mathcal{D}(e^{-a\mathcal{A}})}$$

and

$$c\int_0^T \|Cw\|_V^2 dt \geq \|(w_0,w_1)\|^2_{\mathcal{D}(e^{-b\mathcal{A}})}$$

where

$$\|(w_0,w_1)\|^2_{\mathcal{D}(e^{-\alpha\mathcal{A}})} := \sum_{j\geq 1} e^{-2\alpha\lambda_n}(\lambda_n^2 a_n^2 + b_n^2),$$

then similar to the proof for Theorem 2.6, the lower and upper bound become

$$c_1\|(w_0,w_1)\|^2_{\mathcal{D}(e^{-b\mathcal{A}})} \leq \langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(e^{k\mathcal{A}})',\mathcal{D}(e^{k\mathcal{A}})} \leq c_2\|(w_0,w_1)\|^2_{\mathcal{D}(e^{a\mathcal{A}})}. \tag{2.29}$$

Moreover, we can obtain the following slow decay rate for the system with Riccati-based optimal feedback control (2.25), that is,

$$\|(\hat{w}(t),\hat{w}_t(t))\|^2_{\mathcal{H}} \leq M(t+1)^{-\frac{s}{a+b}}\|(w_0,w_1)\|^2_{\mathcal{D}(e^{(a+b+s)\mathcal{A}})},\ s>0. \tag{2.30}$$

Let $w^T(t)$, $u^T(t)$ be the optimal solution and control to the following problem

$$\min \widetilde{J}_0^T(u) = \frac{1}{2}\int_0^T [\|u(t)\|_U^2 + \|Cw(t)-z\|_V^2]dt,\ u\in L^2(t,T;U) \tag{2.31}$$

where w and u satisfy equation (2.1).

Let $\bar{w},\bar{u}$ be the optimal solution and control to the steady optimal control problem

$$\min_u \widetilde{J}_s = \|u\|_U^2 + \|Cw-z\|_V^2, \tag{2.32}$$

subject to $Aw = Bu$. For the stationary optimal control problem (2.32), $(\bar{u},\bar{w})$ satisfy

$$A\bar{w} = B\bar{u},\ \langle\bar{u},v\rangle + \langle C\bar{w}-z, C\varphi\rangle = 0,\ \text{for every } v,\varphi : A\varphi = Bv.$$

Thus, $C^*(C\bar{w}-z)\in Ker(A)^\perp$ and hence there exists some $\bar{p}$ satisfying

$$A^*\bar{p} = A\bar{p} = C^*(C\bar{w}-z).$$

Hence, by the above and (2.17), we obtain the following OS system

$$\begin{cases} (w^T-\bar{w})_{tt} + A(w^T-\bar{w}) = B(u^T-\bar{u}), \\ (p^T-\bar{p})_{tt} + A(p^T-\bar{p}) = C^*C(w^T-\bar{w}), \\ u^T-\bar{u} = -B^*(p^T-\bar{p}), \\ w^T(0) = w_0, \quad w_t^T(0) = w_1, \\ p^T(T) = p_t^T(T) = 0. \end{cases} \tag{2.33}$$

Let us discuss the turnpike property of the optimal control problem, that is, identifying that to which extent do $u^T(t), w^T(t)$ approximate the stationary ones $\bar{u},\bar{w}$ as $T\to\infty$. In fact, we have the following result.

Theorem 2.10. *Assume that (H1) and (H2) hold true. Then for any $(w_0-\bar{w},w_1)\in\mathcal{D}(\mathcal{A}^k)$, $k>0$ and $\bar{p}\in X_{(k+1)/2}$, $k\geq 1$, it holds that as $T\to\infty$,*

$$\frac{1}{T}\int_0^T (\|C(w^T-\bar{w})\|_H^2 + \|(u^T-\bar{u})\|_H^2)dt \to 0, \tag{2.34}$$

and

$$\|\frac{1}{T}\int_0^T (w^T(t)-\bar{w})dt\|^2_{X_{1/2}} \to 0. \tag{2.35}$$

REMARK 2.11. *We see that when choosing (2.28) instead of (H1) and (H2), i.e., the exact observability of (A, B^*) and (A, C), the result in Theorem 2.10 is consistent with the averaged turnpike property as given in [18]. Thus, in terms of averaged turnpike property, the result in [18] can be considered as a special case of Theorem 2.10 given above.*

REMARK 2.12. *In the above theorem, the averaged turnpike property is obtained under sufficiently smooth initial states $(w_0 - \bar{w}, w_1)$ and $\bar{p}$. Compared to the exponential-type point-wise turnpike property under exact observability of (A, B^*) and (A, C), it is reasonable to expect the polynomial point-wise turnpike property under the weak observability hypothesises (H1) and (H2). However, the point-wise turnpike property is still an open problem. In fact, the proof based on energy estimates along with the properties of Riccati operator proposed in [18] is difficult to apply to the case with weak observability as given in (H1) and (H2) under consideration. The main difficulty in the energy estimates is caused by the slow decay rate which is determined by the weak observability. As we know, the slow decay rate is non-uniform and always dependent on the regularity of the initial states. This is different from the case in [18], where the exact observability is fulfilled and the uniform exponential decay rates always hold for any initial data in state space.*

3. Proof of main results.

3.1. Proof of Proposition 2.5. Following the proof in Lemma 2.4, due to (H1), for $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$, we always can find a control $\widetilde{u} = \begin{cases} \omega_{T_0}(t), & t \in (0, T_0], \\ 0, & t \in (T_0, T], \end{cases}$ such that

$$(w(t), w_t(t)) = 0,\ t \geq T_0$$

and

$$\int_0^T \|\widetilde{u}\|^2_U dt = \int_0^{T_0} \|\omega_{T_0}(t)\|^2_U dt \leq c\|(w_0, w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}.$$

Hence, we get that for $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$,

$$\begin{aligned}\langle \mathcal{E}(T)(w_0, w_1), (w_0, w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} &= \min_u J_0^T \\ &\leq J_0^T(\widetilde{u}) = \frac{1}{2}\int_0^{T_0} [\|\omega_{T_0}\|^2_U + \|Cw(t)\|^2_V]dt \\ &\leq \widetilde{c}\|(w_0, w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}, \quad \text{for all}\, T > T_0. \end{aligned} \tag{3.1}$$

Note that for $(w_0, w_1), (z_0, z_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$, we have

$$\begin{aligned} &2\mathrm{Re}\langle \mathcal{E}(T)(w_0, w_1), (z_0, z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} \\ =& \langle \mathcal{E}(T)(w_0 + z_0, w_1 + z_1), (w_0 + z_0, w_1 + z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} \\ &-\langle \mathcal{E}(T)(w_0, w_1), (w_0, w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} \end{aligned}$$

$$-\langle \mathcal{E}(T)(z_0,z_1),(z_0,z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})},$$

$$\begin{aligned}
&2\mathrm{Im}\langle \mathcal{E}(T)(w_0,w_1),(z_0,z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}\\
&= i\langle \mathcal{E}(T)(w_0+iz_0,w_1+iz_1),(w_0+iz_0,w_1+iz_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}\\
&\quad -\langle \mathcal{E}(T)(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}\\
&\quad -\langle \mathcal{E}(T)(iz_0,iz_1),(iz_0,iz_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})},
\end{aligned}$$

which along with the monotone increasing property of $\mathcal{E}(t)$ and (3.1), yields that the limit $\lim\limits_{T\to\infty}\langle \mathcal{E}(T)(w_0,w_1),(z_0,z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}$ exists for all $(w_0,w_1),(z_0,z_1)\in\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$. By the Uniform Boundedness Theorem, it follows that $\mathcal{E}(\cdot)$ is bounded in $\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$. Thus, for $(w_0,w_1),(z_0,z_1)\in\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$, we define

$$\lim_{T\to\infty}\langle \mathcal{E}(T)(w_0,w_1),(z_0,z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} := \langle \hat{E}(w_0,w_1),(z_0,z_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}.$$

