Time Asymptotics For Heat and Dissipative Wave Equations

Enrique Zuazua *

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*(1) Chair in Applied Analysis, Alexander von Humboldt-Professorship, Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg 91058 Erlangen, Germany. (2) Chair of Computational Mathematics, Fundación Deusto, Av. de las Universidades, 24, 48007 Bilbao, Basque Country, Spain. (3) Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.

1 Introduction

1.1 Examples of Partial Differential Equations

Elliptic equations:

* *Laplace equation*

$$-\Delta_x u = f$$

$$\left[\Delta_x = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i}\right] \qquad N = 1, 2, 3,$$

* *Stokes equation*

$$\begin{cases} -\Delta_x \vec{u} & = & \vec{f} + \vec{\nabla}_x p \\ \text{div}\vec{u} & = & 0 \end{cases}$$

* *Plate equation*

$$\Delta_x^2 u = f$$

* *The Lamé system* is 3-d elasticity

$$\mu \Delta_x \vec{u} - (\lambda + \mu) \vec{\nabla}_x \text{div}_x \vec{u} = \vec{f}$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Elliptic equations describe the stationary (time independent) solutions of evolution equations (hyperbolic and parabolic ones).

Parabolic Equations:

* The *heat equation*:

$$u_t - \Delta_x u = 0\,.$$

* The *Stokes equations*:

$$\begin{cases} \vec{u}_t - \nu \Delta \vec{u} = \vec{\Delta} p \\ \text{div}\, \vec{u} = 0 \end{cases}$$

* *Navier-Stokes equations*:

$$\begin{cases} \vec{u}_t - \nu \Delta \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \vec{\nabla} p \\ \text{div}\, \vec{u} = 0 \end{cases}$$

* *Convection-diffusion equations*:

$$u_t - \Delta u + \text{div}(\vec{f}(u)) = 0$$

Hyperbolic equations:

* The *wave equation*:
$$u_{tt} - \Delta u = 0$$
(arises in the context of acoustic waves, vibrations of strings, membranes).

* The *system of elasticity*:
$$\vec{u}_{tt} - \mu\Delta\vec{u} - (\lambda + \mu)\vec{\nabla}\mathrm{div}\vec{u} = 0\,.$$
(relevant in vitrations of 3-d elastic bodies).

* *Maxwell equations*
$$\begin{cases} \vec{E}_t &= \mathrm{curl}\,\vec{M} \\ \vec{M}_t &= -\mathrm{curl}\,\vec{E} \end{cases}$$

* *Hyperbolic conservation laws*:
$$u_t + \mathrm{div}(\vec{f}(u)) = 0.$$

Other closely related and significant models are also worth mentioning.

* *Schrödinger equation*:
$$iu_t + \Delta_u = 0,$$
(which arises in Quantum Mechanics and Optics).

* *Airy* and *KdV equations*:
$$u_t + u_{xxx} = 0$$
$$u_t + u_{xxx} + u_x + uu_x = 0$$
relevant in the theory of water waves and solitons.

* *Hamilton-Jacobi equations*:
$$u_t + H(\nabla_x u) = 0.$$

1.2 Basic principle and goal of these lectures

Understanding completely the dynamics of evolution processes is too difficult. However, for many relevant systems an asymptotic simplification process occurs and the large time behavior is often given by simpler equations allowing a quite explicit and complete analysis.

The most typical example is that of an evolution equation in which solutions converge, as time tends to infinity, to the solutions of the corresponding elliptic (time-independent) equation.

In these lectures we present some important examples in which this asymptotic simplication occurs and describe some mathematical techniques allowing to explore this issue.

1.3 Some analytical tools

• Fourier & Harmonic Analysis.

• Functional Analysis (spaces, inequalities).

• Semiproup Theory.

• Interpolation.

• Distributions.

. . . . . .

2 The Heat Equation

2.1 The Dirichlet Problem

Given a bounded domain Ω we consider the heat equation with Dirichlet boundary conditions:

$$\begin{cases} u_t - \Delta u = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x, 0) = \varphi(x) & \Omega. \end{cases}$$

Here $u = u(x, t)$ where $x = (x_1, \dots, x_n) \in \mathbb{R}$, $t > 0$ and $\Delta = \Delta_x = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ and $\cdot_t = \frac{\partial \cdot}{\partial t}$.

Fourier Analysis allows obtaining the explicit form of solutions.

Consider first the eigenvalue problem

$$\begin{cases} -\Delta \phi_j = \lambda_j \phi_j & \text{in } \Omega \\ \phi_j = 0 & \text{on } \partial\Omega, j = 1, \dots, \infty \end{cases}$$

The spectral theory for compact self-adjoint operators in Hilbert spaces allows showing that the problem admits an increasing sequence of positive eigenvalues, of finite multilplicity, tending to infinity

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \ldots \to \infty$$

so that the corresponding eigenfunctions $\{\phi_j\}$ constitute an orthonormal basis of $L^2(\Omega)$.

