SHARP NUMERICAL APPROXIMATION OF THE HARDY CONSTANT

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Abstract. We study the P_1 finite element approximation of the best constant in the classical Hardy inequality over bounded domains containing the origin in $\mathbb{R}^N$, for $N \geq 3$.

Despite the fact that this constant is not attained in the associated Sobolev space H^1, our main result establishes an explicit, sharp, and dimension-independent rate of convergence proportional to $1/|\log h|^2$.

The analysis carefully combines an improved Hardy inequality involving a reminder term with logarithmic weights, approximation estimates for Hardy-type singular radial functions constituting minimizing sequences, properties of piecewise linear and continuous finite elements, and weighted Sobolev space techniques.

We also consider other closely related spectral problems involving the Laplacian with singular quadratic potentials obtaining sharp convergence rates.

1. Introduction and main results

The Hardy inequality plays a central role in the analysis of partial differential equations (PDEs) with singular potentials, spectral theory, and the study of critical functional inequalities. In its classical form, it provides a lower bound on the Dirichlet energy in terms of a weighted L^2-norm involving the distance to the origin.

For bounded domains $\Omega \subset \mathbb{R}^N$ containing the origin and with $N \geq 3$, the best Hardy constant is known and sharp but notably not attained in the Sobolev space $H_0^1(\Omega)$. The same occurs in the whole space $\mathbb{R}^N$. This lack of attainability poses significant challenges for the numerical approximation of the constant via variational methods.

In this work, we study the convergence behavior of finite element approximations of the Hardy constant. Specifically, we focus on piecewise linear and continuous P_1 finite elements and establish an explicit convergence rate of order $1/|\log h|^2$ as the mesh size $h \to 0$. Our analysis relies on improved Hardy inequality involving a reminder term with logarithmic weights, carefully constructed singular test functions, weighted Sobolev space estimates, and interpolation error bounds adapted to the singular nature of the Hardy inequality.

To be more precise, given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, containing the origin, let us consider the following minimization problem related with the Hardy inequality

$$\Lambda_N(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}. \tag{1.1}$$

It is well known that this infimum is not attained and is independent of the domain, coinciding with the Hardy constant in the whole space

$$\Lambda_N(\Omega) = \Lambda_N = \frac{(N-2)^2}{4}. \tag{1.2}$$

The minimizer is not achieved in $H_0^1(\Omega)$. For instance, when Ω is a ball, the minimizer should be a radial function of the form, $u(r) = r^{-N/2+1}(a_1 + a_2 \log(r))$ which does not belong to $H_0^1(\Omega)$ but to a larger Hilbert space $\mathcal{H}$, which is essentially the closure of $H_0^1(\Omega)$ with respect to the norm (see Section 2)

$$\|u\|^2_{\mathcal{H}} = \|\nabla u\|^2_{L^2(\Omega)} - \Lambda_N \|\frac{u}{|x|}\|^2_{L^2(\Omega)}. \tag{1.3}$$

Given a P_1 finite-element subspace V_h of $H^1_0(\Omega)$ associated to a finite element mesh in Ω, the Hardy constant can be approximated by the corresponding finite-dimensional minimization problem:

$$\Lambda_h(\Omega) = \inf_{u\in V_h} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}, \tag{1.4}$$

The main result of this paper is the following theorem, which holds under standard assumptions on the finite element mesh, to be detailed below:

Theorem 1.1. *Let Ω be a convex smooth domain of $\mathbb{R}^N$, $N \geq 3$, and V_h the space of P_1 finite elements on Ω. Then*

$$\Lambda_h(\Omega) - \Lambda_N \simeq \frac{1}{|\log h|^2}. \tag{1.5}$$

This result fully clarifies an issue whose analysis was initiated in [11], where the one dimensional case was treated. In [11] the rotational symmetry of the ball was exploited to derive convergence rates in specific configurations by reducing the analysis to the one dimensional case. The approach we introduce here is more flexible and applies to arbitrary smooth convex domains, using that the approximating minimizers have the same singularity at the origin as the radial ones.

Note that the two dimensional case is critical and a inverse square logarithmic correction of the Hardy inequality should be included [1]. This would require further analysis.

Such results are now well established in other related contexts. In particular, the answer is well known for the classical Poincaré inequality, which is directly related to the first eigenvalue of the Dirichlet Laplacian. In that setting, the first continuous eigenvalue and its finite element approximation are known to be h^2-close [8, Prop. 6.30, p. 315], [4, Section 8, p. 700].

It is important to note, however, that the Poincaré inequality differs in two fundamental ways from the Hardy constant under consideration here. First, for the Poincaré constant, the infimum is actually attained, making it a true minimum. Second, this minimum is realized by the first eigenfunction of the Laplacian, which – up to normalization – is unique, belongs to $H^1_0(\Omega)$, and is in fact smooth.

In [13], the same issue was analyzed for the Sobolev constant, yielding polynomial convergence rates, with the order depending on both the dimension N and the p-exponent in $W^{1,p}$. It is worth noting that the Hardy constant differs fundamentally from the Sobolev constant in that, while the Hardy constant is not attained, the Sobolev constant is achieved in the whole space, and the set of minimizers forms a finite-dimensional (of dimension $N+2$) manifold that can be explicitly characterized.

Thus, the Hardy constant represents a new instance, with respect of the existing literature, in the sense that it is not achieved even in the whole space. This explains partially logarithmic and not polynomial approximation rate. The technical reason for the logarithmic rate becomes rather natural within the proof which explores an improved Hardy inequality with a logarithmic correction [9, Th. 2.5.2], which replaces the Sobolev deficit estimates employed [13].

We also consider some closely related spectral problems involving the Laplacian with a singular quadratic potential.

Let us first consider

$$\mu_1(\Omega) = \inf_{u\in H_0^1(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)} - \Lambda_N\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} = \min_{u\in\mathcal{H}} \frac{\|\nabla u\|^2_{L^2(\Omega)} - \Lambda_N\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}. \tag{1.6}$$

Again, the optimal constant is not achieved in $H_0^1(\Omega)$ but in a larger Hilbert space $\mathcal{H}$. Of course, this spectral problem is well-posed thanks to the Hardy inequality, which ensures that the numerator of the Rayleigh quotient is non-negative and coercive in $\mathcal{H}$.

Let us also consider its discrete counterpart, defining $\mu_{1h}(\Omega)$ as the minimum of the same ratio in V_h. In this case we have the following result.

Theorem 1.2. *In the setting of Theorem 1.1*

$$\mu_{1h}(\Omega) - \mu_1(\Omega) \simeq \frac{1}{|\log h|}. \tag{1.7}$$

Note that, although the convergence rate is also logarithmic, its order differs from that established in Theorem 1.1.

