Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks

Summary In this thesis we analyze stationary and dynamic gas flow with uncertain boundary data in networks of pipelines. The gas flow in pipeline networks is modeled by the isothermal Euler equations. The uncertain boundary data is modeled by probability distributions, they represent the a priori unknown gas demand of the consumers. The aim of this work is the analysis of optimization problems with probabilistic constraints in the context of gas transport.

For computing the probability that an uncertain gas demand is feasible we use both, a kernel density estimator approach and the spheric radial decomposition. Feasible in this context means, that the demanded gas can be transported through the network, s.t. bounds for the pressure at the nodes are satisfied. Moreover we discuss advantages and disadvantages of both methods. In the stationary case we extend our model by compressor control and bounds for the pressure at the entry nodes, and we also compute the probability for an uncertain gas demand to be feasible. In the dynamic setting the uncertain gas demand is time dependent, which is modeled by randomized Fourier series.

Further we analyze certain optimization problems with probabilistic constraints, in which the probabilistic constraints are approximated by the kernel density estimator approach. On the one hand we show the existence of optimal solutions for both, the exact and the approximated problems, and on the other hand we show that if the approximation is sufficiently accurate, then the optimal solutions of the approximated problems are close to the solutions of the exact problems. With the approximation of the probabilistic constraints via the kernel density estimator we are able to compute derivatives of the approximated optimization constraints, which allows us to derive necessary optimality conditions for the approximated optimization problems with probabilistic constraints.

Probabilistic Constrained Optimization on Stationary Gas Networks

Consider a connected, directed, tree-structured graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with vertex set $\mathscr{V} = \{v_0, \cdots, v_n\}$ and a set of edges $\mathscr{E} = \{e_1, \cdots, e_n\}$ with $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. Due to the structure, the graph has a unique root v_0. Every edge $e \in \mathscr{E}$ represents a pipe with (positive) length L^e. For $x \in [0, L^e]$ we consider the stationary semi-linear isothermal Euler equations for horizontal pipes and ideal gases

$$\begin{cases} q_x^e(x) = 0, \\ (c^e)^2 p_x^e(x) = -\dfrac{\lambda^e}{2D^e}(R_S T)^2 \dfrac{q^e(x)|q^e(x)|}{p^e(x)}. \end{cases} \tag{sISOstat}$$

Here, p is the pressure, q is the flow and $c^e, \lambda^e, D^e \in \mathbb{R}_{>0}$ denote the speed of sound in the gas, the pipe friction coefficient and the pipe diameter of edge $e \in \mathscr{E}$ respectively. For simplicity we assume that these parameters are space independent. $R_S, T \in \mathbb{R}_{>0}$ are specific gas constant and (constant) temperature, both come from the ideal gas equation. Further p^e and q^e represent the restriction of the pressure and the flow defined over the network to a single edge $e \in \mathscr{E}$. We assume conservation of mass at every node $v \in \mathscr{V} \setminus \mathscr{V}_0$. That means that the gas inflow at every node must be equal to the sum of gas outflow and load of this node, i.e.

$$\sum_{e \in \mathscr{E}_-(v)} q^e \left(\frac{D^e}{2}\right)^2 \pi \;=\; b^v \;+ \sum_{e \in \mathscr{E}_+(v)} q^e \left(\frac{D^e}{2}\right)^2 \pi \quad \forall v \in \mathscr{V} \setminus \mathscr{V}_0.$$

Let the pressure at the inflow node v_0 be given, that is

$$p^e(0) = p_0 \;\in\; \mathbb{R}_{>0} \quad \forall e \in \mathscr{E}_+(v_0).$$

Further we assume continuity in pressure at every inner node, i.e. for all $v \in \mathscr{V}$ with $\mathscr{E}_-(v) \neq \emptyset$ and $\mathscr{E}_+ \neq \emptyset$ holds

$$p^{e_1}(L^{e_1}) = p^{e_2}(0) \quad \forall e_1 \in \mathscr{E}_-(v),\; e_2 \in \mathscr{E}_+. \tag{1}$$

