Fixed Point-Based Optimization Techniques for Image Processing and Restoration

Presentation

By

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Overview

1. Fixed Point Problems and Recursive Schemes
2. Substantial Operators in Fixed Point Theory
3. Substantial Concepts
4. Variational Inequality Problem

Optimization Through Fixed Point Recursive Schemes

Find x (a solution of a problem)

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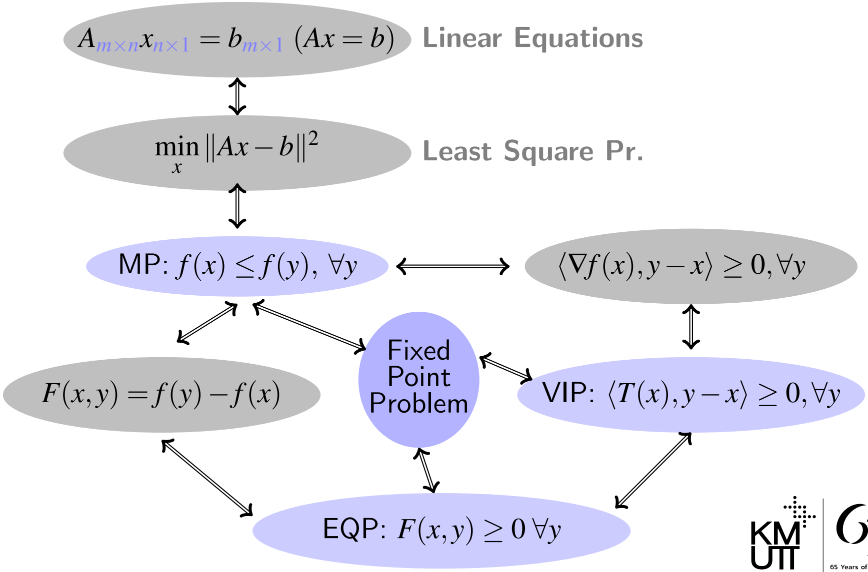

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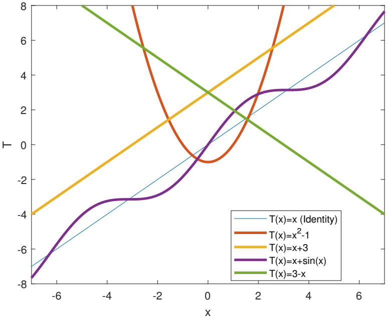

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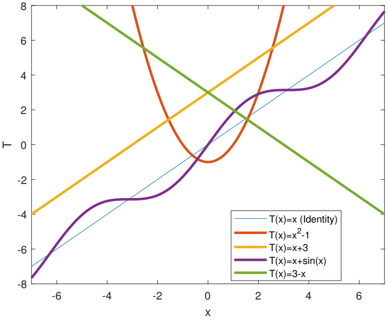

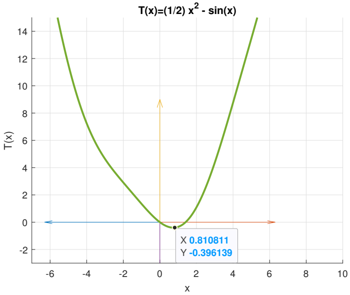

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Substantial Operators in Fixed Point Theory

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Developing Algorithms - Substantial Concepts

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- **Convergence rate:** How fast does it converge?
- **Robustness:** performance under various initial selections and on distinct problems
- **Efficiency:** Ability to find solutions across various problems using few evaluations (total no. of eval to reach solution)
- **Effectiveness:** How many problems can it solve?
- **Accuracy:** Minimal error possible.

The Case of
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- Applications: Machine learning, robotic motion control, signal and image restorations, dynamical systems,...

Fixed Point Techniques for VIP

[1] A. A. Golden, Bullet. Amer. Math. Soc. 70 (1964)
[2] E. S. Levitin, B. T. Polyak, USSR Comput. Math Phys. **6**(1966)
[3] G. M. Korpelevich, Metecon, **12**(1976)
[4] Y. Censor, A. Gibali and S. Reich, Optim. Methods Softw. 26 (2011)

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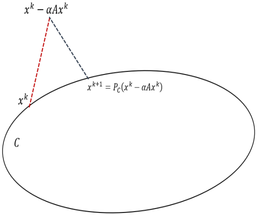