The same problem can be considered in the subcritical case in which the amplitude of the quadratic potential is $0 < \Lambda < \Lambda_N$:

$$\lambda_1(\Omega) = \min_{u\in H_0^1(\Omega)} \frac{\|\nabla u\|^2_{L^2(\Omega)} - \Lambda\|\frac{u}{|x|}\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}. \tag{1.8}$$

Note that, in this case the minimizer is achieved in $H_0^1(\Omega)$ given the coercivity (in $H_0^1(\Omega)$) of the numerator of the Rayleigh ratio.

Minimizing over the finite element subspace V_h we obtain the corresponding FEM approximation $\lambda_{1h}(\Omega)$. In the present case we obtain a polynomial convergence rate:

Theorem 1.3. *In the setting of Theorem 1.1 denoting*

$$m = \sqrt{\Lambda_N - \Lambda} \in \left(0, \frac{N-2}{2}\right] \tag{1.9}$$

the following hold

1. *for* $N \geq 5$

$$\lambda_{1h}(\Omega) - \lambda_1(\Omega) \simeq \begin{cases} h^{2m}, & 0 < m < 1, \\ h^2|\log h|, & m = 1, \\ h^2, & 1 < m \leq \frac{N-2}{2}. \end{cases} \tag{1.10}$$

2. *for* $N = 4$

$$\lambda_{1h}(\Omega) - \lambda_1(\Omega) \simeq \begin{cases} h^{2m}, & 0 < m < 1, \\ h^2, & m = 1, \end{cases}$$

3. *for* $N = 3$

$$\lambda_{1h}(\Omega) - \lambda_1(\Omega) \simeq \begin{cases} h^{2m}, & 0 < m < \frac{1}{2}, \\ h^2, & m = \frac{1}{2}. \end{cases}$$

The main reason we obtain polynomial decay rates rather than logarithmic ones, as in the previous theorems, is that the continuous eigenfunctions are less singular than in the critical case $\Lambda = \Lambda_N$, although they still exhibit a singularity of order $|x|^{-N/2+1+\sqrt{\Lambda_N-\Lambda}}$. However, in this case, the eigenfunction belongs to $H_0^1(\Omega)$ and the error analysis reduces to estimating the $H_0^1(\Omega)$-distance between the eigenfunction and the finite element space.

Note that in the case $\Lambda = 0$, we recover the classical problem of finite element approximation of the Poincaré constant, and our results allow us to retrieve the well-known optimal convergence rate of order h^2.

The paper is organized as follows. We first introduce the functional framework needed to address the Hardy inequality, along with some classical results of finite element theory. We then present the proofs of the main results. The paper concludes with a section on comments and open problems, followed by an appendix containing technical lemmas.

2. Functional framework

As mentioned above, we denote by $\mathcal{H} \subset L^2(\Omega)$, the Hilbert space obtained as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = \int_\Omega \Big(|\nabla u|^2 - \Lambda_N \frac{u^2}{|x|^2}\Big) dx. \tag{2.1}$$

The space $\mathcal{H}$ is isometric with the space

$$W_0^{1,2}(|x|^{-(N-2)}dx, \Omega) = W^{1,2}(d\mu, \Omega), \|v\|_{\tilde{H}} = \Big(\int_\Omega |x|^{-(N-2)} |\nabla v|^2 dx\Big)^{1/2}$$

under the transformation

$$u = Tv = |x|^{-(N-2)/2} v.$$

here and in the sequel the density μ is defined as

$$\mu(x) = |x|^{-(N-2)}. \tag{2.2}$$

It has been proved in [17, 16] that

$$\mathcal{H} = \overline{u = Tv, \ v \in C_c^\infty(\Omega)}^{\|\cdot\|_{\mathcal{H}}}$$

where

$$\|u\|_{\mathcal{H}}^2 = \int_\Omega |x|^{-(N-2)} |\nabla(|x|^{(N-2)/2} u)|^2 dx.$$

Let us introduce the two operators

$$Lu = -\Delta u - \Lambda_N \frac{u}{|x|^2},$$

$$\tilde{L}v = -|x|^{N-2} \nabla \cdot (|x|^{-(N-2)} \nabla v) = -\Delta v + (N-2) \frac{x}{|x|^2} \cdot \nabla v.$$

Under the transformation $u = Tv$ we have

$$Lu = L(Tv) = |x|^{-N/2+1} Lv = T\tilde{L}v,$$

i.e. $LT = T\tilde{L}$.

There exists a sequence of common eigenvalues

$$0 < \mu_1 \le \mu_2 \le \cdots \le \to \infty$$

and the corresponding eigenfunctions are related by $\phi_k = T\psi_k$.

In the particular case when Ω is ball centered at origin – for simplicity we take $\Omega = B_1(0)$ – the eigenvalues can be computed explicitly

$$\psi_{j,n(r,\sigma)} = J_{m_j}(z_{m_j,n} r) f_j(\sigma),$$

where $\{f_j\}_{j\ge 0}$ are the spherical harmonics, $z_{m,n}$ is the nth zero of the Bessel function J_m, $m^2 = j(j + N - 2)$, $j \ge 0$, and $\mu_{j,n} = z_{m_j,n}^2$.

Let us denote by ψ_1 the first eigenfunction, i.e.,

$$\psi_1 = \psi_{0,1} = J_0(z_{0,1}r)$$

and $\phi_1 = T\psi_1$. The properties of the Bessel function guarantee that

$$\psi_1(0) \neq 0, \ \nabla\psi_1(0) = 0.$$

3. Fundamental Tools from Finite Element Analysis

3.1. **Finite element meshes.** Let Ω be a polytope in $\mathbb{R}^N$ (in particular, an interval, a polygonal or a polyhedral domain in dimension one, two and three respectively). For each positive h we construct a partition $\mathcal{T}_h$ (a mesh) of the domain Ω into a finite set of N-simplices or cells (tetrahedrons in dimension $N = 3$) satisfying

(1) $\cup_{T\in\mathcal{T}_h} T = \overline{\Omega}$,
(2) if $T, T' \in \mathcal{T}_h$, $T \neq T'$, then either $T \cap T' = \emptyset$, $T \cap T'$ is a single common vertex or $T \cap T'$ is a whole common facet (point in $N = 1$, edge in $N = 2$, face in $N = 3$).

For each $T \in \mathcal{T}_h$ we denote by ρ_T and h_T the largest ball contained in T and the diameter of T respectively. We set

$$h = h(\mathcal{T}_h) = \max_{T\in\mathcal{T}_h} h_T.$$

We will consider a set of regular meshes $(\mathcal{T}_h)_{h>0}$: there exists $\sigma > 0$, independent of h, such that

$$\frac{h_T}{\rho_T} \leq \sigma, \ \forall\, T \in \mathcal{T}_h, \ \forall\, h > 0. \tag{3.1}$$

Also the mesh is quasi-uniform, i.e.

$$\inf_{h>0} \frac{\min_{T\in\mathcal{T}_h} h_T}{\max_{T\in\mathcal{T}_h} h_T} > 0.$$

Each element of the mesh $\mathcal{T}_h$ is the image of a reference N-simplex through an affine mapping $F_T : \mathbb{R}^N \to \mathbb{R}^N$,

$$F_T(\widehat{x}) = B_T\widehat{x} + b_T,$$

B_T being an invertible $d \times d$ matrix, $b \in \mathbb{R}^N$, i.e.

$$F_T(\widehat{T}) = T, \quad \forall\, T \in \mathcal{T}_h.$$

We recall few properties of matrix $B_T = \nabla F_T$ cf. [3, Lemma 7.4.3, p. 272]

$$\|B_T\| \leq \frac{h_T}{\rho_{\widehat{T}}}, \quad \|B_T^{-1}\| \leq \frac{h_{\widehat{T}}}{\rho_T}$$

and

$$|\det B| = |JF_T| = \frac{|T|}{|\widehat{T}|}.$$

The application F_T will map $\widehat{G}$ the barycenter of $\widehat{T}$ to the one of T.