It follows that for $(w_0,w_1)\in\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$,

$$\lim_{T\to\infty}\langle \mathcal{E}(T)(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} = \langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}.$$

Hence, for $(w_0,w_1)\in\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$, the limit $\lim\limits_{T\to\infty}\mathcal{E}(T)(w_0,w_1)=\hat{E}(w_0,w_1)$ exists in $\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})'$. Then following the proof of Proposition 2.2 in [5] (p.483), we get that $\hat{E}$ satisfies (2.24) and $\hat{E}\le X$ for any solution X of (2.24) and hence $\hat{E}$ is the minimal solution to the algebraic Riccati equation (2.24). The proof is completed. □

3.2. Slow decay rate (Proof of Theorem 2.6). First, by (2.22), we can easily obtain the upper bound as follows.

$$\langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} = \min_u J^\infty \le c_1\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}. \tag{3.2}$$

In order to get the lower bound, the proof is given by the following two steps:
Step 1. Divide w^T in system (2.17) by $w^T=y+z$, where y satisfies

$$\begin{cases} y_{tt}+Ay=Bu^T, \quad 0<t<T,\\ y(0)=0,\quad y_t(0)=0,\\ u^T=-B^*p^T,\end{cases} \tag{3.3}$$

and z satisfies

$$\begin{cases} z_{tt}+Az=0, \quad 0<t<T,\\ z(0)=w_0,\quad z_t(0)=w_1.\end{cases} \tag{3.4}$$

By the hypothesis (H2), we get directly that there exists a constant $c>0$ satisfying

$$\|(w_0,w_1)\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}} \le c\int_0^T \|Cz\|_V^2\,dt, \tag{3.5}$$

where z satisfies (3.4).

We also have the following estimate for (3.3).

$$\|(y, y_t)\|^2_{\mathcal{H}} \le e^t \int_0^t \|Bu^T(s)\|^2_H ds, \quad 0 < t \le T. \tag{3.6}$$

In fact, set the energy function $E(t) = \frac{1}{2}(\|y\|^2_{X_{1/2}} + \|y_t\|^2_H)$. Then, differentiating $E(t)$ by t, together with (3.3), we get

$$\frac{dE(t)}{dt} = \langle y_t, Bu^T \rangle.$$

Hence,

$$\begin{aligned} E(t) &\le \frac{1}{2}\int_0^t (\|Bu^T\|^2_H + \|y_t\|^2_H) ds \\ &\le \frac{1}{2}\int_0^t \|Bu^T(s)\|^2_H ds + \int_0^t E(s) ds, \ \ 0 < t \le T. \end{aligned} \tag{3.7}$$

Using Gronwall's inequality, we obtain

$$E(t) \le \frac{1}{2} e^t \int_0^t \|Bu^T(s)\|^2_H ds, \quad 0 < t \le T, \tag{3.8}$$

which leads to (3.6).

Step 2. we will show that

$$\|(w_0, w_1)\|^2_{X_{1/2-1/(2\eta)} \times X_{-1/(2\eta)}} \le c \langle \hat{E}(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})},$$

where $c > 0$ is some constant.

Note that $w^T = y + z$. Thus,

$$\begin{aligned} \|(w_0, w_1)\|^2_{X_{1/2-1/(2\eta)} \times X_{-1/(2\eta)}} &\le c \int_0^T \|Cz\|^2_V dt \\ &\le c \int_0^T (\|Cw^T\|^2_V + \|Cy\|^2_V) dt \\ &\le c \int_0^T (\|Cw^T\|^2_V + \|y\|^2_{X_{1/2}}) dt \\ &\le c \int_0^T (\|Cw^T\|^2_V + e^t \int_0^t \|Bu^T(s)\|^2_H ds) dt \\ &\le c \cdot \max\{1, e^T - 1\} \int_0^T (\|Cw^T\|^2_V + \|Bu^T\|^2_H) dt \\ &\le c_T \int_0^T (\|Cw^T\|^2_V + \|u^T\|^2_U) dt \\ &= c_T \langle \mathcal{E}(T)(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}', \mathcal{H}}, \end{aligned} \tag{3.9}$$

where $c_T = c \cdot \max\{1, e^T - 1\}$. Thus, by (3.9), we have

$$\langle \mathcal{E}(T)(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{H}', \mathcal{H}} \ge \frac{1}{c_T} \|(w_0, w_1)\|^2_{X_{1/2-1/(2\eta)} \times X_{-1/(2\eta)}}. \tag{3.10}$$

Note that by Proposition 2.5, we know that $\mathcal{E}(T)$ is monotone increasing and bounded in $\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$ and $\lim_{T \to \infty} \mathcal{E}(T)(w_0, w_1) = \hat{E}(w_0, w_1)$ exists in $\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})'$. Hence, for $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})$,

$$\langle \hat{E}(w_0, w_1), (w_0, w_1) \rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})', \mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}$$

$$\geq \langle \mathcal{E}(T)(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}$$
$$\geq \frac{1}{c_T}\|(w_0,w_1)\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}}, \quad \forall T>0. \tag{3.11}$$

Now, we consider the decay rate of the closed-loop system with Riccati-based optimal feedback control (2.25). A direct calculation yields

$$\frac{d\langle \hat{E}(\hat{w},\hat{w}_t),(\hat{w},\hat{w}_t)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}}{dt} = -\|B^*\hat{p}\|^2_{U'} - \|C\hat{w}\|^2_V. \tag{3.12}$$

Integrating the above from 0 to T, we have

$$\langle \hat{E}(\hat{w}(T),\hat{w}_t(T)),(\hat{w}(T),\hat{w}_t(T))\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} - \langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}$$
$$= -\int_0^T (\|B^*\hat{p}\|^2_{U'} + \|C\hat{w}\|^2_V)dt. \tag{3.13}$$

Note from (3.10) that

$$\int_0^T (\|B^*\hat{p}\|^2_{U'} + \|C\hat{w}\|^2_V)dt \geq \langle \mathcal{E}(T)(w_0,w_1),(w_0,w_1)\rangle$$
$$\geq \frac{1}{c_T}\|(w_0,w_1)\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}}. \tag{3.14}$$

Thus,

$$\langle \hat{E}(\hat{w}(T),\hat{w}_t(T)),(\hat{w}(T),\hat{w}_t(T))\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} - \langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}$$
$$\leq -\frac{1}{c_T}\|(w_0,w_1)\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}} \tag{3.15}$$

where $\varrho>0$ and $\eta>0$ is given as (H1) and (H2), respectively.

Then by interpolation, we have

$$\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})} \leq \|(w_0,w_1)\|^{\frac{2s\eta}{1+\frac{1}{\varrho}\eta+s\eta}}_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}} \|(w_0,w_1)\|^{\frac{2(1+\frac{1}{\varrho}\eta)}{1+\frac{1}{\varrho}\eta+s\eta}}_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}, \ s>0,$$

and hence

$$\|(w_0,w_1)\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}} \geq \frac{\|(w_0,w_1)\|^{\frac{2(1+\frac{1}{\varrho}\eta+s\eta)}{s\eta}}_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}}{\|(w_0,w_1)\|^{\frac{2(1+\frac{1}{\varrho}\eta)}{s\eta}}_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}, \ s>0.$$

Thus, by (3.15), we have

$$\langle \hat{E}(\hat{w}(T),\hat{w}_t(T)),(\hat{w}(T),\hat{w}_t(T))\rangle_{\mathcal{H}',\mathcal{H}} - \langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{H}',\mathcal{H}}$$
$$\leq -\frac{1}{c_T}\frac{\|(w_0,w_1)\|^{\frac{2(1+\frac{1}{\varrho}\eta+s\eta)}{s\eta}}_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}}{\|(w_0,w_1)\|^{\frac{2(1+\frac{1}{\varrho}\eta)}{s\eta}}_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}, \ s>0. \tag{3.16}$$