So for every edge $e \in \mathscr{E}$, the full model is given by

$$\begin{cases} q_x^e(x) &= 0 \\ p_x^e(x) &= -\frac{\phi^e}{2}\frac{q^e(x)|q^e(x)|}{p^e(x)} \\ q^e(L^e) &= b^{h(e)} + \sum_{\kappa \in \mathscr{E}_+(h(e))} q^\kappa(0) \\ p^e(0) &= p_0 \qquad\qquad\qquad\qquad e \in \mathscr{E}_+(v_0) \\ p^e(0) &= p^\kappa(L^\kappa) \qquad \kappa \in \mathscr{E}_-(f(e)), \quad e \in \mathscr{E} \backslash \mathscr{E}_+(v_0) \end{cases} \tag{statModel}$$

Let $p \in \mathbb{R}^n$ be the vector of pressures at the outflow nodes, i.e. p_i is the pressure at node v_i $(i = 1, \cdots, n)$. For lower pressure bounds $p_i^{\min} > 0$ and upper pressure bounds $p_i^{\max} \geq p_i^{\min}$, we demand

$$p_i \in \left[p_i^{\min}, p_i^{\max}\right] \quad i = 1, \cdots, n.$$

We define the set of feasible loads as follows: **Definition.** For pressure bounds $p^{\min}, p^{\max} \in \mathbb{R}^n_{\geq 0}$ with $0 < p_i^{\min} \leq p_i^{\max}$ $(i = 1, \cdots, n)$, the set

$$M := \left\{ b \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{l} \text{The solution } (p,q) \in \mathbb{R}^n \times \mathbb{R}^n \text{ of (statModel)} \\ \text{satisfies the box constraints } p \in \left[p^{\min}, p^{\max}\right]. \end{array} \right\} \tag{2}$$

is called the set of feasible loads.

Since the consumers gas demand cannot be known a priori (it depends on various factors like e.g. temperature and gas price), the gas demand can be seen as random variable. For a mean value $\mu \in \mathbb{R}^n_{\geq 0}$ and a positive definite covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$, we define a Gaussian distributed random variable

$$\xi_b \sim \mathscr{N}(\mu, \Sigma),$$

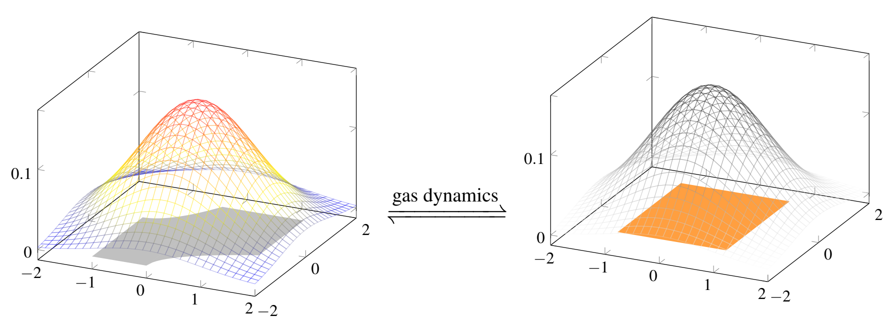
on an appropriate probability space $(\Omega, \mathscr{A}, \mathbb{P})$. We identify the load vector $b \in \mathbb{R}^n$ with the image $\xi_b(\omega)$ for $\omega \in \Omega$ on this probability space.
Our aim is to answer the following question:

***Q_{prob}*:** *For a given inlet pressure, can we guarantee, that every consumer receives their demanded gas, s.t. the gas pressure in the network is neither too high nor too low, in at least α% of all scenarios?*

Since we are in a stationary setting in this chapter, the scenarios are time independent. We want to compute the probability for a random load vector to be feasible, i.e.

$$\mathbb{P}(b \in M). \tag{3}$$

This has also been discussed in [3]. There the authors use both, a kernel density estimator (KDE) approach and the spheric radial decomposition (SRD) to compute the probability (3). On the one hand the set of feasible loads is characterized by using the SRD to compute the desired probability, on the other hand a KDE approach is used to estimate the probability density function of the pressures to compute the desired probability (see *Figure 1*).