Solutions of the heat equation can now be easily developed in Fourier series in this eigenfunctions basis:

$$u(x,t) = \sum_{k=1}^{\infty} \varphi_k \, e^{-\lambda_k t} \, \phi_k(x)$$

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k \, \phi_k(x)$$

$$\varphi_k = \int_{\Omega} \varphi(x) \, \phi_k(x) dx.$$

According to the orthogonality property of eigenfunctions the following holds for the time evolution of the L^2-norm of solutions:

$$||u(t)||^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \varphi_k^2 \, e^{-2\lambda_k t}.$$

This identity illustrates the well-posedness of the system in the forward sense of time and provides an easy bound for the semigroup map $S(t)$ that associates to any initial datum φ the solution $u(t)$ at time t:

$$||S(t)||_{\mathcal{L}(L^2(\Omega))} = e^{-\lambda_1 t}.$$

By the contrary, the same identity illustrates the strong *time-irreversibility* of the system:

$$||u(0)||^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \varphi_k^2 = \sum_{k=1}^{\infty} u_k^2(t) e^{2\lambda_k t}.$$

Time irreversibility and the *smoothing property* of the heat semigroup are strongly linked. The following holds:

$$\varphi = u(0) \in L^2(\Omega) \Rightarrow u(t) \in H^s(\Omega), \ \forall \, s > 0.$$

Indeed,

$$||u(t)||^2_{H^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2s} \, u_k^2(t) = \sum_{k=1}^{\infty} \lambda_k^{2s} \, e^{-2\lambda_k t} \, \varphi_k^2 \le C_s \sum_{k=1}^{\infty} \varphi_k^2 < \infty.$$

Consequently, the heat equation posesses the following main properties:

- Smoothing
- Time Irreversibility
- Dissipation of energy.

The energy dissipation law can also be obtained easily by the *energy method* that consists, roughly, on multiplying the equation by suitable functions of the unknown and integrating by parts. Indeed, integrating with respect to the space variable x in the identity

$$(u_t - \Delta u)u = 0$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2\, dx = 0.$$

In other words,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 = -\int_\Omega |\nabla u|^2\, dx$$

and, using the Poincaré inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \leq -\lambda_1 \int_\Omega u^2$$

and consequently,

$$||u(t)||_{L^2(\Omega)} \leq e^{-\lambda_1 t}||\varphi||_{L^2(\Omega)}.$$

According to this computation solutions of the heat equation on a bounded domain Ω decay exponentially in time with a rate $\lambda_1(\Omega)$. This is in agreement with the prediction we did by using the Fourier expansion of solutions. It is however important to note that the energy method is more flexible and does not require an orthonormal basis of eigenfucntions to be applied.

The computation above is based on the following characterization of the best constant in the Poincaré inequality:

$$\lambda_1(\Omega) = \min_{\varphi \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2\, dx}{\int_\Omega \varphi^2\, dx}$$

which does indeed yield

$$\int_\Omega |\nabla\varphi|^2 \geq \lambda_1 \int_\Omega \varphi^2.$$

In order to better understand the asymptotic behavior of solutions as a function of the domain Ω where the equation holds it is important to take into account that

$$\lambda_1(\Omega) \text{ decreases as } \Omega \text{ increases.}$$

More precisely, by scaling the domain Ω by means of a constant factor $R > 0$

$$\Omega \longrightarrow R\Omega$$

we see that

$$\lambda_1(\Omega) \;\to\; \lambda_1(R\Omega) = \frac{1}{R^2}\lambda_1(\Omega).$$

In particular, the computations above do not yield any decay rate for the *Cauchy problem* since, as $R \to \infty$, $\lambda_1(R\Omega) \to 0$. Thus, when $\Omega = \mathbb{R}^n$, which is a mathematical idealization of a very large domain in which the boundary effects do not have a significant influence on the solution, the computation above does not provide any decay rate..

The following question arises: *Do solutions decay at all when $\Omega = \mathbb{R}^n$?*

2.2 The Cauchy Problem

Consider now the Cauchy problem in the whole space:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = \varphi(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The solution of the Cauchy problem can be explicitly written by *convolution with the gaussian heat kernel*:

$$\boxed{u = G(\cdot,t) \underset{x}{*} \varphi(\cdot)}.$$

More explicitly,

$$u(x,t) = \int_{\mathbb{R}^N} G(x-y,t)\,\varphi(y)\,dy$$

where

$$G(x,t) = (4\pi t)^2 \exp\left(\frac{-|x|^2}{4t}\right)$$

is the fundamental solution of the heat equation, the gaussian kernel.