Therefore,

$$\frac{\langle \hat{E}(\hat{w}(T),\hat{w}_t(T)),(\hat{w}(T),\hat{w}_t(T))\rangle_{\mathcal{H}',\mathcal{H}}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}-\frac{\langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{H}',\mathcal{H}}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}$$
$$\leq -\frac{1}{c_T}\left[\frac{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}\right]^{\frac{1+\frac{1}{\varrho}\eta+s\eta}{s\eta}},\ s>0. \tag{3.17}$$

Thus, by (3.2) and the monotone decreasing property of $\langle \hat{E}(w,w_t),(w,w_t)\rangle$, we have

$$\frac{\langle \hat{E}(\hat{w}(T),\hat{w}_t(T)),(\hat{w}(T),\hat{w}_t(T))\rangle_{\mathcal{H}',\mathcal{H}}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}-\frac{\langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle_{\mathcal{H}',\mathcal{H}}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}$$
$$\leq -\frac{1}{c_2^2c_T}\left[\frac{\langle \hat{E}(\hat{w}(T),\hat{w}_t(T)),(\hat{w}(T),\hat{w}_t(T))\rangle_{\mathcal{H}',\mathcal{H}}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}\right]^{\frac{1+\frac{1}{\varrho}\eta+s\eta}{s\eta}},\ s>0. \tag{3.18}$$

In order to obtain the explicit decay rate of closed-loop system, let us introduce the following result in Ammari and Tucsnak [2].

Lemma 3.1. *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of positive number satisfying*

$$a_{m+1}\leq a_m - C(a_{m+1})^{2+\alpha},\quad \forall m\geq 1, \tag{3.19}$$

for some constants $C>0$ and $\alpha>-1$. Then there exists a positive constant $M_{C,\alpha}$ such that

$$a_m\leq \frac{M_{C,\alpha}}{(m+1)^{\frac{1}{1+\alpha}}}.$$

By the above Lemma, together with (3.18), it is easy to get that

$$\frac{\langle \hat{E}(\hat{w}(t),\hat{w}_t(t)),(\hat{w}(t),\hat{w}_t(t))\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}}{\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})}}\leq M(t+1)^{-\frac{s\eta}{1+\frac{1}{\varrho}\eta}},\ s>0.$$

Note that by (3.11), we know

$$\langle \hat{E}(\hat{w}(t),\hat{w}_t(t)),(\hat{w}(t),\hat{w}_t(t))\rangle_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})',\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}})}\geq c_1\|(\hat{w}(t),\hat{w}_t(t))\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}}.$$

So,

$$\|(\hat{w}(t),\hat{w}_t(t))\|^2_{X_{1/2-1/(2\eta)}\times X_{-1/(2\eta)}}\leq M(t+1)^{-\frac{s\eta}{1+\frac{1}{\varrho}\eta}}\|(w_0,w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{1}{\varrho}+s})},\ s>0,$$

and hence

$$\|(\hat{w}(t),\hat{w}_t(t))\|^2_{\mathcal{H}}\leq M(t+1)^{-\frac{s\eta\varrho}{\varrho+\eta}}\|(w_0,w_1)\|^2_{X_{(1/\varrho+1/\eta+s+1)/2}\times X_{(1/\varrho+1/\eta+s)/2}},\ s>0, \tag{3.20}$$

where $\varrho>0$ and $\eta>0$ is given as in (H1) and (H2), respectively. The proof is completed. □

3.3. Averaged turnpike property (Proof of Theorem 2.10). Let us consider the averaged turnpike property for the LQ optimal control problems with (H1) and (H2) are fulfilled. Firstly, based on the weak observability of (A, B^*), we obtain the following estimate.

LEMMA 3.2. *Assume that (H1) is fulfilled. Then there exists some positive constant $\widetilde{c}$ such that*

$$\|(p(0),p_t(0))\|^2_{\mathcal{D}(\mathcal{A}^k)'} \le \widetilde{c} g_1(T)[\int_0^T \|B^*p\|_H^2 dt + \int_0^T \|f\|^2_{X_{-1/2}} dt + \|p_{0T}\|^2_H + \|p_{1T}\|^2_{X_{-1/2}}], \tag{3.21}$$

where $k>0$, $g_1(T) = \begin{cases} \frac{(T+1)^{-k\varrho+1}-1}{-k\varrho+1} & k\varrho \neq 1, \\ \ln(T+1) & k\varrho = 1, \end{cases}$ *and p is any solution to the following inhomogeneous system*

$$\begin{cases} p_{tt} + Ap = f, \\ p(T) = p_{0T}, \quad p_t(T) = p_{1T}. \end{cases} \tag{3.22}$$

Proof. By duality we get

$$\langle p_t, y\rangle \Big|_0^T - \langle p, y_t\rangle \Big|_0^T + \int_0^T \langle p, y_{tt} + Ay\rangle\, dt = \int_0^T \langle f, y\rangle\, dt. \tag{3.23}$$

Let y be the solution to equation (2.1) with (2.9). Using Hölder's inequality, we get

$$\begin{aligned} |\int_0^T \langle p, y_{tt} + Ay\rangle\, dt| &= |\int_0^T \langle p, -BB^*y_t\rangle\, dt| \\ &\le c(\int_0^T \|B^*p\|_H^2 dt)^{\frac{1}{2}} (\int_0^T \|B^*y_t\|^2_H)^{\frac{1}{2}}. \end{aligned}$$

Thus, by Lemma 2.1, along with (2.11), we get that there exists some constant $c>0$ such that

$$\begin{aligned} &|\int_0^T \langle p, y_{tt} + Ay\rangle\, dt| \\ &\le c(\int_0^T \|B^*p\|^2_H\, dt)^{\frac{1}{2}} (\frac{1}{2}\|(y_0,y_1)\|^2_{\mathcal{H}} + \frac{1}{2}c_1(T+1)^{-k\varrho}\|(y_0,y_1)\|^2_{\mathcal{D}(\mathcal{A}^k)})^{\frac{1}{2}} \\ &\le c(\frac{1}{2} + \frac{1}{2}c_1(T+1)^{-k\varrho})^{\frac{1}{2}} (\int_0^T \|B^*p\|^2_H\, dt)^{\frac{1}{2}} \|(y_0,y_1)\|_{\mathcal{D}(\mathcal{A}^k)}. \end{aligned} \tag{3.24}$$

Besides, we have

$$\begin{aligned} \int_0^T \langle f, y\rangle\, dt &\le (\int_0^T \|f\|^2_{X_{-1/2}})^{\frac{1}{2}} (\int_0^T \|y\|^2_{X_{1/2}})^{\frac{1}{2}} \\ &\le c(\int_0^T \|f\|^2_{X_{-1/2}})^{\frac{1}{2}} (\int_0^T (t+1)^{-k\varrho}\|(y_0,y_1)\|^2_{\mathcal{D}(\mathcal{A}^k)})^{\frac{1}{2}} \\ &\le c(g_1(T))^{\frac{1}{2}} (\int_0^T \|f\|^2_{X_{-1/2}})^{\frac{1}{2}} \|(y_0,y_1)\|_{\mathcal{D}(\mathcal{A}^k)}, \quad k>0. \end{aligned} \tag{3.25}$$

Hence, by (3.23), (3.24) and (3.25), along with Lemma 2.1, we obtain

$$-\langle p_t(0), y_0\rangle + \langle p(0), y_1\rangle$$

$$
\begin{aligned}
&= \int_0^T \langle f, y\rangle dt + \int_0^T \langle p, BB^* y_t\rangle dt + (p_{0T}, y_t(T)) - (p_{1T}, y(T))\\
&\le \int_0^T \langle f, y\rangle dt + \int_0^T \langle p, BB^* y_t\rangle dt + c(T+1)^{-\frac{1}{2}k\varrho}\|(p_{0T}, p_{1T})\|_{\mathcal{H}'}\|(y_0, y_1)\|_{\mathcal{D}(\mathcal{A}^k)}\\
&\le \tilde{c}(g_1(T))^{\frac{1}{2}}[(\int_0^T \|B^* p\|_H^2 dt)^{\frac{1}{2}} + (\int_0^T \|f\|_{X_{-1/2}}^2)^{\frac{1}{2}} + \|(p_{0T}, p_{1T})\|_{\mathcal{H}'}]\|(y_0, y_1)\|_{\mathcal{D}(\mathcal{A}^k)}.
\end{aligned}
$$