(a) Well-known distribution (colored), unknown set of feasible loads (grey)

(b) Unknown distribution (grey), well-known set of feasible pressures (orange)

Figure 1: The two different ways to compute the probability for a random load vector to be feasible

A model extension of (statModel) was discussed in [2]. The focus in the thesis was more on optimization problems with probabilistic constraints. Let convex functions $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be given. For a probability level $\alpha \in (0,1)$ consider the optimization problems

$$\begin{aligned} \min_{p^{\max}\in\mathbb{R}^n} \quad & f_1(p^{\max}) \\ \text{s.t.} \quad & \mathbb{P}\left(b \in M(p^{\max})\right) \geq \alpha, \\ & p^{\max} \geq p^{\min}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \min_{p_0\in\mathbb{R}} \quad & f_2(p_0) \\ \text{s.t.} \quad & \mathbb{P}\left(b \in M(p_0)\right) \geq \alpha, \\ & p_0 \geq 0, \end{aligned} \tag{5}$$

where M is defined in (2). Usually α is chosen large, i.e. close to 1. Consider also the probabilistic constrained optimization problems in which the probability is approximated by the KDE approach, i.e.,

$$\begin{aligned} \min_{p^{\max}\in\mathbb{R}^n} \quad & f_1(p^{\max}) \\ \text{s.t.} \quad & \mathbb{P}_{\text{KDE}}\left(b \in M(p^{\max})\right) \geq \alpha, \\ & p^{\max} \geq p^{\min}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \min_{p_0\in\mathbb{R}} \quad & f_2(p_0) \\ \text{s.t.} \quad & \mathbb{P}_{\text{KDE}}\left(b \in M(p_0)\right) \geq \alpha, \\ & p_0 \geq 0, \end{aligned} \tag{7}$$

where M is defined in (2). Both problems only differ from (4) and (5) in the approximated probabilistic constraint. We state the main results of the thesis. The following lemmata state the existence of optimal solutions:

Lemma. Let a probability level $\alpha \in (0,1)$, an inlet pressure $p_0 \in \mathbb{R}_{\geq 0}$ and a lower pressure bound $p^{\min} \in \mathbb{R}^n_{\geq 0}$ be given. Assume that there exists $x \geq p^{\min}$ (component-by-component), s.t.

$$\int_{p^{\min}}^{x} \rho_p(z)\, dz > \alpha. \tag{8}$$

Let f_1 be strictly monotone increasing in the sense that for all positive directions $d \in \mathbb{R}^n_{\geq 0}$ (with $d > 0$ in at least one component) it follows

$$f_1(p^{\max}) < f_1(p^{\max} + \varepsilon d),$$

for all $\varepsilon > 0$. Then there exists a solution of (4).
Further let $p^{*,\max} \in \mathbb{R}^n$ be a solution of (4). Then the probabilistic constraint is always active, i.e. it is

$$\mathbb{P}(b \in M(p^{\max})) = \alpha.$$

Lemma. Let pressure bounds $p^{\min}, p^{\max} \in \mathbb{R}^n$ with $p^{\max} > p^{\min}$ be given. Assume that there exists $x \geq 0$, s.t.

$$\int_{P^{\max}_{\min}} \rho_{p(x)}(z)\, dz \geq \alpha. \tag{9}$$

Then (5) has at least one solution. If f_2 is strictly monotone (increasing or decreasing), the solution is unique and the probabilistic constraint is always active.

Corrolary. Let a probability level $\alpha \in (0,1)$, an inlet pressure $p_0 \in \mathbb{R}_{\geq 0}$ and a lower pressure bound $p^{\min} \in \mathbb{R}^n_{\geq 0}$ be given. Assume that there exists $x \geq p^{\min}$ (component-by-component), s.t.