For a fixed $\mathcal{T}_h$, we defined the space V_h as

$$V_h = \{f \in C(\overline{\Omega}); f \circ F_T \in \mathbb{P}^1(\widehat{T}), \ \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}^1(\widehat{T})$ is the space of linear polynomials on $\widehat{T}$.

For our approximations we consider the unit ball, $\Omega = B$, and we approximate it with a polygonal domain $B_h \subset B$ as in [2]. With this polygonal domain B_h we introduce the space $V_h \subset H^1(B_h) \cap C(\overline{B}_h)$ to be the space of functions in V_h extended by zero in $B \setminus B_h$.

3.2. **Approximation by piecewise linear functions.** We recall the classical approximation result in Sobolev spaces [5, Th. 4.4.20]: For any polyhedral domain $\Omega \in \mathbb{R}^N$, $N < 2p$, $\mathcal{T}_h$ as above, $1 < p \leq \infty$ or $n \leq 2$ if $p = 1$ (see [5, Th. 4.4.4] for the complete set of restrictions), and $s \in \{0, 1\}$, the global piecewise linear interpolant I^h satisfies

$$\Big(\sum_{T\in\mathcal{T}_h} \|u - I^h u\|^p_{W^{s,p}(T)}\Big)^{1/p} \leq Ch^{2-s}\|u\|_{W^{2,p}(\Omega)},\ \forall u \in W^{2,p}(\Omega). \tag{3.2}$$

The above restrictions are necessarily when one needs to have an estimate for all functions in $W^{2,p}(\Omega)$. When restrict to the class of functions that are smooth, for example $C^2(\overline{\Omega})$ the above restrictions on the dimension can pe relaxed. The restriction $N < 2p$ is exactly the one that guaranteess that $W^{2,p}(\mathbb{R}^N) \subset C^0(\Omega)$, the class where the global linear interpolator I^h is defined.

Lemma 3.1. *Let $\mathcal{T}_h$ a regular mesh on a polyhedral domain $\Omega \in \mathbb{R}^N$, $1 < p \leq \infty$, $s \in \{0, 1\}$. There exists a positive constant $C = C(N, p, s, \sigma)$ such that for all $|\alpha|$=s and $1 \leq p < \infty$:*

$$\Big(\sum_{T\in\mathcal{T}_h} \|D^\alpha(u - I^h u)\|^p_{L^p(T)}\Big)^{1/p} \leq Ch^{2-s}\Big(\sum_{T\in\mathcal{T}_h} |T|\|D^2u\|^p_{L^\infty(T)}\Big)^{1/p},\ \forall u \in C^2(\overline{\Omega}), \tag{3.3}$$

and

$$\max_{T\in\mathcal{T}_h} \|D^\alpha(u - I^h u)\|_{L^\infty(T)} \leq Ch^{2-s} \max_{T\in\mathcal{T}_h} \|D^2u\|_{L^\infty(T)},\ \forall u \in C^2(\overline{\Omega}). \tag{3.4}$$

The proof is a slightly modification of the one in [5, Th. 4.4.4, Chapter 4]. The proof can be extended to more general finite elements and $u \in C^m(\overline{\Omega})$ but this is out of the scope of this paper. For a proof see [13].

In fact the above estimates are sharp in the case of functions that are uniform convex in one direction.

Lemma 3.2. *For any $p \in (1, \infty)$ there exists a positive constant $C(p)$ such that for any $T \in \mathcal{T}_h$ and any $u \in C^2(T)$*

$$\min_{A\in\mathbb{R}^N} \int_T |Du - A|^p dx \geq C(p)\rho_T^{N+p} \max_{\xi\in\mathbf{S}^{N-1}} \min_{x\in\overline{\Omega}} |\xi^T D^2u(x)\xi|^p.$$

The same holds under the assumption that the function is uniformly convex in one direction, i.e. $\inf_T |\partial^2_{x_k} u(x)| > 0$ *for some x_k.*

3.3. **Finite element eigenvalue approximation.** Let us now recall the classical theory for eigenvalue approximation, [4]. Here we present it in the simplest case. Following the notations in [4, Section 8, p. 697], let V a real Hilbert space and $a(\cdot, \cdot)$ be a symmetric continuous and coercive bilinear form on V. Let H be another Hilbert space such that $V \subset H$ with compact embedding, b a symmetric continuous bilinear form on H, such that $b(u, u) > 0$, for all $u \in V$, $u \neq 0$. Let $V_h \subset V$ be a family of finite-dimensional spaces of V.

Let λ_1 be the first eigenvalue of *the form a relative to the form* b, i.e. the smallest λ_1 so that there exists a non-trivial $u_1 \in V$ such that

$$a(u_1, v) = \lambda_1 b(u, v), \forall\ v \in V.$$

In a similar way one define λ_{1h} in the space V_{1h}: there exists $u_{1h} \in V_h$ such that

$$a(u_1, v) = \lambda_{1h} b(u, v), \forall\ v_h \in V_h.$$

A fundamental result in the theory of eigenvalue approximation is the following, originally established in [8, Prop. 6.30, p. 315]; see also [4, p. 700] for a more direct statement:

$$C_1\varepsilon_h^2 \le \lambda_{1h} - \lambda_1 \le C_2\varepsilon_h^2 \tag{3.5}$$

where

$$\varepsilon_h = d(u_1, V_h) = \inf_{v_h\in V_h} \|u_1 - v_h\|_V.$$

The upper bound together with the trivial estimate $0 \le \lambda_{1h} - \lambda_1$, can be also found in [15, Th. 6.4-2].

This analysis will be useful in the proofs of Theorem 1.2 and Theorem 1.3, which are effectively related to eigenvalue problems. However, it does not apply to Theorem 1.1, which falls outside this framework and requires an independent analysis.

4. Proof of Theorem 1.1

Let us consider Ω a convex smooth domain such that $0 \in \Omega \subset \overline{\Omega} \subset B_R$ for some $R > 0$. Without loss of generality assume that $R = 1$. We approximate the domain Ω by a polygonal domain $\Omega_h \subset \Omega$ as in [2]. We introduce the finite element subspace $V_h \subset H^1(\Omega_h) \cap C(\overline{\Omega}_h)$ extended by zero to $\Omega \setminus \Omega_h$.