By Riesz representation theorem, we can choose $(y_0, y_1) \in \mathcal{D}(\mathcal{A}^k)$ such that $\|(y_0, y_1)\|_{\mathcal{D}(\mathcal{A}^k)} = \|(p(0), p_t(0))\|_{\mathcal{D}(\mathcal{A}^k)'}$ and

$$
-\langle p_t(0), y_0\rangle + \langle p(0), y_1\rangle = \|(p(0), p_t(0))\|_{\mathcal{D}(\mathcal{A}^k)'}^2.
$$

Thus,

$$
\|(p(0), p_t(0))\|_{\mathcal{D}(\mathcal{A}^k)'}^2 \le \tilde{c} g_1(T)[\int_0^T \|B^* p\|_H^2 dt + \int_0^T \|f\|_{X_{-1/2}}^2 dt + \|(p_{0T}, p_{1T})\|_{\mathcal{H}'}^2] \tag{3.26}
$$

where $k > 0$ and $\mathcal{H}' = H \times X_{-1/2}$ is the dual space of $\mathcal{H}$. The proof is completed. □

By the similar discussion, together with (H2), we obtain that

Lemma 3.3. *Assume that (H2) holds true. Then there exists a constant* $\widetilde{c} > 0$ *satisfying*

$$
\|(w(T), w_t(T))\|_{\mathcal{D}(\mathcal{A}^k)'}^2 \le \tilde{c} g_2(T)[\int_0^T \|Cw\|_H^2 dt + \int_0^T \|f\|_{X_{-1/2}}^2 dt + \|(w_0, w_1)\|_{H\times X_{-1/2}}^2]. \tag{3.27}
$$

where $k \ge 1$, $g_2(T) = \begin{cases} \frac{(T+1)^{-k\eta+1}-1}{-k\eta+1} & k\eta \ne 1, \\ \ln(T+1) & k\eta = 1, \end{cases}$ *and w is any solution to the following inhomogeneous system*

$$
\begin{cases} w_{tt} + Aw = f, \\ w(0) = w_0, \quad w_t(0) = w_1. \end{cases} \tag{3.28}
$$

Proof. Similar to the proof for Lemma 3.2, by duality we have

$$
\langle w_t, \phi\rangle \Big|_0^T - \langle w, \phi_t\rangle \Big|_0^T + \int_0^T \langle w, \phi_{tt} + A\phi\rangle\, dt = \int_0^T \langle f, \phi\rangle\, dt \tag{3.29}
$$

where ϕ is the solution to equation (2.13). Using Hölder's inequality, we get

$$
\begin{aligned}
|\int_0^T \langle w, \phi_{tt} + A\phi\rangle dt| &= |\int_0^T \langle w, C^* C\phi_t\rangle dt|\\
&\le c(\int_0^T \|Cw\|_H^2 dt)^{\frac{1}{2}}(\int_0^T \|C\phi_t\|_H^2)^{\frac{1}{2}}.
\end{aligned}
$$

Then, due to Lemma 2.2 and (2.15), we get that there exists some constant $c > 0$ such that

$$
\begin{aligned}
&|\int_0^T \langle w, \phi_{tt} + A\phi\rangle\, dt|\\
&\le c(\int_0^T \|Cw\|_H^2\, dt)^{\frac{1}{2}}(\frac{1}{2}\|(\phi_0^T, \phi_1^T)\|_{\mathcal{H}}^2 + \frac{1}{2}c_1(T+1)^{-k\eta}\|(\phi_0^T, \phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}^2)^{\frac{1}{2}}
\end{aligned}
$$

$$\le c(\frac{1}{2}+\frac{1}{2}c_1(T+1)^{-k\eta})^{\frac{1}{2}}(\int_0^T \|Cw\|_H^2\,dt)^{\frac{1}{2}}\|(\phi_0^T,\phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)},\quad k\ge 1. \tag{3.30}$$

We also have

$$\begin{aligned}\int_0^T \langle f,\phi\rangle\,dt &\le (\int_0^T \|f\|_{X_{-1/2}}^2 dt)^{\frac{1}{2}}(\int_0^T \|\phi\|_{X_{1/2}}^2 dt)^{\frac{1}{2}}\\ &\le c(\int_0^T \|f\|_{X_{-1/2}}^2 dt)^{\frac{1}{2}}(\int_0^T (T-t+1)^{-k\eta}\|(\phi_0^T,\phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}^2 dt)^{\frac{1}{2}}\\ &\le c(g_2(T))^{\frac{1}{2}}(\int_0^T \|f\|_{X_{-1/2}}^2 dt)^{\frac{1}{2}}\|(\phi_0^T,\phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)},\ k\ge 1. \end{aligned}\tag{3.31}$$

Hence, by (3.29), (3.30) and (3.31), along with Lemma 2.2, we obtain

$$\begin{aligned}&\langle w_t(T),\phi_0^T\rangle-\langle w(T),\phi_1^T\rangle\\ =&\int_0^T\langle f,\phi\rangle dt-\int_0^T\langle w,C^*C\phi_t\rangle dt+\langle w_t(0),\phi(0)\rangle-\langle w(0),\phi_t(0)\rangle\\ \le&\int_0^T\langle f,\phi\rangle dt-\int_0^T\langle w,C^*C\phi_t\rangle dt+c(T+1)^{-\frac{1}{2}k\eta}\|(\phi_0^T,\phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}\|(w_0,w_1)\|_{H\times X_{-1/2}}\\ \le& c(g_2(T))^{\frac{1}{2}}[(\int_0^T \|Cw\|_H^2 dt)^{\frac{1}{2}}+(\int_0^T \|f\|_{X_{-1/2}}^2)^{\frac{1}{2}}+\|(w_0,w_1)\|_{H\times X_{-1/2}}]\|(\phi_0^T,\phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}.\end{aligned}$$

By Riesz representation theorem, we can choose $(\phi_0^T,\phi_1^T)\in\mathcal{D}(\mathcal{A}^k)$ such that $\|(\phi_0^T,\phi_1^T)\|_{\mathcal{D}(\mathcal{A}^k)}=\|(w(T),w_t(T))\|_{\mathcal{D}(\mathcal{A}^k)'}$ and

$$\langle w_t(T),\phi_0^T\rangle-\langle w(T),\phi_1^T\rangle=\|(w(T),w_t(T))\|_{\mathcal{D}(\mathcal{A}^k)'}^2.$$

Thus,

$$\begin{aligned}&\|(w(T),w_t(T))\|_{\mathcal{D}(\mathcal{A}^k)'}^2\\ \le&\tilde{c}g_2(T)[\int_0^T \|Cw\|_H^2 dt+\int_0^T \|f\|_{X_{-1/2}}^2 dt+\|(w_0,w_1)\|_{H\times X_{-1/2}}^2],\quad k\ge 1.\end{aligned}$$

The proof is completed. □

Proof of Theorem 2.10. Taking the dual product of the first and second equations in (2.33) with $p^T-\bar{p}$ and $w^T-\bar{w}$, respectively, we obtain

$$\int_0^T\langle (p^T-\bar{p})_{tt},w^T-\bar{w}\rangle dt+\int_0^T\langle A(p^T-\bar{p}),w^T-\bar{w}\rangle dt=\int_0^T\langle C^*C(w^T-\bar{w}),w^T-\bar{w}\rangle dt, \tag{3.32}$$

and

$$\int_0^T\langle (w^T-\bar{w})_{tt},p^T-\bar{p}\rangle dt+\int_0^T\langle A(w^T-\bar{w}),p-\bar{p}\rangle dt=\int_0^T\langle B(u^T-\bar{u}),p^T-\bar{p}\rangle dt. \tag{3.33}$$