$$\int_{p^{\min}}^{x} \rho_{p,N_{\text{KDE}}}(z)\, dz > \alpha. \tag{10}$$

Let f_1 be strictly monotone increasing in the sense that for all positive directions $d \in \mathbb{R}^n_{\geq 0}$ (with $d > 0$ in at least one component) it follows

$$f_1(p^{\max}) < f_1(p^{\max} + \varepsilon d),$$

for all $\varepsilon > 0$. Then there exists a solution of (6).
Further let $p^{*,\max} \in \mathbb{R}^n$ be a solution of (6). Then the probabilistic constraint is always active, i.e. it is

$$\mathbb{P}_{\text{KDE}}(b \in M(p^{\max})) = \alpha.$$

Corollary. Let pressure bounds $p^{\min}, p^{\max} \in \mathbb{R}^n$ with $p^{\max} > p^{\min}$ be given. Assume that there exists $x \geq 0$, s.t.

$$\int_{P^{\max}_{\min}} \rho_{p(x),N_{\text{KDE}}}(z)\, dz \geq \alpha. \tag{11}$$

Then (7) has at least one solution. If f_2 is strictly monotone (increasing or decreasing), the solution is unique and the probabilistic constraint is always active.

The next theorems are one of the key results of the thesis. They state that under slight assumptions the optimal solution of the approximated problems is close to the optimal solution of the original problems if the sampling size is sufficiently large. For $\alpha \in (0,1)$ we define $g_1^\alpha, g_{1,N_{\text{KDE}}}^\alpha : \mathbb{R}^n \to \mathbb{R}$ with

$$g_1^\alpha : p^{\max} \mapsto \alpha - \mathbb{P}(b \in M(p^{\max})),$$
$$g_{1,N_{\text{KDE}}}^\alpha : p^{\max} \mapsto \alpha - \mathbb{P}_{N_{\text{KDE}}}(b \in M(p^{\max})),$$

and $g_2^\alpha, g_{2,N_{\text{KDE}}}^\alpha : \mathbb{R} \to \mathbb{R}$ with

$$g_2^\alpha : p_0 \mapsto \alpha - \mathbb{P}(b \in M(p_0)),$$
$$g_{2,N_{\text{KDE}}}^\alpha : p_0 \mapsto \alpha - \mathbb{P}_{N_{\text{KDE}}}(b \in M(p_0)).$$

Further we define

$$\mathscr{X} := \{x \in \mathbb{R}^n \mid x \geq p^{\min} \text{ and } g_1^\alpha(x) = 0\ \},$$
$$\mathscr{X}_{N_{\text{KDE}}} := \{x \in \mathbb{R}^n \mid x \geq p^{\min} \text{ and } g_{1,N_{\text{KDE}}}^\alpha(x) = 0\ \},$$

as set of all points, for which the probabilistic constraint in (4) and the approximated probabilistic constraint in (6) are active, i.e. $\mathscr{X}$ contains the roots of g_1^α and $\mathscr{X}_{\text{KDE}}$ contains the roots of $g_{1,N_{\text{KDE}}}^\alpha$.

Theorem. Let a probability level $\alpha \in (0,1)$, an inlet pressure $p_0 \in \mathbb{R}_{\geq 0}$ and a lower pressure bound $p^{\min} \in \mathbb{R}^n_{\geq 0}$ be given. Assume that there exist $x \geq p^{\min}$ (component-by-component), s.t. (8) and (10) hold.
Let f_1 be strictly monotone increasing in the sense of *Lemma* and *Corollary* .
Assume that for all $x \in \mathscr{X}$ there exist $d_1, d_2 \in \mathbb{R}^n \backslash \{\nvdash_n\}$, s.t. for all $\tau \in (0,1)$ it holds

$$g_1^\alpha(x + \tau d_1) < 0 \quad \text{and} \quad g_1^\alpha(x + \tau d_2) > 0. \tag{12}$$

Let $p^{*,\max}$ be a solution of (4). Assume that the solution is unique. Further assume that there exist $\delta, \varepsilon > 0$, s.t. for $p \in \mathscr{X}$ with

$$\|p^{*,\max} - p\| > \frac{\delta}{2},$$

it holds

$$|f(p^{*,\max}) - f(p)| > \varepsilon. \tag{13}$$

Then there exist a sufficiently large number N_{KDE}, s.t. the solution $p_{N_{\text{KDE}}}^{*,\max}$ of (6) is close to $p^{*,\max}$ in the sense that we have

$$\|p^{*,\max} - p_{N_{\text{KDE}}}^{*,\max}\| < \delta \quad \mathbb{P}-\text{almost surely}. \tag{14}$$