The proof of Theorem 1.1 (1.5) treats separately the lower and upper bound.

The following Hardy inequality with a logarithmic reminder term [14, Th. 2.5.2, p. 25] will play a key role: There exists a positive constant $C = C(N,\Omega)$ such that for all $\phi \in C_c^\infty(\Omega)$ it holds

$$\int_\Omega |\nabla\phi|^2 dx - \Lambda_N \int_\Omega \frac{\phi^2}{|x|^2} dx \ge C \int_\Omega |\nabla\phi|^2 \Big(\log\frac{1}{|x|}\Big)^{-2} dx. \tag{4.1}$$

By density it also holds for any $\phi \in H_0^1(\Omega)$. We also recall that (see [14, (2.5.7), p. 25-26])

$$\inf_{\phi\in C_c^\infty(\Omega),\phi\neq 0} \frac{\int_\Omega |\nabla\phi|^2 dx - \Lambda_N \int_\Omega \frac{\phi^2}{|x|^2} dx}{\int_\Omega |\phi|^2|x|^{-2}\big(\log\frac{1}{|x|}\big)^{-2} dx} = \frac{1}{4}. \tag{4.2}$$

In dimension $N \ge 3$, the infimum above, 1/4, can be achieved along a minimizing sequence, by taking regularizations $u_\varepsilon = \eta_\varepsilon \tilde{u}_2$ (see Section 4.2 for more details) of

$$\tilde{u}_2(x) = \tilde{u}_2(|x|),\ \tilde{u}_2(r) = |r|^{-\frac{N-2}{2}}\Big(\log\frac{1}{|r|}\Big)^{1/2}, \tag{4.3}$$

which is the distributional solution of

$$-\Delta w - \Lambda_N \frac{w}{|x|^2} = \frac{1}{4}\frac{w}{|x|^2}\Big(\log\frac{1}{|x|}\Big)^{-2}.$$

This regularization consists, roughly speaking, in truncating the function $\tilde{u}_2$ outside a suitably chosen ball that contains the singularity a $x = 0$.

It is worth to mention that functions $u(x) = |x|^{-(N-2)/2}v(x)$ where $v(x) \simeq (\log(1/x))^\alpha$ as $x \to 0$ belong to space $\mathcal{H}$ if $\alpha \in [0, 1/2)$ but fail to be in $\mathcal{H}$ for $\alpha \ge 1/2$, see [16].

Since u_ε has the suport outside the ball of radius ε^2 we have

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx - \Lambda_N \int_\Omega u_\varepsilon^2|x|^{-2}dx}{\int_\Omega |u_\varepsilon|^2|x|^{-2}} \simeq \frac{1}{4}\frac{\int_\Omega |u_\varepsilon|^2|x|^{-2}\Big(\log\frac{1}{|x|}\Big)^{-2}dx}{\int_\Omega |u_\varepsilon|^2|x|^{-2}} \lesssim |\log\varepsilon|^{-2}.$$

The family u_ε may be used as a minimizing sequence for Λ_N, with a controlled error of order $|\log\varepsilon|^{-2}$. The proof of Theorem 1.1 proceeds by a careful finite element approximation of this minimising sequence.

4.1. **The lower bound.** Although it is briefly presented in [11], we include a sketch here for completeness.

Let $v_h\in V_h$ the minimizer corresponding to Λ_h:

$$\Lambda_h=\frac{\int_\Omega |\nabla v_h|^2dx}{\int_\Omega v_h^2|x|^{-2}dx}.$$

It follows that

$$\Lambda_h-\Lambda_N=\frac{\int_\Omega |\nabla v_h|^2dx-\Lambda_N\int_\Omega v_h^2|x|^{-2}dx}{\int_\Omega v_h^2|x|^{-2}dx}\geq C(N,\Omega)\frac{\int_\Omega |\nabla v_h|^2|\log|x||^{-2}dx}{\int_\Omega v_h^2|x|^{-2}dx}.$$

Using that ∇v_h is constant in each triangle and Lemma 8.2 in the Appendix below we get

$$\begin{aligned}\int_\Omega |\nabla v_h|^2|\log|x||^{-2}dx&=\sum_{T\in\mathcal{T}_h}\int_T|\nabla v_h|^2|\log|x||^{-2}dx=\sum_{T\in\mathcal{T}_h}|\nabla v_h|^2\int_T|\log|x||^{-2}dx\\&\gtrsim\frac{1}{|\log h|^2}\sum_{T\in\mathcal{T}_h}|T||\nabla v_h|^2dx=\frac{1}{|\log h|^2}\int_\Omega|\nabla v_h|^2\\&=\frac{\Lambda_h}{|\log h|^2}\int_\Omega v_h^2|x|^{-2}dx.\end{aligned}$$

This shows that $\Lambda_h-\Lambda_N\gtrsim\Lambda_h|\log h|^{-2}\geq\Lambda_N|\log h|^{-2}$.

4.2. **The upper bound.** It is sufficient to prove the upper bound

$$\Lambda_h-\Lambda_N\lesssim\frac{1}{|\log h|^2},$$

in the case when Ω is a ball centered at origin B. Indeed, given that the constant Λ_N is independent of the bounded domain under consideration, the upper holds in the general case by comparison.

To simplify the presentation, and without loss of generality, we consider the case when $\Omega=B_1$ the unit ball.

As mentioned above, the main idea on the upper bound is to use an approximating sequence u_ε of

$$\tilde u_2(r)=r^{-N/2+1}\left(\log\frac{1}{r}\right)^\alpha$$

and project it on the space V_h. In the following we make this construction more precise.

Along the proof we will also employ the function

$$u_2(r)=r^{-N/2+1}.$$

We consider the following cut-off function inspired by [12, proof of Corol. VIII.6.4] and [6]:

$$\eta_\varepsilon(x)=\begin{cases}0, & |x|<\varepsilon^2,\\ \xi(\frac{\log(|x|/\varepsilon^2)}{\log(1/\varepsilon)}), & |x|\in(\varepsilon^2,\varepsilon),\\ 1, & |x|>\varepsilon,\end{cases}$$

where $\xi:[0,1]\to[0,1]$ is a smooth function such that for some $\mu\in(0,1)$, $\xi=0$ on $[0,\mu]$ and $\xi=1$ on $[1-\mu,1]$.

The function η_ε satisfies the following properties:

(1) $\eta_\varepsilon(r) = 0$ on $0 < r < \varepsilon^{2-\mu}$,

(2) $|\eta_\varepsilon'(r)| \le \frac{\|\xi\|_\infty}{r|\log\varepsilon|} \lesssim \frac{1}{r|\log\varepsilon|}$, $\varepsilon^2 \le r \le \varepsilon$,

(3) $|\eta''(r)| \le \frac{\|\xi''\|_\infty}{r^2|\log\varepsilon|^2} + \frac{\|\xi'\|_\infty}{r^2|\log\varepsilon|} \lesssim \frac{1}{r^2|\log\varepsilon|}$, $\varepsilon^2 < r < \varepsilon$.