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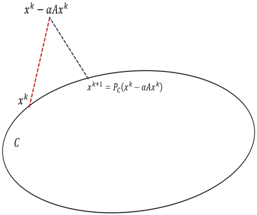

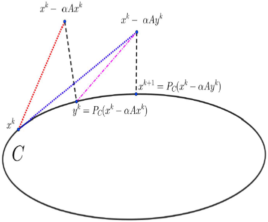

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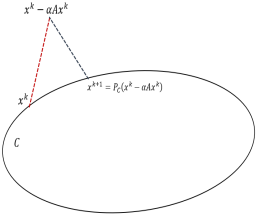

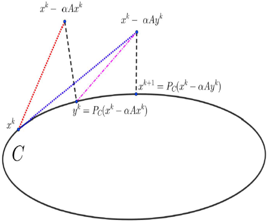

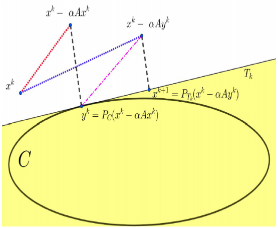

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$$\begin{cases} x^1 \in C, \\ y^k = P_C(I - \alpha A)\left(x^k\right) \\ d\left(x^k, y^k\right) \equiv \left(x^k - y^k\right) - \lambda\left(Ax^k - Ay^k\right) \\ x^{k+1} = x^k - \gamma\beta^k d\left(x^k, y^k\right) \end{cases}$$

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Golden Ratio[7]

$$\begin{cases} x^1 \in C, \quad \phi^2 = 1 + \phi \\ y^k = \dfrac{(\phi - 1)x^k + y^{k-1}}{\phi} \\ x^{k+1} = \operatorname{prox}_{\lambda g}(y^k - \lambda A x^k) \end{cases}$$

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Approximation Schemes for VIP

Projection Contraction Method[5]

$$\begin{cases} x^1 \in C, \\ y^k = P_C(I-\alpha A)\left(x^k\right) \\ d\left(x^k, y^k\right) \equiv \left(x^k - y^k\right) - \lambda\left(Ax^k - Ay^k\right) \\ x^{k+1} = x^k - \gamma\beta^k d\left(x^k, y^k\right) \end{cases}$$

Tseng Method[6]

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- General concerns about iterative techniques: Fastness in the number of iterations, CPU-time, Computational complexity, and expansiveness.
- Metric projection P_C is meaningful when C is closed and convex. However, the computation of the metric projection is tedious in many closed convex sets. So there need to be closed and convex sets with explicit forms of computing metric projection or less computation of it in approximation techniques. E.g., Projection onto balls, half-spaces, hyperplanes, boxes, and so on are quite welcome.

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Image Restoration Problem

Image Recovery Problems

Figure: Blurred image

Image Recovery Problems

Original Image

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$$z = \begin{bmatrix} z_{1,1} & \cdots & z_{1,n_2} \\ \vdots & \ddots & \vdots \\ z_{n_1,1} & \cdots & z_{n_1,n_2} \end{bmatrix}_{n_1 \times n_2}$$

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Problem: How can we recover the original image z?

Image Recovery: Operator Perspective

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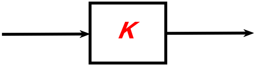

Blurring operator

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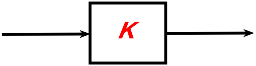

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Image Recovery: Operator Perspective

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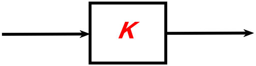

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  Various operators can be employed, for example:

$$K_1 u = z, \quad K_2 u = z, \quad \ldots, \quad K_? u = z$$

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Some Sources of Image Noise

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where $\|\cdot\|_2$ denotes the Euclidean norm, $\alpha > 0$ is a regularization parameter, and $\mathscr{R}(u)$ is the regularization function.

Regularizations and Image Restoration
Via
Optimization Algorithms

Image Restoration Via Algorithms

Solve an optimization problem to restore noisy images while preserving edges.