It satisfies the following properties:

- $G_t - \Delta G = 0, \quad$ in $\mathbb{R}^N \times (0,\infty)$
- $G(t) \to \delta_0 =$ Dirac delta at $x = 0$, as $t \to 0^+$;
- Conservation of mass:
$$\int G(x,t)dx = 1, \quad \forall t > 0.$$
- L^p-decay
$$||G(\cdot,t)||_{L^p(\mathbb{R}^N)} = C_p t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}.$$
  In particular
$$||G(t)||_{L^1} = \int G\,dx = 1, \quad ||G(t)||_{L^\infty} \le ct^{-\frac{N}{2}}.$$

- L^p-decay of derivatives.

$$||D_x^\alpha G(\cdot,t)||_{L^p(\mathbb{R}^N)} = C_{p\alpha} t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{|\alpha|}{2}}$$

- We also observe that that the fundamental solution has the *self-similar form*:

$$G = t^{-\frac{N}{2}} f\left(\frac{x}{\sqrt{t}}\right), \quad f(z) = (4\pi)^{-\frac{N}{2}} \exp\left(-\frac{z^2}{4}\right).$$

The fundamental solution of the heat equation can be computed explicitly in several forms. One of them is by direct application of the basic properties of the *Fourier transform*. The fundamental solution solves the following system in the physical space:

$$\begin{cases} u_t - \Delta u = 0 \\ u(0) = \delta_0. \end{cases}$$

Applying Fourier transform in space we get:

$$\begin{cases} \widehat{u}_t + |\xi|^2 \widehat{u} = 0 \\ \widehat{u}(0) \equiv 1. \end{cases}$$

The solution in the Fourier variables can then be obtained explicitly:

$$\widehat{u}(\xi,t) = e^{-|\xi|^2 t}.$$

Finally, applying the inverse Fourier transform we easily obtain that

$$u(x,t) = G(x,t) = (4\pi t)^{-N/2} \exp\left(\frac{-|x|^2}{4t}\right).$$

But, the Fourier analysis provides more information than the explicit form of the solution. Indeed, it indicates that the heat equation dissipates exponentially each Fourier component of the solution with a rate $|\xi|^2$. Consequently,

→ The dissipation rate is very high for the high frequencies.

→ It is very slow for the low ones.

Of course, there is no unifom exponential decay rate. However, one can get a compromise between the strong decay of high frequencies and the weak decay rate of the low ones by introducing extra assumptions on the integrability of the initial datum.

It is important to note that, if no additional assumption on the initial datum is imposed one can not obtain any decay rate. More precisely, the norm of the semigroup $S(t)$

associated with the Cauchy problem as a bounded linear operator form $L^2(\mathbb{R}^N)$ into itself is one for all $t > 0$.

This fact corresponds to a general result in the context of dissipative semigroups in Banach spaces

$$S(t) = e^{At} : X \to X$$

guaranteeing that either:

$$||S(t)||_{\mathcal{L}(X,X)} = 1, \quad \forall t > 0$$

or

$$||S(t)||_{\mathcal{L}(H,H)} \leq C\, e^{-\omega t} \qquad \forall t > 0$$

for suitable constants $C, \omega > 0$.

In particular, if we have a polynomial decay rate of the form

$$||S(t)||_{\mathcal{L}(X,X)} \leq c\, t^{-\sigma},$$

then, necesarilly, the semigroup decays exponentially as well.

In the particular case of the semigroup associated with the Cauchy problem for the heat equation, the semigroup is of unit norm. This fact is closely related to the fact that the gaussian heat kernel $G(t)$ is of unit norm in $L^1(\mathbb{R}^N)$ for all $t > 0$ as well. Indeed, according to Youngs's inequality we have

$$||S(t)\varphi||_{L^2(\mathbb{R}^N)} = ||G(t) * \varphi||_{L^2(\mathbb{R}^N)} \leq ||G(t)||_{L^1(\mathbb{R}^N)} ||\varphi||_{L^2(\mathbb{R}^N)} = ||\varphi||_{L^2(\mathbb{R}^N)}$$

and this bound, which does not provide any decay rate, is sharp.

We have discussed the main properties of the gaussian heat kernel. But, in fact, all solutions of the heat equation have similar properties and they can be obtained by the energy method, i.e. multiplying the equation by functions of the unknown and integrating by parts:

- *Conservation of mass.*
  Integrating the heat equation with respect to x we get:

$$0 = \int_{\mathbb{R}^N} u_t\, dx - \int_{\mathbb{R}^N} \Delta_x\, u\, dx = \frac{d}{dt} \int u\, dx.$$

  Consequently,

$$\int_{\mathbb{R}^N} u(x,t)\, dx = \int_{\mathbb{R}^N} \varphi(x)\, dx, \qquad \text{for all} \quad t > 0.$$

- *Energy dissipation law.*
  Multiplying by u and integrating w.r.t. x we get:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} u^2\, dx = - \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

- L^p-decay.