Integrating the above by parts, we obtain

$$\begin{aligned}&\int_0^T\|C(w^T-\bar{w})\|_H^2 dt-\int_0^T\langle B(u^T-\bar{u}),p^T-\bar{p}\rangle dt\\ =&\langle (p^T-\bar{p})_t(T),(w^T-\bar{w})(T)\rangle-\langle (p^T-\bar{p})_t(0),(w^T-\bar{w})(0)\rangle\\ &-\langle (p^T-\bar{p})(T),(w^T-\bar{w})_t(T)\rangle+\langle (p^T-\bar{p})(0),(w^T-\bar{w})_t(0)\rangle\end{aligned}$$

$$= -\langle (p^T-\bar p)_t(0),(w_0-\bar w)\rangle + \langle \bar p,(w^T-\bar w)_t(T)\rangle + \langle (p^T-\bar p)(0),w_1\rangle. \quad (3.34)$$

By Lemma 3.2 along with (2.33), we get

$$\begin{aligned}&\|(p^T(0)-\bar p,(p_t^T(0)-\bar p)_t)\|^2_{\mathcal{D}(\mathcal{A}^k)'}\\ \le\ &\widetilde{c}g_1(T)[\int_0^T \|B^*(p^T-\bar p)\|_H^2dt+\int_0^T \|C^*C(w^T-\bar w)\|^2_{X_{-1/2}}dt+\|\bar p\|_H^2]. \quad (3.35)\end{aligned}$$

Similarly, by Lemma 3.3 along with (2.33), we have

$$\begin{aligned}&\|(w(T)-\bar w,w_t(T))\|^2_{X_{-k/2}\times X_{-(k+1)/2}}\\ \le\ &\widetilde{c}g_2(T)[\int_0^T \|B(u^T-\bar u)\|^2_{X_{-1/2}}dt+\int_0^T \|C(w^T-\bar w)\|^2_Hdt\\ &+\|w_0-\bar w\|_H^2+\|w_1\|^2_{X_{-1/2}}],\ k\ge 1. \quad (3.36)\end{aligned}$$

Thus, by (3.35), it holds that

$$\begin{aligned}&\langle (p^T-\bar p)_t(0),(w_0-\bar w)\rangle+\langle (p^T-\bar p)(0),w_1\rangle\\ \le\ &\langle \|(p^T-\bar p)(0),(p^T-\bar p)_t(0))\|_{\mathcal{D}(\mathcal{A}^k)'}\|(w_0-\bar w,w_1)\|_{\mathcal{D}(\mathcal{A}^k)}\\ \le\ &\widetilde{c}(g_1(T))^{\frac12}[\int_0^T \|B^*(p^T-\bar p)\|_H^2dt+\int_0^T \|C^*C(w^T-\bar w)\|^2_{X_{-1/2}}dt+\|\bar p\|_H^2]^{\frac12}\\ &\cdot\|(w_0-\bar w,w_1)\|_{\mathcal{D}(\mathcal{A}^k)}, \quad (3.37)\end{aligned}$$

and by (3.36), we get

$$\begin{aligned}&\langle \bar p,(w^T-\bar w)_t(T)\rangle \le \|\bar p\|_{X_{(k+1)/2}}\|(w^T-\bar w)_t(T)\|_{X_{-(k+1)/2}}\\ \le\ &\widetilde{c}(g_2(T))^{\frac12}[\int_0^T \|B(u^T-\bar u)\|^2_{X_{-1/2}}dt+\int_0^T \|C(w^T-\bar w)\|_H^2dt+\|w_0-\bar w\|_H^2+\|w_1\|^2_{X_{-1/2}}]^{\frac12}\\ &\cdot\|\bar p\|_{X_{(k+1)/2}},\ k\ge 1. \quad (3.38)\end{aligned}$$

Substituting (3.37) and (3.38) into (3.34) yields

$$\begin{aligned}&\int_0^T \|C(w^T-\bar w)\|_H^2dt-\int_0^T\langle B(u^T-\bar u),p^T-\bar p\rangle dt\\ \le\ &\widetilde{c}(g_1(T))^{\frac12}[\int_0^T \|B^*(p^T-\bar p)\|_H^2dt+\int_0^T \|C^*C(w^T-\bar w)\|^2_{X_{-1/2}}dt+\|\bar p\|_H^2]^{\frac12}\\ &\cdot\|(w_0-\bar w,w_1)\|_{\mathcal{D}(\mathcal{A}^k)}\\ &+\widetilde{c}(g_2(T))^{\frac12}[\int_0^T \|B(u^T-\bar u)\|^2_{X_{-1/2}}dt+\int_0^T \|C(w^T-\bar w)\|_H^2dt+\|w_0-\bar w\|_H^2+\|w_1\|^2_{X_{-1/2}}]^{\frac12}\\ &\cdot\|\bar p\|_{X_{(\widetilde{k}+1)/2}},\quad k>0,\ \widetilde{k}\ge 1, \quad (3.39)\end{aligned}$$

and hence, due to $C^*\in\mathcal{L}(H,X_{-1/2})$ and the third equation in (2.33), we get that there exist some constants $c_j, j=1,2$ such that

$$\begin{aligned}&\int_0^T \|C(w^T-\bar w)\|_H^2dt-\int_0^T\langle B(u^T-\bar u),p^T-\bar p\rangle dt\\ \le\ &c_1g_1(T)\|(w_0-\bar w,w_1)\|^2_{\mathcal{D}(\mathcal{A}^k)}+c_2g_2(T)\|\bar p\|^2_{X_{(\widetilde{k}+1)/2}} \quad (3.40)\end{aligned}$$

and so,

$$\frac1T\int_0^T \|C(w^T-\bar w)\|_H^2dt-\frac1T\int_0^T\langle B(u^T-\bar u),p^T-\bar p\rangle dt$$

$$\leq \frac{c_1 g_1(T)}{T}\|(w_0-\bar{w},w_1)\|^2_{\mathcal{D}(\mathcal{A}^k)}+\frac{c_2 g_2(T)}{T}\|\bar{p}\|^2_{X_{(\widetilde{k}+1)/2}}\to 0,\ as\ k>0,\widetilde{k}\geq 1. \quad (3.41)$$

Integrating the first equation in (2.33) from 0 to T yields

$$\int_0^T A(w-\bar{w})dt=\int_0^T B(u-\bar{u})dt-w_t(T)+w_1.$$

So,

$$\|\frac{1}{T}\int_0^T A(w-\bar{w})dt\|_{X_{-1/2}}\leq \|\frac{1}{T}\int_0^T B(u-\bar{u})dt\|_{X_{-1/2}}+\frac{1}{T}(\|w_t(T)\|_{X_{-1/2}}+\|w_1\|_{X_{-1/2}}).$$

Thus,

$$\begin{aligned}
&\|\frac{1}{T}\int_0^T (w-\bar{w})dt\|_{X_{1/2}}\\
&\sim \|\frac{1}{T}\int_0^T A(w-\bar{w})dt\|_{X_{-1/2}}\\
&\leq \|\frac{1}{T}\int_0^T B(u-\bar{u})dt\|_{X_{-1/2}}+\frac{1}{T}(\|w_t(T)\|_{X_{-1/2}}+\|w_1\|_{X_{-1/2}}). \quad (3.42)
\end{aligned}$$

By (3.41), along with the Hölder's inequality, we get

$$\begin{aligned}
&\|\frac{1}{T}\int_0^T (w-\bar{w})dt\|^2_{X_{1/2}}\\
&\leq \frac{1}{T}\int_0^T \|(u-\bar{u})\|^2_H dt+\frac{c}{T^2}\\
&\leq \frac{c_1 g_1(T)}{T}\|(w_0-\bar{w},w_1)\|^2_{\mathcal{D}(\mathcal{A}^k)}+\frac{c_2 g_2(T)}{T}\|\bar{p}\|^2_{X_{(\widetilde{k}+1)/2}}+\frac{c}{T^2}\to 0,\ as\ k>0,\ \widetilde{k}\geq 1.
\end{aligned}$$

The proof is completed. □

4. Examples. This section is devoted to presenting some examples on hyperbolic LQ optimal control problems with weak controllability and observability. By using the abstract results obtained in this work, we can identify the explicit slow decay rates and turnpike property for these examples.