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- Note that the TV norm is equivalent to the vector 1-norm of the gradient:

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Mathematical Approach

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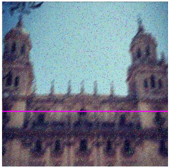
Alternating Direction Method (ALTV-l_1-l_2)

$$
\begin{cases}
u^{k+1} =_u \Big\{ \dfrac{\alpha_1}{2}\|Ku - z\|_2^2 - \langle \lambda_1^k, g^k - (Ku - z)\rangle \\
\qquad\qquad + \dfrac{\gamma_1}{2}\|g^k - (Ku - z)\|_2^2 - \langle \lambda_2^k, v^k - \nabla u\rangle \\
\qquad\qquad + \dfrac{\gamma_2}{2}\|v^k - \nabla u\|_2^2 + \dfrac{\alpha_3}{2}\|\nabla u\|_2^2 \Big\}, \\
v^{k+1} =_v \Big\{ \|v\|_1 - \langle \lambda_2^k, v - \nabla u^{k+1}\rangle + \dfrac{\gamma_2}{2}\|v - \nabla u^{k+1}\|_2^2 \Big\}, \\
g^{k+1} =_g \Big\{ \dfrac{\alpha_2}{2}\|g\|_1 - \langle \lambda_1^k, g - (Ku^{k+1} - z)\rangle \\
\qquad\qquad + \dfrac{\gamma_1}{2}\|g - (Ku^{k+1} - z)\|_2^2 \Big\}, \\
\lambda_1^{k+1} = \lambda_1^k - \gamma_1\Big(g^{k+1} - (Ku^{k+1} - z)\Big), \\
\lambda_2^{k+1} = \lambda_2^k - \gamma_2\Big(v^{k+1} - \nabla u^{k+1}\Big).
\end{cases}
$$

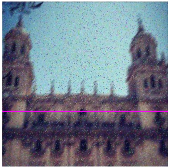
Degraded and Recovered Images

Degraded and Recovered Images



(a) degraded

Degraded and Recovered Images


(a) degraded


(b) Chan et al.

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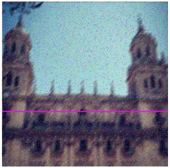
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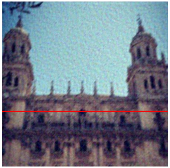
(c) Cai et al.

Degraded and Recovered Images

(a) degraded (b) Chan et al. (c) Cai et al. (d) ALTV-l_1-l_2

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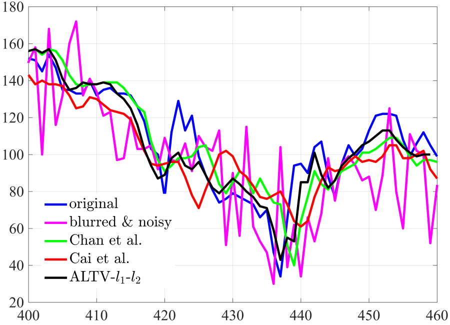
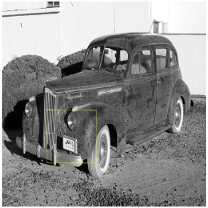


Image Restoration

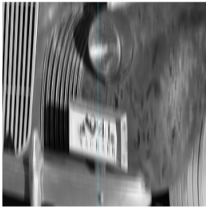
The original, observed and restored images are given in Figure 4.1-4.2.



(a) original image

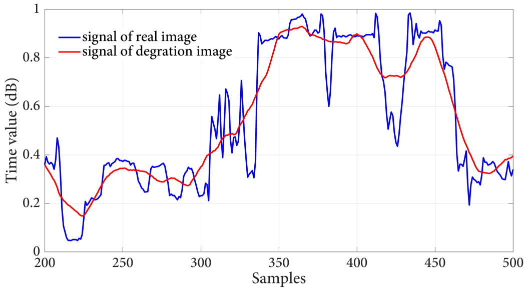
(b) zoom image

(c) zoom image and focus (:,401)



(d) degradation image and focus (:,401)

Image Restoration



(a) signal image and focus (:,401)

Figure 4.1: Figure (a) shows the original image "car", figure (b) shows the zoom original image size 604×856 figure (c) shows the zoom original image focus at point $(:, 401)$ and figure (d) shows the zoom degradation image by a motion blur and random noise focus at point $(:, 401)$ and figure (e) show signal of zoom original and degradation image.