  Applying Young's inequality in the representation formula of the solution as convolution of the heat kernel with the initial datum we obtain:

$$||u(t)||_{L^p(\mathbb{R}^N)} = ||G(t) * \varphi||_{L^p(\mathbb{R}^N)} \leq ||G((t)||_{L^p(\mathbb{R}^N)}||\varphi||_{L^1(\mathbb{R}^N)}$$
$$\leq C_p||\varphi||_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}.$$

  This is true for all $p \geq 1$. The maximal decay rate is achieved for $p = \infty$, the decay rate being $N/2$. When $p = 1$ the L^1 norm does not decay. This is in agreement with the property of conservation of mass.

  Note however that the L^p decay property is guaranteed by assuming that the initial datum is in $L^1(\mathbb{R}^N)$. The semigroup does not decay as a bounded linear operator from $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$.

- *Comparison of solutions*: Using the positivity of the heat kernel G it can also be easily seen that $\varphi \geq \widehat{\varphi}$ implies that the associated solutions are ordered as well: $u \geq \widehat{u}$.

  This property may also be obtained by mutiplying the equation by $\text{sgn}^-(u - \widehat{u})$. Indeed,

$$0 = \int_{\mathbb{R}^N} (u_t - \widehat{u}_t)\text{sgn}^-(u - \widehat{u}) - \underbrace{\int_{\mathbb{R}^N} \Delta(u - \widehat{u})\, \text{sgn}^-(u - \widehat{u})\, dx}$$

$$\geq 0$$
$$\geq \frac{d}{dt}\int_{\mathbb{R}^N} \left|[u - \widehat{u}]^-\right| dx.$$

  Here sgn^- stands for the function taking value -1 for $s \leq 0$ and value 0 for $s \geq 0$. On the other hand, $[s]^-$ is the negative part function taking value s for $s \leq 0$ and 0 for $s \geq 0$.

  Consequently, taking into account that $\varphi \geq \widehat{\varphi}$ we have

$$\int_{\mathbb{R}^N} |[\varphi - \widehat{\varphi}]^-| dx = 0$$

  and we deduce immediatley that

$$\int_{\mathbb{R}^N} |[u - \widehat{u}]^-|\, dx = 0$$

  for all $t > 0$ which is equivalent to the fact that $u \geq \widehat{u}$.

In order to justify this computation and to avoid the technical difficulties related with the lack of smoothness of the sgn^- function one has to use a regular approximation β_ε of the sgn^- function, such that β_ε is of class C^1, vanishes at the origin, and is non-decreasing. Using the Green formula one then gets

$$-\int_{\mathbb{R}^N} \Delta w\, \beta_\varepsilon(w)dx \;=\; \int_{\mathbb{R}^N} \nabla w \cdot \beta_\varepsilon'(w)\nabla w = \int_{\mathbb{R}^N} \beta_\varepsilon'(w)|\nabla w|^2 \geq 0.$$

In this way we conclude the decreasing character of the integral

$$\int_{\mathbb{R}^N} B_\varepsilon(u - \widehat{u})dx$$

where $B_\epsilon(s) = \int_0^s \beta_\varepsilon(z)dz$. Passing to the limit as ε tends to zero we deduce that the integral $\int_{\mathbb{R}^N} |[u - \widehat{u}]^-|\, dx$ decreases as well.

We have seen that most of the properties of solutions of the heat equation can be obtained in two different ways: Using the explicit expression of solutions by convolution with the keat kernel or by integration by parts. The proofs based on integration by parts are much more robust than those that use the explicit representation formula of solutions.

The following is an interesting alternative proof of the L^p-decay property. This method applies to nonlinear parabolic equations and is also useful to obtain L^p estimates for solutions of semilinear elliptic equations. As far as we know, the application of this technique in the context of parabolic equations is due to L. Véron [20].

- Step 1: Multiply the equation by $|u|^{p-2}u$. We get

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} (p-1)\ |\nabla u|^2\ |u|^{p-2}dx = 0 \\ \Rightarrow\ &\frac{d}{dt}\int_{\mathbb{R}^N} |u|^p dx \quad \leqslant \quad 0 \\ \Rightarrow\ &\|u(t)\|_{L^p(\mathbb{R}^N)} \quad \leqslant \quad \|u_0\|_{L^p(\mathbb{R}^N)}, \quad \forall t > 0. \end{aligned}$$

- Step 2: Use of Sobolev's inequality. For instance, in three space dimensions ($N = 3$) the Sobolev inequality reads.

$$\left(\int_{\mathbb{R}^N} |u|^6 \right)^{1/6} \leqslant\ c \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2}.$$

On the other hand, from the identities of the first step we have

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + (p-1)\int |\nabla u|^2\ |u|^{p-2}dx = 0,$$

and, taking into account that

$$\int |\nabla u|^2 \; |u|^{p-2} dx = \frac{4}{p^2} \int |\nabla [(u)^{p/2}]|^2 dx$$

we get

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{4(p-1)}{p^2}\int |\nabla (u)^{p/2}|^2 dx = 0.$$