As a consequence we get

(1) $|\nabla\eta_\varepsilon(x)| \le \frac{1}{|x||\log\varepsilon|}$ if $\varepsilon^2 < |x| < \varepsilon$ and vanishes otherwise,

(2) $|D^2\eta_\varepsilon(x)| \lesssim \frac{1}{|x|^2|\log\varepsilon|}$, if $\varepsilon^2 < |x| < \varepsilon$ and vanishes otherwise.

With this function η_ε we introduce

$$u_\varepsilon(x) = \tilde{u}_2(|x|)\eta_\varepsilon(|x|)\psi(|x|) \in C_c^\infty(\Omega),$$

where $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi \equiv 1$ for $|r| \le 1/4$ and $\psi \equiv 0$ for $|r| > 1/2$.

The following lemma provides quantitative estimates for this sequence, viewed as a minimizing sequence for the Hardy constant.

Lemma 4.1. *Let $\alpha \ge 0$. The family of functions u_ε satisfies the following estimates*

$$A_\varepsilon = \int_\Omega |\nabla u_\varepsilon|^2 dx - \Lambda_N \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \lesssim \begin{cases} |\log\varepsilon|^{2\alpha-1}, & \alpha > 1/2, \\ \log|\log\varepsilon|, & \alpha = 1/2, \\ 1, & \alpha \in [0,1/2), \end{cases} \tag{4.4}$$

$$B_\varepsilon = \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \simeq |\log\varepsilon|^{2\alpha+1}, \tag{4.5}$$

and

$$\|u_\varepsilon\|^2_{H^2(\Omega)} \lesssim \frac{|\log\varepsilon|^{2\alpha}}{\varepsilon^{4-2\mu}}. \tag{4.6}$$

Remark 4.2. As a consequence we get that the quotient $A_\varepsilon/B_\varepsilon$ satisfies

$$\frac{A_\varepsilon}{B_\varepsilon} \lesssim \begin{cases} \frac{1}{|\log\varepsilon|^2}, & \alpha > \frac{1}{2}, \\ \frac{\log|\log\varepsilon}{|\log\varepsilon|^2}, & \alpha = \frac{1}{2}, \\ \frac{1}{|\log\varepsilon|^{2\alpha+1}}, & \alpha \in [0, \frac{1}{2}). \end{cases} \tag{4.7}$$

Proof. Using that $u_\varepsilon(x) = \tilde{u}_2(|x|)\eta_\varepsilon(r)\psi(r)$, we obtain

$$B_\varepsilon \le \int_{\varepsilon^2}^{1/2} r^{N-3}|\tilde{u}_2(r)|^2\eta_\varepsilon^2(r)dr \lesssim \int_{\varepsilon^2}^{1/2} \frac{|\log r|^{2\alpha}}{r} dr \lesssim |\log\varepsilon|^{2\alpha+1}$$

and

$$B_\varepsilon \gtrsim \int_\varepsilon^{1/4} \frac{|\log r|^{2\alpha}}{r} dr \gtrsim |\log\varepsilon|^{2\alpha+1},$$

and then (4.5).

Estimate (4.4) is more delicate.

Let us recall that for any $u \in \mathcal{H}$,

$$\|u\|_H^2 = \int_\Omega |x|^{-(N-2)} \left|\nabla\left(\frac{u}{u_2}\right)\right|^2 dx = \int_\Omega |\nabla u|^2 dx - \int_\Omega \Lambda_N \frac{u^2}{|x|^2} dx.$$

For u_ε it holds

$$A_\varepsilon=\int_\Omega\Big|\nabla u_\varepsilon-u_\varepsilon\frac{\nabla u_2}{u_2}\Big|^2dx=\int_\Omega|u_2\nabla\theta_\varepsilon|^2dx,$$

where

$$\theta_\varepsilon(r)=\Big(\log\frac{1}{r}\Big)^\alpha\eta_\varepsilon(r)\psi(r). \tag{4.8}$$

Note that

$$\nabla\theta_\varepsilon=\Big(\log(\frac{1}{|x|})^\alpha\nabla\eta_\varepsilon-\alpha(\log\frac{1}{|x|})^{\alpha-1}\frac{x}{2|x|^2}\eta_\varepsilon\Big)\psi+(\log(\frac{1}{|x|})^\alpha)\nabla\psi,$$

and, according to the properties of ψ and η_ε,

$$\begin{aligned}\int_\Omega|u_2\nabla\theta_\varepsilon|^2dx&\lesssim\int_{|x|<1/2}\Big(u_2^2|\nabla\eta_\varepsilon|^2|\log|x||^{2\alpha}+u_2^2\eta_\varepsilon^2\frac{|\log|x||^{2\alpha-2}}{|x|^2}\Big)dx+\int_{1/4<|x|<1/2}|\log|x||^{2\alpha}\eta_\varepsilon^2dx\\&\lesssim\int_{\varepsilon^2}^{\varepsilon}r|\log r|^{2\alpha}\frac{dr}{r^2|\log\varepsilon|^2}+\alpha^2\int_{\varepsilon^2}^{r_0}\frac{|\log r|^{2\alpha-2}dr}{r}+\int_{1/4}^{1/2}\frac{|\log r|^{2\alpha}dr}{r}\\&\lesssim\begin{cases}|\log\varepsilon|^{2\alpha-1}, & \alpha>1/2,\\ 1+\log|\log\varepsilon|, & \alpha=1/2,\\ 1+|\log\varepsilon|^{2\alpha-1}, & \alpha\in[0,1/2).\end{cases}\end{aligned}$$

This completes the proof of (4.4).

Let us now estimate the $H^2(\Omega)$ norm of u_ε. For simplicity assume that $\psi\equiv1$ since the main contribution comes from the singularity near the origin. As above we have

$$\|u_\varepsilon\|_{L^2}^2=\int_{\varepsilon^2}^1r^{N-1}|\tilde u_2(r)|^2\eta_\varepsilon^2(r)dr\lesssim\int_{\varepsilon^2}^1r|\log r|^{2\alpha}dr\lesssim1.$$

Using the expression of u_ε and the properties of η_ε we have

$$\begin{aligned}|u_\varepsilon'(r)|&\lesssim r^{-N/2}|\log r|^\alpha\eta_\varepsilon(r)+r^{-N/2}|\log r|^{\alpha-1}\eta_\varepsilon(r)+r^{-N/2+1}|\log r|^\alpha\eta_\varepsilon'(r)\\&\lesssim r^{-N/2}|\log r|^\alpha\eta_\varepsilon(r)+r^{-N/2+1}|\log r|^\alpha\eta_\varepsilon'(r)\\&\lesssim r^{-N/2}|\log r|^\alpha+r^{-N/2}\frac{|\log r|^\alpha}{|\log\varepsilon|}\lesssim r^{-N/2}|\log r|^\alpha.\end{aligned}$$