Theorem. Consider $\delta > 0$ and let pressure bounds $p^{\min}, p^{\max} \in \mathbb{R}^n$ with $p^{\max} > p^{\min}$ be given. Assume that there exists $x \geq 0$, s.t. (9) and (11) hold. Assume that f_2 is strictly monotone (increasing or decreasing). Let p_0^* be the solution of (5) and let $p_{0,N_{\text{KDE}}}^*$ be the solution of (7). If for all $x \geq 0$ with $g_2^\alpha(x) = 0$ there exists $\varepsilon > 0$, s.t.

$$\text{sgn}(g_2^\alpha(x - \tau)) = -\text{sgn}(g_2^\alpha(x + \tau)) \quad \forall \tau \in (0, \varepsilon),$$

where sgn is the sign function, then for N_{KDE} sufficiently large it follows

$$p^*_{0,N_{\text{KDE}}} \in (p^*_0 - \delta, p^*_0 + \delta).$$

Last we state necessary optimality conditions for the approximated probabilistic constrained optimization problems.

Corollary. Let $p^{*,\max} \in \mathbb{R}^n$ be a (local) optimal solution of (6). Since the LICQ holds in $p^{*,\max}$ (cf. *Remark* **??**), there exists a multiplier $\mu^* \geq 0$, s.t.

$$\begin{aligned} \nabla f_1(p^{*,\max}) + \mu^* \nabla g_1^\alpha(p^{*,\max}) &= 0, \\ g_1^\alpha(p^{*,\max}) &\leq 0, \\ \mu^* g_1^\alpha(p^{*,\max}) &= 0. \end{aligned}$$

Thus $(p^{*,\max}, \mu^*) \in \mathbb{R}^n \times \mathbb{R}$ is a Karush-Kuhn-Tucker point.

Corollary. Let $p^*_0 \in \mathbb{R}$ be a (local) optimal solution of (7). Since the LICQ holds in p^*_0 (cf. *Remark* **??**), there exists a multiplier $\mu^* \geq 0$, s.t.

$$\begin{aligned} f'(p^*_0) + \mu^* (g_2^\alpha)'(p^*_0) &= 0, \\ g_2^\alpha(p^*_0) &\leq 0, \\ \mu^* g_2^\alpha(p^*_0) &= 0. \end{aligned}$$

Thus $(p^*_0, \mu^*) \in \mathbb{R} \times \mathbb{R}$ is a Karush-Kuhn-Tucker point.

The computation of the gradients, the proofs of the theorems and lemmata and interpretations of the assumptions and the results can be found in [1]. The necessary optimality conditions of the Corollaries above can be used to characterize the solution of the approximated probabilistic constrained optimization problems as it is shown in *Figure 2*.

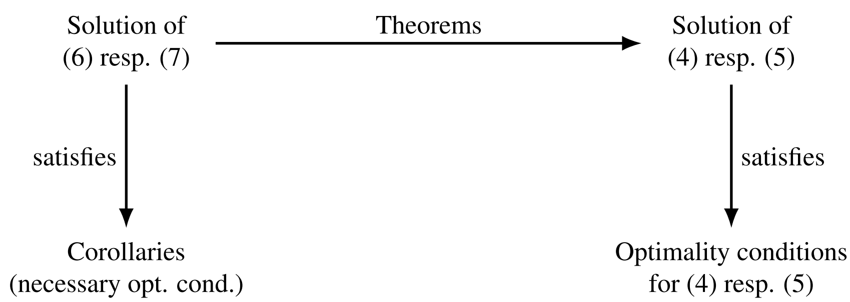


Figure 2: Scheme on the relation of the necessary optimality conditions

Probabilistic Constrained Optimization on Dynamic Gas Networks

Time dependent probabilistic constraints: For a time dependent uncertain boundary function $b(t)$ and a (time dependent) feasible set $M(t)$, the formulation of time dependent probabilistic constraints that we will use later is