4.1. Wave networks. Wave networks have been studied by many researchers (see [9], [13], [14], [27], [29] and the references therein). Here we consider a simple case: star-shaped wave networks.

The setting as given in [9] is chosen to form the wave networks. A 1-d wave network $\mathcal{R}:=\cup_{j=1}^N e_j$ is formed by N wave equations on the curve $e_j,\ j=1,2\cdots,N$ with interval $(0,\ell_j)$. For $k\neq j$, $\overline{e_j}\cap\overline{e_k}$, where $\overline{e_i},\ i=1,2,\cdots,N$ is denoted by the closure of e_i, is either empty or a common end called a vertex or a node. Assume that the wave equation arises on the intervals $(0,\ell_j),\ j=1,2,\cdots,N$ in the network with state $(w_j,w_{j,t})$, respectively.

Let $G=\{e_j,\ j=1,2,\cdots N\}$ be the set of e_j of $\mathcal{R}$, and $\mathcal{V}$ be the set of vertices of $\mathcal{R}$. Denote by $G_v=\{j=\{1,2,\cdots,N\},\ v\in\overline{e_j}\}$ the set of edges having v as a vertex. Denote by $card(G_v)$ the number of edges that meet at v. We call v is an exterior node if $card(G_v)=1$, the set of which is denoted by $\mathcal{V}_{int}$, while if $card(G_v)\geq 2$, the node v is called an interior node and the set of them is $\mathcal{V}_{ext}$.

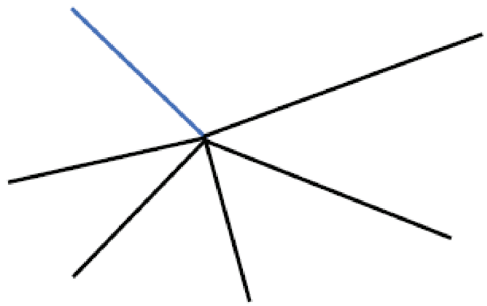

FIG. 4.1. *Star-shaped network of wave equations*

Assume that the Dirichlet conditions are fulfilled at the exterior nodes and the geometrical continuity is satisfied at the interior nodes of the network. The control is assumed to be located only at one edge. For convenience, we set the index of the controlled edge is 1 and j_0 is the index of the observed edge. Then we get the following wave equations on a network (see Fig. 4.1 for instance):

$$\begin{cases} w_{1,tt}(x,t) - w_{1,xx}(x,t) = u(x,t),\ x \in (0,\ell_1),\ t>0, \\ w_{j,tt}(x,t) - w_{j,xx}(x,t) = 0,\ x \in (0,\ell_j),\ j=2,...N,\ t>0, \\ w_\ell(v,t) = w_j(v,t),\ \forall \ell, j \in G_v, v \in \mathcal{V}_{int},\ t>0, \\ \sum\limits_{j\in G_v} w_{j,x}(v,t) = 0,\ \forall v \in \mathcal{V}_{int},\ t>0, \\ w_j(v,t) = 0,\ \forall j \in G_v, v \in \mathcal{V}_{ext},\ t>0, \\ w_j(t=0) = w_j^0,\ w_{j,t}(t=0) = w_j^1, \quad j = 1,2,\cdots,N, \end{cases} \tag{4.1}$$

where $((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N)$ is the given initial state.

Consider the infinite-horizon optimal control problem of quadratic type

$$\min J^\infty(u) = \frac{1}{2}\int_0^\infty [\|u(x,t)\|_U^2 + \|w_{j_0,x}(x,t)\|^2_{L^2(0,\ell_{j_0})}]dt,\ u \in L^2(0,T;U), \tag{4.2}$$

We define the Hilbert space

$$L^2(\mathcal{R}) = \{f = (f_j)_{j=1}^N | f_j \in L^2(0,\ell_j),\ \forall j = 1,2,\cdots,N\}$$

and

$$V^m(\mathcal{R}) = \{f = (f_j)_{j=1}^N | f_j \in H^m(0,\ell_j), f_j(\ell_j) = 0, f_j(0) = f_i(0), \forall i,j = 1,2,\cdots,N\}.$$

Define the operator A in $L^2(\mathcal{R})$ as

$$A(w_j)_{j=1}^N := -(w_{j,xx})_{j=1}^N$$

with domain $\mathcal{D}(A) = \left\{ (w_j)_{j=1}^N \in V^2(\mathcal{R}) \,\middle|\, \sum\limits_{j=1}^N w_{j,x}(0) = 0 \right\}$. Thus, the star-shaped networks (4.1) can be rewritten as the abstract form (2.1).

Set the state space $\mathcal{H}$ as follows:

$$\mathcal{H} = V^1(\mathcal{R}) \times L^2(\mathcal{R})$$

equipped with inner product: for $W = ((w_j)_{j=1}^N, (z_j)_{j=1}^N)$, $\widetilde{W} = ((\widetilde{w})_{j=1}^N, (\widetilde{z})_{j=1}^N) \in \mathcal{H}$,

$$(W, \widetilde{W})_{\mathcal{H}} = \sum_{k=1}^N \int_0^{\ell_j} w_{k,x}\overline{\widetilde{w}_{j,x}}dx + \sum_{j=1}^N \int_0^{\ell_j} z_j\overline{\widetilde{z}_j}dx.$$

It is easy to check that $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Hilbert space. The system operator can be set as $\mathcal{A} := \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}$ in $\mathcal{H}$.

By [9], we have the following weak observability estimate for system (4.1).

Proposition 4.1. *There exists a positive constant $T > 0$ such that*

$$c_1 \int_0^T \int_0^{\ell_1} |w_{1,t}|^2 dxdt \geq \sum_{n\geq 1} \gamma_n^2 [\lambda_n^2 a_n^2 + b_n^2] \tag{4.3}$$

and

$$c_2 \int_0^T \int_0^{\ell_{j_0}} |w_{j_0,x}|^2 dxdt \geq \sum_{n\geq 1} \widetilde{\gamma}_n^2 [\lambda_n^2 a_n^2 + b_n^2] \tag{4.4}$$

where $c_j > 0$, $j = 1, 2$ are some constant, λ_n^2 is the eigenvalue of the operator A corresponding to system (4.1), a_n, b_n are the Fourier coefficients given as in (2.5), and γ_n^2, $\widetilde{\gamma}_n^2 > 0$ are the weights, which are determined by the lengths of edges involved in the network.

In general, we just know that $\gamma_n \to 0$ as n goes to infinity and can not get a better estimate for the decay rate of γ_n. However, following the proof for Theorem 2.6, as well as Remark 2.9, if the estimates $\gamma_n \geq \Phi_1(\lambda_n)$ and $\widetilde{\gamma}_n \geq \Phi_2(\lambda_n)$ hold, we can get

$$\widetilde{c}_1 \|((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N)\|^2_{\mathcal{D}(\Phi_2(\mathcal{A}))} \leq \langle \hat{E}((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N), ((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N)\rangle, \tag{4.5}$$

$$\langle \hat{E}((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N), ((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N)\rangle \leq c_2 \|((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N)\|^2_{\mathcal{D}(\Phi_1^{-1}(\mathcal{A}))}, \tag{4.6}$$

where c_1, c_2 are some positive constants, $\mathcal{D}(\Phi_1^{-1}(\mathcal{A}))$ is denoted by the space satisfying $\sum_{n\geq 1} \Phi_1^{-2}(\lambda_n)[\lambda_n^2 a_n^2 + b_n^2] < \infty$, and a_n, b_n are the Fourier coefficients given as in (2.5).