It implies that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2=\int_{\varepsilon^{2-\mu}}^1r^{N-1}|u_\varepsilon'(r)|^2dr\lesssim\int_{\varepsilon^{2-\mu}}^1r^{-1}|\log r|^{2\alpha}dr\lesssim|\log\varepsilon|^{2\alpha+1}.$$

The second order derivatives of u_ε satisfies

$$\begin{aligned}|D^2u^\varepsilon(x)|\sim|u_\varepsilon''(r)|\lesssim\ &\eta_\varepsilon(r)r^{-N/2-1}\Big(|\log r|^\alpha+|\log r|^{\alpha-1}+|\log r|^{\alpha-2}\Big)\\&+\eta_\varepsilon'(r)r^{-N/2}\Big(|\log r|^\alpha+|\log r|^{\alpha-1}\Big)\\&+\eta_\varepsilon''(r)r^{-N/2+1}|\log r|^\alpha\\\lesssim\ &\eta_\varepsilon(r)r^{-N/2-1}|\log r|^\alpha+\eta_\varepsilon'(r)r^{-N/2}|\log r|^\alpha+\eta_\varepsilon''(r)r^{-N/2+1}|\log r|^\alpha.\end{aligned}$$

This implies that

$$\begin{aligned}\int_\Omega |\partial_{ij}u_\varepsilon|^2dx &\lesssim \int_{\varepsilon^{2-\mu}}^1 r^{N-1}\Big(|u_\varepsilon''(r)|^2+\frac{|u_\varepsilon'(r)|^2}{r^2}\Big)dr\\ &\lesssim \int_{\varepsilon^{2-\mu}}^1 \Big(\eta_\varepsilon^2(r)r^{-3}|\log r|^{2\alpha}+|\eta_\varepsilon'(r)|^2r^{-1}|\log r|^{2\alpha}+|\eta_\varepsilon''|^2r|\log r|^{2\alpha}\Big)dr\\ &\lesssim \int_{\varepsilon^{2-\mu}}^1 r^{-3}|\log r|^{2\alpha}dr+\int_{\varepsilon^{2-\mu}}^\varepsilon r^{-1}|\log r|^{2\alpha}\frac{dr}{r^2|\log\varepsilon|^2}+\int_{\varepsilon^{2-\mu}}^\varepsilon r|\log r|^{2\alpha}\frac{dr}{r^4|\log\varepsilon|^2}\\ &\lesssim \frac{|\log\varepsilon|^{2\alpha}}{\varepsilon^{4-2\mu}}+\frac{|\log\varepsilon|^{2\alpha-2}}{\varepsilon^{4-2\mu}}\lesssim\frac{|\log\varepsilon|^{2\alpha}}{\varepsilon^{4-2\mu}}.\end{aligned}$$

The proof of the Lemma is now complete. □

Let us now prove the desired upper bound in Theorem 1.1. For simplicity we treat the case $N=3$. The general case $N\geq 4$ can be dealt with as in [13] by using Lemma 3.2.

We consider

$$\tilde u_2(r)=r^{-N/2+1}\left(\log\frac{1}{r}\right)^\alpha,\alpha>\frac{1}{2},$$

and the corresponding u_ε constructed previously

$$u_\varepsilon(x)=\tilde u_2(|x|)\eta_\varepsilon(|x|)\psi(|x|).$$

Let us denote $\Pi_h u_\varepsilon$ the projection of u_ε on the space V_h. In view of the classical estimate (3.2) it satisfies

$$\|\nabla(\Pi_h u_\varepsilon-u_\varepsilon)\|_{L^2(B)}\lesssim h\|u_\varepsilon\|_{H^2(B)}.$$

Since Π_h is the projection on V_h it follows that

$$\int_B|\nabla(\Pi_h u_\varepsilon)|^2dx=\int_B|\nabla u_\varepsilon|^2dx-\int_B|\nabla(u_\varepsilon-\Pi_h u_\varepsilon)|^2dx.$$

Also

$$\begin{aligned}\int_B\frac{|\Pi_h u_\varepsilon|^2}{|x|^2}dx&\geq\int_B\frac{u_\varepsilon^2}{|x|^2}dx-\frac14\int_B\frac{|u_\varepsilon-\Pi_h u_\varepsilon|^2}{|x|^2}dx\\ &\geq\int_B\frac{u_\varepsilon^2}{|x|^2}dx-\frac{1}{4\Lambda_N}\int_B|\nabla(u_\varepsilon-\Pi_h u_\varepsilon)|^2dx.\end{aligned}$$

Let us denote

$$R_{\varepsilon,h}=\frac{\int_B|\nabla(u_\varepsilon-\Pi_h u_\varepsilon)|^2dx}{\int_B u_\varepsilon^2|x|^{-2}dx},\quad Q_\varepsilon=\frac{\int_B|\nabla u_\varepsilon|^2dx}{\int_B u_\varepsilon^2|x|^{-2}dx}.$$

We know from Lemma 4.1 that

$$Q_\varepsilon=\frac{A_\varepsilon}{B_\varepsilon}=\Lambda_N+O(|\log\varepsilon|^{-2})$$

and

$$R_{\varepsilon,h}\lesssim\frac{h^2\|u_\varepsilon\|^2_{H^2(B)}}{|\log\varepsilon|^{2\alpha+1}}\lesssim\frac{h^2}{|\log\varepsilon|^{2\alpha+1}}\frac{|\log\varepsilon|^{2\alpha}}{\varepsilon^{4-2\mu}}=\frac{h^2}{\varepsilon^{4-2\mu}|\log\varepsilon|}.$$

Under the assumption that $R_{\varepsilon,h}$ is small enough we get

$$\begin{aligned}
S_{1h} &= \min_{v\in V_h} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2|x|^{-2}dx} \le \frac{\int_B |\nabla(\Pi_h u_\varepsilon)|^2 dx}{\int_B |\Pi_h u_\varepsilon|^2|x|^{-2}dx} \le \frac{\int_B |\nabla u_\varepsilon|^2 dx + \int_B |\nabla(u_\varepsilon - \Pi_h u_\varepsilon)|^2 dx}{\int_B u_\varepsilon^2|x|^{-2}dx - \frac{\Lambda_N}{4}\int_B |\nabla(u_\varepsilon - \Pi_h u_\varepsilon)|^2 dx}\\
&= \frac{Q_\varepsilon + R_{\varepsilon,h}}{1-\frac{\Lambda_N}{4}R_{\varepsilon,h}} \le (Q_\varepsilon + R_{\varepsilon,h})(1+2R_{\varepsilon,h}) \le Q_\varepsilon + O(R_{\varepsilon,h})\\
&\le \Lambda_N + O\Big(\frac{1}{|\log\varepsilon|^2}\Big) + O(\frac{h^2}{\varepsilon^{4-2\mu}|\log\varepsilon|}).
\end{aligned}$$

Taking

$$\varepsilon^2 \simeq h \tag{4.9}$$

we have that

$$R_{\varepsilon h} = O(\frac{h^\mu}{|\log h|}), \quad \mu\in(0,1), \tag{4.10}$$

and we get the desired estimate

$$\Lambda_h \le \Lambda + O(\frac{1}{|\log h|^2}).$$

5. Proof of Theorem 1.2

Classical arguments on the finite element approximation of eigenvalues show that

$$\mu_1(\Omega) - \mu_{1h} \sim \inf_{v_h\in V_h} \|\phi_1 - v_h\|^2_{\mathcal{H}} \tag{5.1}$$

where ϕ_1 is the corresponding first eigenfunction associated to $\mu_1(\Omega)$. In the case where Ω is the unit ball we have

$$\phi_1(x) = \phi_1(|x|) = |x|^{-(N-2)/2} J_0(z_{0,1}|x|). \tag{5.2}$$

When Ω is a smooth domain containing the origin, one can show, using a cut-off argument, that ϕ_1 exhibits the same type of singularity near the origin as in the case of the unit ball.