The application of the Sobolev inequality in this case yields:

$$\int |\nabla (u)^{p/2}|^2 dx \geqslant c \left(\int |u|^{3p} \right)^{1/3}$$

and therefore

$$\frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{c(p-1)}{p^2} \left(\int |u|^{3p} \right)^{1/3} \leqslant 0.$$

- Step 3: We now use classical interpolation inequalities (which are in fact a consequence of Hölder's inequality):

$$\|u\|_{L^p} \;\leqslant\; \|u\|_{L^1}^{2p/[3p-1]} \; \|u\|_{L^{3p}}^{(p-1)/[3p-1]}.$$

Then, using the fact that the L^1-norm decreases,

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int |u|^p dx + \frac{c(p-1)}{p^2}\left[\frac{\|u\|_{L^p}}{\|u\|_{L^1}^{2p/(3p-1)}} \right]^{p(3p-1)/3(p-1)} &\leqslant \; 0 \\ \Rightarrow \; \frac{d}{dt}\int |u|^p dx + \frac{c(p-1)}{p^2 \|\varphi\|_{L^1}^{2p^2/(p-1)}} \left(\int |u|^p dx \right)^{(3p-1)/3(p-1)} &\leqslant \; 0. \end{aligned}$$

Consequently, the function

$$\phi(t) = \int_{\mathbb{R}^N} |u|^p dx$$

satisfies

$$\frac{d\phi}{dt} + C_p(\|\varphi\|_{L^1}) \phi^{(3p-1)/3(p-1)} \;\leq\; 0.$$

Solving this differential ineguality we get:

$$\phi(t) \;\leq\; C_p t^{-\frac{3}{2}(p-1)} \Rightarrow \;\; \|u(t)\|_{L^p} \;\leq\; C_p \|\varphi\|_{L^1} t^{-\frac{3}{2}(1-\frac{1}{p})}.$$

Note that this corresponds, when $N = 3$, to the estimate we got using Young's ineguality in the convolution identity.

This method of proving L^p estimates applies to more general nonlinear evolution equations like, for instance,

$$u_t - \Delta u + \operatorname{div}(\vec{f}(u)) = 0\,.$$

Indeed,

$$\int_{\mathbb{R}^N} \operatorname{div}(\vec{f}(u))|u|^{p-2}u dx = -\int_{\mathbb{R}^N} (p-1)\vec{f}(u)|u|^{p-2}\nabla u = -\int_{\mathbb{R}^N} \operatorname{div}(\vec{F}_p(u))dx = 0,$$

where

$$\vec{F}_p(z) = \int_0^z \vec{f}(s)|s|^{p-2}(p-1)ds$$

and therefore all the contributions coming from the nonlinear term cancel in this computation, leading to the same result as for the linear heat equation.

2.3 Development of functions in the Dirac basis. Large time behavior

Acoording to the estimates above, when the initial datum φ of the Cauchy problem belongs to $L^1(\mathbb{R}^N)$, $t^{\frac{N}{2}(1-\frac{1}{p})}u(t)$ is a bounded trajectory in $L^p(\mathbb{R}^N)$ is $t \to \infty$.

The following question then arises naturally: *Does the limit*

$$\lim_{t\to\infty} t^{\frac{N}{2}(1-\frac{1}{p})}u(t)$$

exist in $L^p(\mathbb{R}^N)$*, and, if yes, can we compute it explicitly.*

There are, at least, three methods to answer to these questions:

- Developing functions in the basis of delta functions.
- Self-similar variables.
- Scaling.

Let us discuss the first method, based on the possibility of developing functions on the basis constituted by the Dirac delta and its derivatives, the coefficients being the momenta of the function to be developed.

The following holds:

Decomposition Lemma. ([7]). *Assume that* $f \in L^1(\mathbb{R}^N; 1+|x|)$*, i.e.*

$$\int_{\mathbb{R}^N} |f(x)|(1+|x|)dx < \infty\,.$$

Then, there exists $\vec{F} \in (L^1(\mathbb{R}^N))^N$ *such that*

$$f \;=\; \int_{\mathbb{R}^N} f(x)dx\delta_0 + \operatorname{div}(\vec{F})$$

and

$$\|\vec{F}\|_{L^1(\mathbb{R}^N)} \;\leq\; C_N \| \, |x| f \, \|_{L^1(\mathbb{R}^N)}.$$