$$\mathbb{P}(\ b \in M(t)\ \forall t \in [0,T]\ )\ \geq \alpha. \tag{15}$$

This is a strong condition. It means, we want to guarantee, that a percentage α of all possible random boundary functions (in an appropriate probability space $(\Omega, \mathscr{A}, \mathbb{P})$) is feasible in every point in time $t \in [0,T]$. So we do not allow any violation of the feasibility for most of the random boundary functions. In the context of gas transport this is a reasonable condition since the pressure bounds are fix and do not allow violations. In fact the formulation (15) is a so-called probust constraint, i.e. a constraint that contains both, a probabilistic and a robust part.

Uncertain time dependent functions: Let a function $f \in L^2([0,T])$ with $f(0) = 0$ be given. For $m = 0, 1, 2, \cdots$, we define the orthonormal series

$$\psi_m(t) := \frac{\sqrt{2}}{\sqrt{T}} \sin\left(\left(\frac{\pi}{2} + m\pi\right)\frac{t}{T}\right), \tag{16}$$

and the coefficients

$$\mathfrak{a}_m^0(f) := \int_0^T f(t)\psi_m(t)\,dt. \tag{17}$$

Then the Fourier series representation of $f(t)$ is given by

$$f(t) = \sum_{m=0}^{\infty} \mathfrak{a}_m^0(f)\psi_m(t). \tag{18}$$

Since it is $\psi_m(0) = 0$ and thus,

$$\sum_{m=0}^{\infty} \mathfrak{a}_m^0(f)\psi_m(0) = 0,$$

we also assume $f(0) = 0$. For the implementation we truncate the Fourier series after $N_F \in \mathbb{N}$ terms, i.e.

$$f_{N_F}(t) := \sum_{m=0}^{N_F} \mathfrak{a}_m^0 \psi_m(t). \tag{19}$$

Consider a sequence of identically distributed random variables $(\xi_m)_{m\geq 0}$ on an appropriate probability space $(\Omega, \mathscr{A}, \mathbb{P})$. For $\omega \in \Omega$ consider the random function

$$f^{\omega}(t) = \sum_{m=0}^{\infty} \xi_m(\omega)\mathfrak{a}_m^0 \psi_m(t).$$

Then according to an extension of the Paley-Zygmund Theorem it follows that if $f \in L^2([0,T])$, then f^{ω} converges almost surely in L^2.

Mathematical modelling: Consider a connected, directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the vertex set $\mathscr{V} = \{v_0, \cdots, v_n\}$ and a set of edges $\mathscr{E} = \{e_1, \cdots, e_m\} \subseteq \mathscr{V} \times \mathscr{V}$ $(n, m \in \mathbb{N})$. Every edge $e \in \mathscr{E}$ represents a pipe with (positive) length L^e. For $(t,x) \in [0,T] \times [0, L^e]$ we consider the isothermal Euler equations for horizontal pipes and ideal gases

$$\begin{cases} \rho_t^e + q_x^e = 0, \\ q_t^e + \left((c^e)^2 \rho^e + \dfrac{(q^e)^2}{\rho^e} \right)_x = -\dfrac{\lambda_F^e}{2D^e} \dfrac{q^e|q^e|}{\rho^e} \end{cases} \tag{qISO}$$

with density ρ^e and flow q^e. The constants $c^e, \lambda_F^e, D^e \in \mathbb{R}_{\geq 0}$ denote the speed of sound in the gas, the pipe friction coefficient and the pipe diameter of the edge $e \in \mathscr{E}$ respectively. For simplicity we assume that these parameters are constant on every edge. Further ρ^e and q^e represent the restriction of the pressure and the flow (defined over the network) to a single edge $e \in \mathscr{E}$.