Lasso Models for Image Restoration

Denoising Problem

Minimize the functional:

$$\min_{u} \frac{1}{2}\|Ku-\xi\|_2^2+\lambda\|Wu\|_1$$

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$$\min_{u} \frac{1}{2}\|Ku-\xi\|_2^2+\lambda\|Wu\|_1$$

Terminologies

- u : Restored image (vectorized form)
- ξ : Observed noisy image
- K : Degradation operator (e.g., blur, downsampling)
- W : Transform operator (e.g., wavelet, DCT)
- λ : Regularization parameter

Lasso Models for Image Restoration

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- Data fidelity term: $\frac{1}{2}\|Ku - \xi\|_2^2$
- Sparsity-promoting term: $\lambda\|Wu\|_1$

A derivative-free Scheme for Nonlinear Equations

Ibrahim, A.H., Kumam, P., Abubakar, A.B. and Abubakar, J.,A derivative-free projection method for nonlinear equations with non-Lipschitz operator: application to LASSO problem, Math. Methods Appl. Sci. **46** (2023), no. 8, 9006–9027; MR4589854.

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- The method is applied to the LASSO problem in image recovery, demonstrating its practical utility.
- Numerical experiments illustrate competitive performance and favorable comparisons with related algorithms.

- Let $u_0 \in \mathbb{R}^n$, and parameters $\zeta \in (0,1)$, $\mathrm{Tol} \in (0,1)$, $b \in (0,1)$, $r \in (0,1)$, $\rho \in (0,2)$.
- Compute the search direction using

$$p_k = -\mathscr{B}(u_k) + \frac{\left\|\mathscr{B}(u_k)\right\|^2}{\max\left\{p_{k-1}^T y_{k-1},\, t\left\|p_{k-1}\right\| \left\|\mathscr{B}(u_k)\right\|\right\}} p_{k-1}, \quad k \geq 1,$$

where $p_0 = -\mathscr{B}(u_0), \quad t > 0, \quad y_{k-1} = \mathscr{B}(u_k) - \mathscr{B}(u_{k-1})$.
- Choose $t_k = b\, r^i$, where i is the least nonnegative integer satisfying

$$-B\left(u_k + b\, r^i p_k\right)^T p_k \geq \zeta\, b\, r^i \|p_k\|^2.$$

- If $v_k := u_k + t_k p_k \in \mathscr{C}$ and $\|B(v_k)\| \leq \mathrm{Tol}$, **stop**. Otherwise, compute the next iterate by

$$u_{k+1} = P_{\mathscr{C}}\left(u_k - \rho\left(\frac{-B(v_k)^T(t_k p_k)}{\|B(v_k)\|^2}\right) B(u_k + t_k p_k)\right)$$

- Set $k \leftarrow k+1$ and reiterate.

Relaxed Forward-backward Proximal for Variational

Kratuloek, K., Kumam, P., Sriwongsa, S. and Abubarkar, J., A relaxed splitting method for solving variational inclusion and fixed point problems, Comput. Appl. Math. **43** (2024), no. 1, Paper No. 70, 18 pp.; MR4698053

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- Efficient performance in image recovery and superior performance compared with two existing algorithms.

Hybrid BFGS-CG for Monotone Equations

Abubakar, A.B., Kumam, P., Mohammad, H., Ibrahim, A.H., Seangwattana, T. and Hassan, B.A., A hybrid BFGS-like method for monotone operator equations with applications, J. Comput. Appl. Math. **446** (2024), Paper No. 115857, 23 pp.; MR4718321

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Edge-Preserving Variational Denoising Model

Denoising Problem

Minimize the functional:

$$\min_{u=\{u_{ij}\}} \mathscr{U}(u) = \min_{u} \left(\sum_{(i,j)\in\Omega} \phi_0(u_{ij} - \xi_{ij}) + \frac{1}{2} \sum_{(i,j)\in\Omega} \sum_{(k,l)\in N_{(i,j)}} \phi_\alpha(u_{ij} - u_{kl}) \right)$$

Note that the functional is differentiable and convex

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Note that the functional is differentiable and convex

Terminologies

$$\phi_0(v) = \sqrt{v^2 + a}, \quad \phi_\alpha(v) = \sqrt{v^2 + \alpha}$$

- $\xi = \{\xi_{ij}\}$: Observed noisy image over domain Ω
- $N_{(i,j)}$: Neighborhood of pixel (i,j)
- a, α: Constants controlling the degree of smoothness and edge preservation.

Efficient Spectral RMIL Conjugate Gradient

Salihu, N., Kumam, P., Awwal, A.M., Sulaiman, I.M. and Seangwattana, T., The global convergence of spectral RMIL conjugate gradient method for unconstrained optimization with applications to robotic model and image recovery. Plos one **18** (2023), p.e0281250.

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- Applicable to image restoration.
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Newton-like Conjugate Gradient for Optimization

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- Proven applicability to image reconstruction models.

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