Moreover, if

$$f \in L^1(\mathbb{R}^N; 1 + |x|^2)$$

then,

$$f = \int f(x)dx\delta_0 - \int f(x)xdx \cdot \nabla\delta_0 + \sum_{|\alpha|=2} D^\alpha F_\alpha$$

with

$$\sum_{|\alpha|=2} \|F_\alpha\|_{L^1} \leq C_N \| \, |x|^2 f \|_{L^1} \, .$$

This decompostion formula can be easily used to analyze the asymptotic behavior of solutions. Recall that

$$u(x,t) = [G(\cdot,t) * \varphi(\cdot)] \, (x).$$

On the other hand, according to the decomposition formula,

$$\varphi = \int_{\mathbb{R}^N} \varphi dx\delta_0 + \operatorname{div}(\vec{\phi})$$

and, consequently,

$$\begin{aligned} u &= G * \left[\int_{\mathbb{R}^N} \varphi dx \, \delta_0 \right] + G * \operatorname{div}(\vec{\phi}) = \int_{\mathbb{R}^N} \varphi dx \, [G * \delta_0] + \nabla G * \vec{\phi} \\ &= \int_{\mathbb{R}^N} \varphi dx \, G + \nabla G * \vec{\phi}. \end{aligned}$$

In other words,

$$u \;-\; \int_{\mathbb{R}^N} \varphi dx \, G \;=\; \nabla G * \vec{\phi}$$

and therefore

$$\|u - \int_{\mathbb{R}^N} \varphi dx G\|_{L^p} \;\leq\; \|\nabla G\|_{L^p} \|\vec{\phi}\|_{L^1} \leq C \|\nabla G\|_{L^p} \|x\varphi\|_{L^1}.$$

But, taking into account that,

$$G = t^{-N/2} f(x/\sqrt{t}) \;\Rightarrow\; \nabla_x G = t^{-(N+1)/2} F(x/\sqrt{t})$$

we have

$$\|\nabla_x G\|_{L^p} \;\le\; C_p t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

and therefore

$$t^{\frac{N}{2}(1-\frac{1}{p})}\|u - \int_{\mathbb{R}^N} \varphi dx\, G\|_{L^p} \;\le\; C_p t^{-1/2}\,.$$

We have proved the following result:

Theorem.

- *If* $\varphi \in L^1(\mathbb{R}^N; 1+|x|)$, *then*

$$\|u - \int_{\mathbb{R}^N} \varphi dx\, G\|_{L^p} t^{\frac{N}{2}(1-\frac{1}{p})} \;\le\; C_p t^{-1/2}\,.$$

- *If* $\varphi \in L^1(\mathbb{R}^N)$, *then*

$$t^{\frac{N}{2}(1-\frac{1}{p})}\|u - \int_{\mathbb{R}^N} \varphi dx\, G\|_{L^p} \to 0\,,\; t \to \infty\,.$$

Summarizing, roughly speaking, we can say that

$$u \;\sim\; \int_{\mathbb{R}^N} \varphi dx\, G\,, \quad \text{as} \quad t \to \infty\,.$$

Proof.

- The first statement has been proved before.
- The second result can be proved by density. Indeed,

$$\exists u_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^N) : u_{0,\varepsilon} \xrightarrow[\epsilon \to 0]{L^1} u_0; \int u_{0,\varepsilon} = \int u_0.$$

Using this approximating sequence and applying the previous result we have:

$$\|u - \int u_0\, dx\, G\|_{L^p}\, t^{\frac{N}{2}(1-\frac{1}{p})} \leqslant \|u_\varepsilon - \int u_{0,\varepsilon}\, dx\, G\|_{L^p}\, t^{\frac{N}{2}(1-\frac{1}{p})} + \|u_\varepsilon - u\|_{L^p}\, t^{\frac{N}{2}(1-\frac{1}{p})}$$

$$\leqslant C_\varepsilon t^{-1/2} + C\|u_0 - u_{0,\varepsilon}\|_{L^1} \leqslant \frac{\delta}{2} + \frac{\delta}{2}.$$

The latter can be guaranteed by taking ε small enough (depending on δ), and then, the first one, once ε is fixed, taking t large enough.

Using more terms of the development on the Dirac basis we can obtain more terms on the asymptotic development of the solution as $t \to \infty$ as well. In this way we can show, as $t \to \infty$,

$$u \sim \int_{\mathbb{R}^N} u_0 dx\, G - \int_{\mathbb{R}^N} x u_0 dx \cdot \nabla G + \ldots$$

It is important to observe that every time we add a term in the asymptotic expansion (containing the higher order momentum of the initial datum and the corresponding derivative of the Gaussian kernel) we obtain an extra decay rate of the order of $t^{-\frac{1}{2}}$.

Proof of the Decomposition Lemma. We have to show that

$$\boxed{u_0 = \int u_0 dx \delta_0 + \operatorname{div}(\vec{V_0}).}$$

In one space dimension ($N = 1$) we can compute V_0 explicitly as follows:

$$V_0(x) = \begin{cases} -\int_x^\infty u_0\, dx, & x > 0 \\ \int_{-\infty}^x u_0\, dx, & x < 0. \end{cases}$$