Although it is unknown on the estimate of γ_n for general networks, it can be better estimated for some special cases. For instance, by [9], we see that for star-shaped networks, the weights γ_n is determined by the edge-length ratios $\frac{\ell_i}{\ell_j}$, where $i, j = 2, \cdots, N$, $i \neq j$ and $\widetilde{\gamma}_n$ is determined by the ratios $\frac{\ell_i}{\ell_j}$, where $i, j = 1, 2, \cdots, N$, $i \neq j$, $i \neq j_0$, $j \neq j_0$. Specifically, if $\frac{\ell_i}{\ell_j}$ belongs to some special irrational sets(see [21]), we can obtain that there always exist some constants $\zeta, \xi > 0$ and $c > 0$ such that $\gamma_n \geq \frac{c}{\lambda_n^\zeta}$ and $\gamma_n \geq \frac{c}{\lambda_n^\xi}$. Based on it, together with Theorem 2.6, we can obtain the following slow decay rate of the closed-loop system with Riccati-based optimal feedback control.

$$\|((\hat{w}_j)_{j=1}^N, (\hat{w}_{jt})_{j=1}^N)\|^2_{\mathcal{H}} \leq M(t+1)^{-\frac{s}{\zeta+\xi}} \|((w_j^0)_{j=1}^N, (w_j^1)_{j=1}^N)\|^2_{\mathcal{D}(\mathcal{A}^{\zeta+\xi+s})}, \quad s > 0.$$

FIG. 4.2. *Wave equation on rectangular domain with local control*

Besides, if $((w_j^0 - \bar{w}_j)_{j=1}^N, (w_j^1)_{j=1}^N) \in \mathcal{D}(\mathcal{A}^k),\ k > 0$ and $(\bar{p}_j)_{j=1}^N \in V_2(\mathcal{R})$ hold, the averaged turnpike property (2.34), (2.35) hold for such kind of networks, that is,

$$\|\frac{1}{T}\int_0^T (u^T - \bar{u})dt\|_{L^2(0,\ \ell_1)} \to 0,\ as\ \ T \to \infty$$

and

$$\|\frac{1}{T}\int_0^T ((w_j^T(t))_{j=1}^N - (\bar{w}_j)_{j=1}^N)dt\|_{V^1(\mathcal{R})} \to 0,\ as\ \ T \to \infty.$$

4.2. Wave equation without GCC: Rectangular domain. Consider a wave equation on a square $\Omega = (0,\pi) \times (0,\pi)$ with local control(see Fig. 4.2):

$$\begin{cases} w_{tt} = \Delta w(x,t) - \chi_{\Omega_0} u(x,t), \\ w|_{\partial\Omega} = 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1, \end{cases} \tag{4.7}$$

where $\Omega_0 = \{(x_1,x_2)|a < x_1 < b,\ 0 < x_2 < \pi\}$ is a strip-type subdomain parallel to one boundary. Meanwhile, assume that

$$Cw = \nabla w\ in\ \Omega,$$

Obviously, (A, C) is exactly observable.

The stability of wave equation in rectangular domain with locally viscous damping was ever considered by [4] and [23], where they obtained that under $u(x,t) = \alpha w_t(x,t)$, the system can be stabilized polynomially, and the optimal decay rate is given as follows:

$$\|(w,w_t)\|_{\mathcal{H}}^2 \le C_k(t+1)^{-\frac{4}{3}k}\|(w_0,w_1)\|_{\mathcal{D}(\mathcal{A}^k)}^2, \quad k > 0, \tag{4.8}$$

in which the state space is chosen as $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ and $\mathcal{A} = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix}$.

Choose the following infinite-horizon quadratic cost performance index:

$$\min J^\infty(u) = \frac{1}{2}\int_0^\infty [\|u(x,t)\|_{L^2(\Omega_0)}^2 + \|\nabla w(x,t)\|_{L^2(\Omega)}^2]dt,\ u \in L^2(0,T;U). \tag{4.9}$$

By (4.8), together with the proof of Theorem 2.6, we can get the bounds related to the corresponding algebraic Riccati operator, that is, there exist constant $c_j > 0,\ j = 1,2$ satisfying

$$c_1\|(w_0,w_1)\|_{\mathcal{H}}^2 \le \langle \hat{E}(w_0,w_1),(w_0,w_1)\rangle \le c_2\|(w_0,w_1)\|_{\mathcal{D}(\mathcal{A}^{\frac{3}{4}})}^2. \tag{4.10}$$

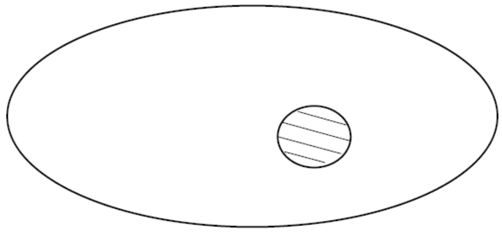

FIG. 4.3. *Wave equation on general domain with local control*

Thus, by Theorem 2.6, there exists a positive constant $M > 0$ such that

$$\|(\hat{w}(t), \hat{w}_t(t))\|_{\mathcal{H}}^2 \leq M(t+1)^{-\frac{4s}{3}} \|(w_0, w_1)\|^2_{\mathcal{D}(\mathcal{A}^{\frac{3}{4}+s})}, \quad s > 0, \tag{4.11}$$

where $(\hat{w}(t), \hat{w}_t(t))$ is the solution to system (4.7) under the Riccati-based optimal feedback control law $u(t) = -\chi_{\Omega_0}\hat{E}(w, w_t)$.

In terms of turnpike property, by Theorem 2.10, we further obtain that for any $(w_0 - \bar{w}, w_1) \in \mathcal{D}(\mathcal{A}^k)$, $k > 0$ and $\bar{p} \in H^2(\Omega)$, the averaged turpike property holds, that is, as $T \to \infty$,

$$\|\frac{1}{T}\int_0^T (u^T - \bar{u})dt\|^2_{L^2(\Omega_0)} \to 0, \quad \|\frac{1}{T}\int_0^T (w^T(t) - \bar{w})dt\|^2_{H^1(\Omega)} \to 0.$$

REMARK 4.2. *In this example, although there is no weak observability estimate (H1) being fulfilled directly, by the proof in Theorem 2.6, the slow decay rate of the system with locally viscous damping given as in (4.8) is enough to help us obtain the upper bound in (4.10). In fact, the slow decay rate is "almost" equivalent to the weak observability estimate (H1).*

It should be noted that some more general energy decay rates for multi-dimensional wave equation on partially rectangular or torus were obtained in [3], [6] and [17], based on which, the slow decay rates and turnpike properties for such infinite-horizon LQ optimal control problems can be also estimated similarly from Theorem 2.6 and 2.10, respectively.

4.3. Wave equation without GCC: General case. Consider the wave equation on domain Ω with local control (see Fig. 4.3).

$$\begin{cases} w_{tt} - \Delta w + u\chi_{\mathcal{O}} = 0, & \text{in } (0,T)\times\Omega, \\ w = 0, & \text{on } (0,T)\times\partial\Omega, \\ w(0) = w_0\,,\ w_t(0) = w_1, & \end{cases} \tag{4.12}$$

where u is the control input, $\mathcal{O}$ is a subset of the whole domain Ω. Suppose that $Cw = \chi_{\widetilde{\mathcal{O}}}\nabla w$, where $\widetilde{\mathcal{O}}$ is another subset of Ω.

Choose the following infinite-horizon quadratic cost performance index:

$$\min J^\infty(u) = \frac{1}{2}\int_0^\infty [\|u(x,t)\|^2_{L^2(\mathcal{O})} + \|\nabla w(x,t)\|^2_{L^2(\widetilde{\mathcal{O}})}]dt,\ u \in L^2(0,T;U). \tag{4.13}$$

In [18], Porretta and Zuazua showed that the solution to system (4.12) with the Riccati-based optimal feedback control can be stabilized exponentially in the energy space $H^1(\Omega)\times L^2(\Omega)$, provided that the subset $\mathcal{O}\subset\Omega$ and $\widetilde{\mathcal{O}}\subset\Omega$ verifies the GCC. It is well-known that the GCC guarantees the exact observability of (A,B^*) and (A,C).