It remains to quantify the above distance in the particular case of the unit ball.

Let us first prove the upper bound. A similar computations as in Lemma 4.1 with $\alpha = 0$ gives us

$$\begin{aligned}
\inf_{v_h\in V_h} \|\phi_1 - v_h\|^2_{\mathcal{H}} &\lesssim \|\phi_1 - \phi_{1,\varepsilon}\|^2_{\mathcal{H}} + \|\phi_{1,\varepsilon} - I^h\phi_{1,\varepsilon}\|^2_{\mathcal{H}}\\
&\lesssim \frac{1}{|\log\varepsilon|} + \|\nabla(\phi_{1,\varepsilon} - I^h\phi_{1,\varepsilon})\|^2_{L^2(\Omega)}\\
&\lesssim \frac{1}{|\log\varepsilon|} + \frac{h^2}{\varepsilon^{4-2\mu}}.
\end{aligned}$$

Taking $\varepsilon^2 \simeq h$ we obtain the desired upper bound.

Let us now prove that for any $v_h \in V_h$

$$\|\phi_1 - v_h\|^2_{\mathcal{H}} \gtrsim \frac{1}{|\log h|}.$$

Using the Hardy inequality with logarithmic correction in Lemma 8.1 we obtain

$$\|\phi_1 - v_h\|^2_{\mathcal{H}} \gtrsim \int_\Omega \frac{|\nabla\phi_1 - \nabla v_h|^2 dx}{|\log(1/|x|)|^2} = \sum_{T\in\mathcal{T}_h}\int_T \frac{|\nabla\phi_1 - \nabla v_h|^2 dx}{|\log(1/|x|)|^2}.$$

It is interesting to notice that for all the triangles situated outside the ball of radius h the contribution if of order $1/|\log h|^2$. The main contribution comes from the triangle T_0 containing the origin or the finitely many triangles for which the origin lies on their boundaries.

To simplify the presentation, let us consider the case in which the origin is the in center of T_0, and a ball of radius $\rho_{T_0} \simeq h$ centered at $x = 0$ is contained in T_0. Otherwise we can perform a similar computation choosing a conical subset.

In this case, this main contribution is given by

$$I_h = \int_{|x|\leq h} \frac{|\nabla\phi_1 - \nabla v_h|^2 dx}{|\log(1/|x|)|^2} \geq \inf_{A\in\mathbb{R}^N} \int_{|x|\leq h} \frac{|\nabla\phi_1 - A|^2 dx}{|\log(1/|x|)|^2}.$$

Since

$$\nabla\phi_1(x) = \phi_1'(|x|)\frac{x}{|x|}$$

we obtain that

$$|\nabla\phi_1 - A| \geq ||\phi_1'(r)| - |A||$$

and we have to prove that for any $A \geq 0$

$$f(A) = A^2 \int_0^h \frac{r^{N-1}dr}{|\log(1/|x|)|^2} - 2A\int_0^h \frac{r^{N-1}|\phi'(r)|dr}{|\log(1/|x|)|^2} + \int_0^h \frac{r^{N-1}|\phi'(r)|^2 dr}{|\log(1/|x|)|^2} \gtrsim \frac{1}{|\log h|},$$

where $\phi_1(r) = r^{-(N-1)/2}J_0(z_{01}r)$.

Denoting $f(A) = A^2 a_h - 2Ab_h + c_h$ we know that the minimum is taken for $A = b_h/a_h$ and

$$\min_{A\in\mathbb{R}} f(A) = f(\frac{b_h}{a_h}) = c_h - \frac{b_h^2}{a_h}.$$

We use that for any the following expansion as $h \to 0$, valid for $\alpha \geq -1$:

$$\int_0^h \frac{r^\alpha}{|\log r|^2} \simeq \begin{cases} \frac{h^{\alpha+1}}{|\log h|^2}, & \alpha > -1, \\ \frac{1}{|\log h|}, & \alpha = -1. \end{cases}$$

Since $\phi_1'(r) \simeq r^{-N/2}$ as $r \to 0$ we obtain that

$$c_h \simeq |\log h|^{-1}, \quad a_h \simeq r^{N/2}|\log h|^{-2}, \quad b_h \simeq r^{N/2}|\log h|^{-2}.$$

Then

$$\min_{A\in\mathbb{R}} f(A) \simeq \frac{1}{|\log h|}.$$

6. Proof of Theorem 1.3

We consider the case where the domain Ω is the unit ball since the singularity analysis near the origin is the same for all domains containing $x = 0$.

In the present case, the norm introduced by the bilinear form is equivalent to the $H_0^1(B)$-one. Thus, it is sufficient to provide sharp upper bounds on

$$d(\phi_1, V_h) \simeq \inf_{v_h\in V_h} \|\nabla\phi_1 - \nabla v_h\|_{H_0^1(B_1)},$$

where ϕ_1 is the first eigenfunction.

Let us set

$$m = \sqrt{\Lambda_N - \Lambda} \in \Big(0, \frac{N-2}{2}\Big].$$

Recall that in the case of the ball the first eigenfunction can be explicitly computed (see [17]), $\phi_1(x) = \phi_1(|x|)$,

$$\phi_1(r) = r^{-N/2+1}J_m(j_{m,1}r) \simeq r^{-N/2+1+m}, r \simeq 0,$$

where J_m is the Bessel function of order m and $j_{m,1}$ is its first zero.

This eigenfunction ϕ_1 belongs to $H^{1+s}(B_1)$ for all $0<s<m$ if $m<(N-2)/2$ and it is $C^\infty(B_1)$ if $m=(N-2)/2$.

Let us prove the upper bound. We take a function v_h that vanishes in the triangle T_0 containing the origin. We split in two parts:

$$\|\nabla\phi_1-\nabla v_h\|^2_{L^2(B_1)}\leq \int_{|x|\lesssim h}|\nabla\phi_1|^2dx+h^2\int_{|x|\gtrsim h}|D^2\phi_1|^2dx:=I_1+I_2.$$

When $m<\frac{N}{2}-1$ we have

$$I_1\leq\int_0^h r^{N-1}r^{2(-N/2+m)}dr\simeq h^{2m}$$

while for $m=\frac{N}{2}-1$ we have $I_1\simeq h^N$ since ϕ_1 is C^∞ in this case.