For this function V_0 we have

$$\frac{dV_0}{dx} = u_0 \quad \text{for } x > 0,\, x < 0$$

and

$$[V_0]_{x=0} = -\int u_0\, dx.$$

Consequently,

$$\frac{dV_0}{dx} = u_0 - \int u_0\, dx\, \delta_0.$$

In several space dimensions the same can be proved integrating along rays. We then obtain

$$\vec{V_0} = -x \int_1^\infty t^{N-1}\, u_0(tx) dt.$$

■

The same formula allows to get a L^p-version of the decomposition result. Applying Minkowski's inequality we get

$$\|\vec{V_0}\|_{L^p_x} \leqslant \int_1^\infty t^{N-1} \|x u_0(tx)\|_{L^p_x} dt.$$

Taking into account that

$$\|x u_0(tx)\|_{L_p} = t^{-1-N/p} \|x u_0(x)\|_{L^p_x} \leqslant \|x u_0(x)\|_{L^p_x} \int_1^\infty t^{N(1-\frac{1}{p})-2} dt$$

we obtain

$$\|\vec{V_0}\|_{L^p_x} \leqslant \|x u_0(x)\|_{L^p_x} \int_1^\infty t^{N(1-\frac{1}{p})-2} dt.$$

The last integral converges if and only if

$$N(1 - \frac{1}{p}) - 2 < -1 \Leftrightarrow N(\frac{p-1}{p}) - 2 < 1 \Leftrightarrow p < N/(N-1).$$

This yields the following result:

Theorem 2.1. *([7])*

- *Assume that* $1 \le p < \frac{N}{N-1}$, *and* $f \in L^1(\mathbb{R}^N)$, $|x|f \in L^p(\mathbb{R}^N)$ *then there exists* $\vec{F} \in (L^p(\mathbb{R}^N))^N$ *such that*

$$f = \int_{\mathbb{R}^N} f(x) dx \delta_0 + \mathrm{div}\vec{F}.$$

- *If* $\frac{N}{N-1} < p \le \infty$, *under the assumption that* $|x|f \in L^p(\mathbb{R}^N)$ *there exists* $\vec{F} \in L^p(\mathbb{R}^n)$ *such that*

$$f \;=\; \mathrm{div}(\vec{F})\,.$$

Proof. We indicate the main steps of the proof in the first case. The second one is left as an exercise.

- **Case 1.** $1 \le p < \frac{N}{N-1}$. We set

$$F_j \;=\; -\int_0^1 x_j f\left(\frac{x}{t}\right) \frac{1}{t^{N+1}} dt.$$

Then,

$$\begin{aligned}\|F_j\|_p &\le \int_0^1 \|x_j f\left(\frac{x}{t}\right)\|_p \frac{1}{t^{N+1}}\, dt = \int_0^1 t^{1+\frac{N}{p}} \|xf\|_p \frac{1}{t^{N+1}}\, dt \\ &= \|xf\|_p \int_0^1 t^{N(\frac{1}{p}-1)}\, dt \le C(p,N)\|xf\|_p\end{aligned}$$

with $C(p,N) = \frac{p}{|N-(N-1)p|} < \infty \iff \frac{1}{p} - 1 > -\frac{1}{N} \iff p < \frac{N}{N-1}$.

The definition of the remainder term F_j can be motivated as follows:

$$\langle f - \int_{\mathbb{R}^N} f\, dx \delta_0, \varphi\rangle \;=\; \int_{\mathbb{R}^N} f(x)\,[\varphi(x) - \varphi(0)]\; dx$$

$$= \int_{\mathbb{R}^N} f(x) \int_0^1 x \cdot \nabla_x \varphi(tx)\, dt\, dx = -\int_{\mathbb{R}^N} \varphi(x) \mathrm{div} \underbrace{\left[x \int_0^1 f\left(\frac{x}{t}\right) \frac{dx}{t^{N+1}} \right]}_{F}.$$

■

Higher order generalizations can also be easily obtained. In particular, it follows that

$$f \;=\; \sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \int_{\mathbb{R}^N} f(x) x^\alpha\, dx D^\alpha \delta_0 \;\;+\;\; \sum_{|\alpha|=k+1} D^\alpha F_\alpha,$$

with F_α belonging to L^1, under the assumption that $f \in L^1\left(1+|x|^{k+1}\right)$.

Applying this decomposition results one can get the asymptotic expansion of solutions as $t \to \infty$.

Consider the Cauchy problem

$$u_t - \Delta u = 0$$

with initial datum $\varphi = \varphi(x)$.

We have, $u = G * \varphi$ with, as indicated above, $G = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$. We know that

$$G * D^\alpha \delta_0 = D^\alpha \left[G * \delta_0\right] = D^\alpha G.$$

Applying then the decomposition formula

$$\varphi = \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{|\alpha|!} \int_{\mathbb{R}^N} \varphi(x) x^\alpha \, dx D^\alpha \delta_0 \quad + \sum_{|\alpha|=k+1} D^\alpha F_\alpha \, .$$

we obtain

$$u = \sum_{|x| \leq k} \frac{(-1)^{|\alpha|}}{|\alpha|!} \int_{\mathbb{R}^N} \varphi(x) x^\alpha \, dx D^\alpha G \quad + \underbrace{\sum_{|\alpha|=k+1} F_\alpha * D^\alpha G}_{R}$$

with

$$\|R\|_{L^p(\mathbb{R}^N)} \quad \leq \quad C_p \| |x|^{k+1} \varphi \|_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}.$$

The later estimate is sharp.