Whenever $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are general open nonempty subsets of Ω, not necessarily satisfying the GCC, the exact observability of (A,B^*) and (A,C) cannot be fulfilled. However, some weak observability estimates still hold for (A,B^*) and (A,C). In fact, from [30], we have that for system (4.12), there exist some constants $a>0$, $c>0$ and $T>0$ satisfying

$$c\int_0^T\int_{\mathcal{O}}|w_t|^2dxdt\geq\|(w_0,w_1)\|^2_{\mathcal{D}(e^{-a\mathcal{A}})}\quad ((A,B^*)\ \textit{weakly obserbvable}),$$

and the following logarithmic decay rate holds under feedback control $u(t)=\chi_{\mathcal{O}}w_t$.

$$\int_\Omega[|w_t(t)|^2+|\nabla w(t)|^2]dx\leq\frac{C_0}{[\log(2+t)]^2}\Big(\|w_0\|^2_{H^2(\Omega)}+\|w_1\|^2_{H^1(\Omega)}\Big)\qquad\forall t>0\,. \tag{4.14}$$

Similarly, there exist some constants $b>0$, $c>0$ and $T>0$ satisfying

$$c\int_0^T\int_{\widetilde{\mathcal{O}}}|\nabla w|^2dxdt\geq\|(w_0,w_1)\|^2_{\mathcal{D}(e^{-b\mathcal{A}})},\quad ((A,C)\ \textit{weakly obserbvable}).$$

Thus, as it was presented in Remark 2.9, the lower and upper bounds can be derived by (2.29). Moreover, we can get the slow decay rate for system (4.12) with Riccati-based optimal feedback control, as given in (2.30).

The averaged turnpike property still holds for any $(w_0-\bar{w},w_1)\in\mathcal{D}(\mathcal{A})$ and $\bar{p}\in H^2(\Omega)$. Indeed, based on (4.14), along with the proof for Lemma 3.2 and 3.3, we can see that the results in Lemma 3.2 and 3.3 still hold with some $g_1(T)$ (resp. $g_2(T)$), which is determined by $\int_0^T\frac{1}{[\log(2+t)]^2}dt$ (resp. $\int_0^T\frac{1}{[\log(2+T-t)]^2}dt$) (see (3.25), (3.31)).

Note that by mean value theorem of integrals, we have

$$\int_0^T\frac{1}{[\log(2+t)]^2}dt=\int_0^T\frac{1}{[\log(2+T-t)]^2}dt=\frac{T}{[\log(2+\alpha_TT)]^2}, \tag{4.15}$$

where $\alpha_TT\to\infty$ as $T\to\infty$. Thus, choosing $g_1(t)=g_2(t)=\frac{T}{[\log(2+\alpha_TT)]^2}$ and following the proof of Theorem 2.10, we finally obtain from (3.41) with $k=\widetilde{k}=1$ that

$$\begin{aligned}&\frac{1}{T}\int_0^T\|\nabla(w^T-\bar{w})\|^2_{L^2(\Omega)}dt+\frac{1}{T}\int_0^T\|u^T-\bar{u}\|^2_{L^2(\mathcal{O})}dt\\ &\leq\frac{c_1g_1(T)}{T}\|(w_0-\bar{w},w_1)\|^2_{\mathcal{D}(\mathcal{A})}+\frac{c_2g_2(T)}{T}\|\bar{p}\|^2_{H^2(\Omega)}\\ &=\frac{1}{[\log(2+\alpha_TT)]^2}(c_1\|(w_0-\bar{w},w_1)\|^2_{\mathcal{D}(\mathcal{A})}+c_2\|\bar{p}\|^2_{H^2(\Omega)})\to 0.\end{aligned} \tag{4.16}$$

It should be noted that the estimate (4.16) is consistent with the one in Section 4.3 in Porretta and Zuazua [18], where the LQ optimal control problem related to the multi-dimensional wave equation without GCC was considered.

5. Conclusions. In this work, we considered the slow decay rate and turnpike property of the hyperbolic LQ optimal control problems and mainly obtained the following results:

1. The slow decay rate of infinite-horizon hyperbolic LQ optimal control problems was considered. Under the weak observability of (A, B^*) and (A, C), the lower and upper bounds of the corresponding algebraic Riccati operator were estimated, respectively. Then the explicit slow decay rate of the closed-loop system with Riccati-based optimal feedback control was estimated, which is a key step to further discuss the turnpike properties of the LQ optimal control problems if lacking of exact controllability or observability.

2. Under weak observability of (A, B^*) and (A, C) hypothesises, the averaged turnpike property for the LQ optimal control problems was proved under certain conditions on the regularity of the initial state, the stationary optimal state and its dual. This result is consistent with the slow turnpike in average identified in section 4.3 in [18] which can be considered as one special case of our work (see the example in Section 4.3 in our work).

Besides the averaged turnpike property, we would like to point out that the starting point of our work was to see whether there were some kinds of point-wise slow turnpike properties holding for the LQ optimal control problems under weak controllability or observability hypothesises. In fact, it can be seen from [18] and [31] that the exponential decay of the closed-loop systems with Riccati-based optimal feedback control always leads to the exponential-type point-wise turnpike property when exact controllability and observability hypothesises are fulfilled, that is, there exist $\lambda > 0$ and $K > 0$ such that

$$\|w^T(t) - \bar{w}\|_{X_{1/2}}^2 + \|u^T(t) - \bar{u}\|_H^2 \leq K(e^{-\lambda t} + e^{-\lambda(T-t)}) \qquad \forall t \in [0, T], \tag{5.1}$$

where (w^T, u^T) and $(\bar{w}, \bar{u})$ are the optimal trajectories and controls for the optimal control problem (2.31) and for the stationary problem (2.32), respectively. Thus, based on the slow decay rate obtained in our work, it is reasonable to predict that there should be some kinds of slow point-wise turnpike properties holding under weak controllability or observability.

However, note that the energy estimate method based on the properties of Riccati operator proposed in [18] can not work well for this issue under consideration. Indeed, when doing so, there is an inevitable process to estimate $\|w^T - \bar{w}\|_{X_{1/2}}$ by using Duhamel's formula along with Gronwall's lemma. This process is tough to tackle for the LQ optimal control problems under weak observability of (A, B^*) and (A, C) like hypothesises (H1) and (H2), because that under these weak hypothesises, the corresponding closed-loop systems with Riccati-based optimal controls always achieve non-uniform slow decay rates which depending on the regularity of the initial states, while it can be carried out effectively for the case of exact controllability and observability due to the uniform exponential decay rates always hold for all initial states and this uniformity causes the Gronwall's lemma easy to be used. So, it is still an open problem that whether there are some kinds of slow point-wise turnpike property holding for such hyperbolic LQ problems. Compared to (5.1), based on the weak observation of (A, B^*) and (A, C) as given in (H1) and (H2), it seems that the slow point-wise turnpike property could hold and have the following form:

$$\|w^T - \bar{w}\|_{X_{1/2}}^2 + \|u^T(t) - \bar{u}\|_H^2 \leq K(t+1)^{-k_1} \|(w_0, w_1)\|_{\mathcal{D}(\mathcal{A})}^2 + (T - t + 1)^{-k_2} \|\bar{p}\|_{X_1}^2.$$

It could be verified from the view of frequency domain by Riesz basis representation, but careful estimates are still needed, which is an interesting issue and worth investigating in future.

The same problems are also worth discussing for linear parabolic systems or some non-linear systems such as semilinear wave equations, nonlinear models of fluid mechanics and so on. In addition, the case of weak boundary observability or controllability is another interesting issue. Some new techniques could be involved due to the unboundedness of the boundary control or observation operators.

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