For the second term we first consider the case $N\geq 5$. When $m=\frac{N-2}{2}$ clearly we have $I_2\lesssim h^2$. For $0<m<\frac{N-2}{2}$ a similar argument as for I_1 yields

$$I_2\lesssim h^2\int_h^1 r^{N-1}r^{2(-N/2-1+m)}dr=h^2\int_h^1 r^{2m-3}dr\lesssim\begin{cases}h^{2m}, & 0<m<1,\\ h^2|\log h|, & m=1,\\ h^2, & m\in(1,\frac{N-2}{2}).\end{cases}$$

Putting together the result for I_1 and I_2 leads to the desired result.

When $N=4$ we obtain, in a similar way, that $I_2\leq h^2$ for $m=\frac{N-2}{2}=1$ and

$$I_2\leq h^2\int_h^1 r^{2m-3}dr\simeq h^{2m},\ 0<m<1.$$

In dimension $N=3$ we get $I_2\leq h^2$ when $m=\frac{1}{2}=\frac{N-2}{2}$ and

$$I_2\leq h^2\int_{|x|\gtrsim h}|D^2\phi_1|^2dx\leq h^2\int_h^1 r^{2m-3}dr\simeq h^{2m},\ 0<m<1/2.$$

To obtain the lower bound in (1.10) we divide the integral in two parts, the first one as in the proof of Theorem 1.2 and the second one integrating outside the ball of radius h:

$$\|\nabla\phi_1-\nabla v_h\|^2_{L^2(B_1)}=\int_{|x|<h}|\nabla\phi_1-\nabla v_h|^2dx+\int_{h<|x|<1}|\nabla\phi_1-\nabla v_h|^2dx=I_1+I_2.$$

Let us consider the case $N\geq 5$ since the others are similar. For I_1 the same arguments as in the proof of Theorem 1.2 give us that $I_1\gtrsim h^{2m}$ for $0<m<\frac{N-2}{2}$ or $I_1\gtrsim h^{2N}$ if $m=\frac{N-2}{2}$. Using Lemma 3.2 as in [13] we get

$$I_2\gtrsim\begin{cases}h^{2m}, & 0<m<1,\\ h^2|\log h|, & m=1,\\ h^2, & m>1.\end{cases}$$

The proof is now complete.

7. Conclusions and Open Problems

In this work, we have analyzed the finite element approximation of the best Hardy constant in bounded domains containing the origin for dimensions $N \geq 3$. Despite the absence of minimizers for the Hardy inequality in the standard Sobolev space $H_0^1(\Omega)$, we have rigorously established that the first eigenvalue of the corresponding discrete eigenvalue problem converges to the continuous Hardy constant as the mesh size $h \to 0$.

Our main result provides an explicit convergence rate of order $1/|\log h|^2$, independent of the spatial dimension, and reflects the singular nature of the underlying functional inequality. The analysis demonstrates how the finite element method is capable of capturing the concentration phenomena inherent in the minimization sequences that saturate the Hardy inequality.

Wee have also analysed similar approximation issues for two closely related eigenvalue problems.

Several open problems naturally arise from this study:

- *The 2-dimensional case:* As mentioned in the introduction, the two-dimensional case is critical and requires an inverse-square logarithmic correction to the Hardy inequality, as shown in [1]. This case would therefore require a more detailed analysis.
- *Other Hardy inequalities:* The literature on Hardy inequalities is extensive; we refer the reader to the monograph [14] for a comprehensive overview. The finite element approximation techniques developed in this paper could, in principle, be systematically applied to the many existing variants of the Hardy inequality.
- *Weighted spectral problems:* Using that

$$\mu_1(\Omega) = \min_{u \in \mathcal{H}} \frac{\|u\|_{\mathcal{H}}^2}{\|u\|_{L^2(\Omega)}^2} = \min_{v \in W_0^{1,2}(|x|^{-(N-2)}dx, \Omega)} \frac{\int_\Omega |x|^{-(N-2)} |\nabla v|^2 dx}{\int_\Omega |x|^{-(N-2)} v^2 dx}$$

  one can consider introducing the finite element approximation in this weighted setting

$$\tilde{\mu}_{1h}(\Omega) = \min_{v \in V_h} \frac{\int_\Omega |x|^{-(N-2)} |\nabla v|^2 dx}{\int_\Omega |x|^{-(N-2)} v^2 dx}$$

  and analyze the approximation rates.
- *Higher-Order Finite Elements:* It would be of interest to investigate whether similar convergence rates can be established for higher-order finite element spaces and other finite-element variants, like discontinuous Galerkin approximations.
- *Adaptive Mesh Refinement:* Given the concentration of minimizing sequences near the singularity at the origin, adaptive mesh refinement strategies could improve the numerical approximation. Theoretical analysis of such adaptive methods in this singular context remains an open challenge.
- *Extension to Non-Radial Settings and General Domains:* While our analysis is presented for general bounded domains containing the origin, extensions to more complex geometries or domains with additional singularities could reveal new phenomena. The same can be said in the context of multi-polar Hardy inequalities (see [7] for the analysis in the continuous setting).
- *Other Singular Inequalities:* A natural direction for future work is to study the finite element approximation of best constants in other critical inequalities, such as Rellich or Hardy-Sobolev inequalities, where similar non-attainability issues arise. Similar issues also arise in the fractional setting, see [10].

We hope that this contribution stimulates further research on the numerical analysis of critical inequalities and the development of numerical methods tailored for singular variational problems.

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8. Appendix

Lemma 8.1. *Let be Ω a smooth domain such that $0 \in \Omega \subset \overline{\Omega} \subset B_R$ for some $R > 0$. There exists a positive constant $C = C(N, \Omega)$ such that for all $u \in \mathcal{H}$ we have*

$$\|u\|_{\mathcal{H}}^2 \geq C(N,\Omega) \int_\Omega \frac{|\nabla u|^2 dx}{|\log(R/|x|)|^2}. \tag{8.1}$$

Proof. The inequality holds for $u \in C_c^\infty(\Omega)$ as proved in [14, Th. 2.5.2, p. 25]

$$\|u\|_{\mathcal{H}}^2 = \int_\Omega |\nabla u|^2 dx - \Lambda_N \int_\Omega \frac{u^2}{|x|^2} dx \geq C \int_\Omega |\nabla u|^2 \Big(\log \frac{R}{|x|} \Big)^{-2} dx.$$

By density the inequality extends to functions u in $\mathcal{H}$. □

Lemma 8.2. *For any $T \in \mathcal{T}_h$ the following holds:*

$$\int_T \frac{dx}{|\log|x||^2} \gtrsim \frac{|T|}{|\log h|^2}. \tag{8.2}$$

The proof is elementary and is left to the reader.

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