We state that all nodes $v \in \mathscr{V}$ with $|\mathscr{E}_0(v)| > 1$ are inner nodes and that all nodes $v \in \mathscr{V}$ with $|\mathscr{E}_0(v)| = 1$ are terminal nodes. We assume w.l.o.g. that gas inflow and gas outflow only occurs at terminal nodes. Next we define coupling conditions for all inner nodes of the graph. Therefor for $v \in \mathscr{V}$ and $e \in \mathscr{E}_0(v)$ we define

$$x^e(v) := \begin{cases} 0 & \text{if } e \in \mathscr{E}_+(v), \\ L^e & \text{if } e \in \mathscr{E}_-(v), \end{cases}$$

as the end of edge e, that corresponds to node v. Then we assume conservation of mass for all inner nodes, i.e. for all nodes $v \in \mathscr{V}$ with $|\mathscr{E}_0(v)| > 1$ we have

$$\sum_{e \in \mathscr{E}_0(v)} (D^e)^2 q^e(t, x^e(v)) = 0 \quad \forall t \in [0,T]. \tag{20}$$

Further we assume continuity in density for all inner nodes, i.e. for all nodes $v \in \mathscr{V}$ with $|\mathscr{E}_0(v)| > 1$ we have

$$\rho^e(t, x^e(v)) = \rho^f(t, x^f(v)) \quad \forall e, f \in \mathscr{E}_0(v). \tag{21}$$

Since we assume ideal gas, the continuity in density conforms with the continuity in pressure. Next we define boundary conditions for terminal nodes, i.e. for all nodes $v \in \mathscr{V}$ with $|\mathscr{E}_0(v)| = 1$ we define

$$\rho^e(t,0) := \rho_0(t) \quad \text{if } e \in \mathscr{E}_+(v), \tag{22}$$

and

$$q^e(t,L^e) := b(t) \quad \text{if } e \in \mathscr{E}_-(v). \tag{23}$$

As in the stationary case the function $\rho(\cdot)$ states the (time dependent) inlet density and $b(\cdot)$ states the (time dependent) gas outflow.

Last we define initial conditions for the density and for the flow, i.e. for all $e \in \mathscr{E}$ we define

$$\rho^e(0,x) := \rho^e_{\text{ini}}(x), \tag{24}$$

and

$$q^e(0,x) = q^e_{\text{ini}}(x). \tag{25}$$

So the full model is given by the isothermal Euler equations (qISO), coupling conditions (20) and (21), boundary conditions (22) and (23) and initial conditions (24) and (25):

$$\begin{cases} \text{For all } e \in \mathscr{E} \text{ we have} \\ \quad \rho^e_t + q^e_x = 0, \\ \quad q^e_t + \left((c^e)^2\rho^e + \frac{(q^e)^2}{\rho^e}\right)_x = -\frac{\lambda^e_F}{2D^e}\frac{q^e|q^e|}{\rho^e}, \\ \quad \rho^e(0,x) = \rho^e_{\text{ini}}(x), \\ \quad q^e(0,x) = q^e_{\text{ini}}(x), \\ \text{for all } v \in \mathscr{V} \text{ with } |\mathscr{E}_0(v)| > 1 \text{ we have} \\ \quad \sum_{e\in\mathscr{E}_0(v)} (D^e)^2 q^e(t,x^e(v)) = 0, \\ \quad \rho^e(t,x^e(v)) = \rho^f(t,x^f(v)) \qquad e,f \in \mathscr{E}_0(v), \\ \text{for all } v \in \mathscr{V} \text{ with } |\mathscr{E}_0(v)| = 1 \text{ we have} \\ \quad \rho^e(t,0) = \rho^e_0(t) \qquad e \in \mathscr{E}_+(v), \\ \quad q^e(t,L^e) = b^e(t) \qquad e \in \mathscr{E}_-(v). \end{cases} \tag{dynModel}$$

Let $q^v(t)$ be the flow of the gas at node $v \in \mathscr{V}$ and let $\rho^v(t)$ be the density of the gas at node $v \in \mathscr{V}$. Let bounds for the density $\rho^{\min}, \rho^{\max} \in \mathbb{R}^{|\mathscr{V}_{\text{out}}|}$ with $0 < \rho^{\min} \leq \rho^{\max}$ (component-by-component) be given. For $t^* \in [0,T]$, we demand

$$p^v(t^*) \in \left[\rho^{v,\min}, \rho^{v,\max}\right] \quad \forall v \in \mathscr{V}_{\text{out}},$$

where $\rho^{v,\min}$ and $\rho^{v,\max}$ are the bounds for the density at node $v \in \mathscr{V}_{\text{out}}$. We define the set of feasible loads for the dynamic setting as follows:

Definition. For density bounds $\rho^{\min}, \rho^{\max} \in \mathbb{R}^{|\mathscr{V}_{\text{out}}|}$ with $0 < \rho^{v,\min} \leq \rho^{v,\max}$ ($v \in \mathscr{V}_{\text{out}}$) and for a point in time $t^* \in [0,T]$, the set

$$M(t^*) := \left\{ \begin{array}{c} b : [0,T] \to \mathbb{R}^{|\mathscr{V}_{\text{out}}|} \\ b_i \in \text{Lip}([0,T]) \end{array} \middle| \begin{array}{l} \text{The solution } \rho^v(t), q^v(t) \text{ of (dynModel)} \\ \text{satisfies the box constraints} \\ \rho^v(t^*) \in \left[\rho^{v,\min}, \rho^{v,\max}\right] \text{ at time } t^* \end{array} \right\} \tag{26}$$

is called the set of feasible loads.

Since we are in a dynamic setting in this chapter, the scenarios depend on the time. We want to compute the probability that a random boundary function is feasible, i.e.

$$\mathbb{P}\big(\, b \in M(t) \; \forall t \in [0,T] \,\big). \tag{27}$$

As in the stationary setting, the load $b(t)$ is considered to be random but we have no information about the structure of the set of feasible loads $M(t)$. The densities $\rho^v(t)$ at the nodes $v \in \mathscr{V}_{\text{out}}$ are also random but the set of feasible densities

$$\mathscr{P}^{\max}_{\min} := \otimes_{v\in\mathscr{V}_{\text{out}}} \left[\rho^{v,\min}, \rho^{v,\max}\right], \tag{28}$$

is a well-known cuboid. For $t \in [0,T]$ let $\rho_{\text{out}}(t)$ be the densities at the nodes $v \in \mathscr{V}_{\text{out}}$. Then we have

$$\mathbb{P}\big(\ b \in M(t)\ \forall t \in [0,T]\ \big) = \mathbb{P}\big(\ \rho_{\text{out}}(t) \in \mathscr{P}_{\min}^{\max}\ \forall t \in [0,T]\ \big). \tag{29}$$

For $i = 1, \cdots, N_{\text{KDE}}$ we define

$$\underline{\rho}_{\text{out}}(b^{\omega,i}) := \min_{t \in [0,T]} \rho_{\text{out}}(t, b^{\omega,i}),$$

and

$$\overline{\rho}_{\text{out}}(b^{\omega,i}) := \max_{t \in [0,T]} \rho_{\text{out}}(t, b^{\omega,i}),$$

as minimal and maximal gas densities of ρ_{out} on $[0,T]$. The minimal and the maximal gas densities have to be understood component-by-component and they exist due to the continuity of ρ^v and the compactness of $[0,T]$. For $i = 1, \cdots, N_{\text{KDE}}$ it follows

$$\begin{gathered} \rho_{\text{out}}(t, b^{\omega,i}) \in \mathscr{P}_{\min}^{\max}\ \forall t \in [0,T] \\ \Updownarrow \\ \underline{\rho}_{\text{out}}(b^{\omega,i}), \overline{\rho}_{\text{out}}(b^{\omega,i}) \in \mathscr{P}_{\min}^{\max}. \end{gathered} \tag{30}$$

We can now approximate the probability density function of the minimal and maximal gas densities, which leads to a $(2n_{\text{out}})$-dimensional (time independent) probability density function $\rho_{\rho,\text{KDE}}$. Since there is no time dependency anymore we can use the theory from the stationary case. Then also the results from the probabilistic constrained optimization problems can be applied to the dynamic case. More information , details and a detailed numerical example on the dynamic case can be found in the thesis [1].

References

[1] M. Schuster: *Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks*. Dissertation (2021), `https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/410`

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