The decomposition results we have proved on the basis of the Diract delta and its derivatives are closely related with the Taylor power series expansion of the Fourier transform of the function to be developed. Indeed, the values of the Fourier transform and its derivatives at the origin can be easily computed

$$\begin{aligned} \widehat{f}(\xi) &= \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot \xi} \, dx \\ \widehat{f}(0) &= \int_{\mathbb{R}^N} f(x) \, dx \\ D\widehat{f}(0) &= -2\pi i \int_{\mathbb{R}^N} x f(x) \, dx \\ &\cdots \end{aligned}$$

Therefore, provided $\widehat{f}$ is real analytic we have

$$\widehat{f}(\xi) = \sum_{k=0}^{\infty} \frac{D^k \widehat{f}(0)}{k!} \xi^k.$$

We get

$$f(x) = \sum_{k=0}^{\infty} \frac{D^k \widehat{f}(0)}{k!} \mathcal{F}^{-1}\left(\xi^k\right)$$

which, taking into account that $\widehat{\delta}_0 = 1$, i.e. $\delta_0 = \mathcal{F}^{-1}(1)$, and the resulting expressions for $\mathcal{F}^{-1}(\xi^k)$, yields the infinite order asymptotic expansion we have obtained rigorously at finite order in the previous results.

The decomposition formulas obtained above are also closely related with *Hardy inequalities*. Indeed, observe that, for instance, the decomposition formula

$$f \ = \ \int_{\mathbb{R}^N} f\, dx \delta_0 - \mathrm{div}(\vec{F})$$

(which holds, according to the previous results, when $f \ \in \ L^1$, $|x|f \ \in \ L^p$, $1 \le p < \frac{N}{N-1}$ with $\vec{F} \in (L^p)^N$) yields

$$\left| \int_{\mathbb{R}^N} |x| f(x) \frac{[\varphi(x) - \varphi(0)]}{|x|} dx \right| = \left| - \int_{\mathbb{R}^N} -\vec{F} \cdot \nabla\varphi \right| \le \|\vec{F}\|_{L^p} \|\nabla\varphi\|_{L^{p'}} \le C\| \, |x| f\|_{L^p} \|\nabla\varphi\|_{L^{p'}}$$

if and only if $1 \le p < N/(N-1)$. By duality this yields the Hardy inequality,

$$\|\frac{\varphi(x) - \varphi(0)}{|x|}\|_{L^{p'}(\mathbb{R}^N)} \le C_{p'} \|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)} \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N) \qquad N < p' \le \infty.$$

The reverse is also true the extremal case $p = 1$ being excluded.

By the contrary, the decompostion formula

$$f = \mathrm{div}(\vec{F}), \quad p > N/(N-1)$$

yields

$$\left| \int_{\mathbb{R}^N} |x| f(x) \frac{\varphi(x)}{|x|} dx \right| = \left| - \int_{\mathbb{R}^N} \vec{F}, \nabla\varphi\, dx \right| \leqslant \ \|\vec{F}\|_{L^p} \ \|\nabla\varphi\|_{L^{p'}} \leqslant \ c_p \| \, |x| \, f\|_{L^p} \ \|\nabla\varphi\|_{L^{p'}}$$

and, consequently,

$$\|\frac{\varphi(x)}{|x|}\|_{L^{p'}(\mathbb{R}^N)} \leqslant \ C_{p'} \|\nabla\varphi\|_{L^{p'}(\mathbb{R}^N)}$$

which holds, when

$$\frac{N}{N-1} < p \leqslant \infty \Leftrightarrow 1 \leqslant \ p' < N.$$

This establishes the connection between the decomposition formulas on the Dirac basis and Hardy inegualities. Note however that, as we mentioned before, it is better to use the explicit formulas obtained above, without using Hardy inequalities and duality, to avoid the problems related with the extremal cases $p = 1, \infty$.

The problem of obtaining sharp properties of the function f guaranteeing that the infinite order expansion

$$f = \sum_{\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} \int_{\mathbb{R}^N} f(x) x^\alpha \, dx \, D^\alpha \delta_0$$

holds true is open.

2.4 Scaling: A basic tool for computing asymptotics.

We have shown that, using multipliers and integration by parts, one can get sharp $L^p(\mathbb{R}^N)$-estimates and that this method applies as well for a number of nonlinear parabolic equations (see [[18]). However, obviously, in the nonlinear context, there is no explicit convolution formula for solutions allowing to obtain the asymptotic expansion. In this case scaling arguments are very useful although they give, in principle, only the first term of the expansion. Note also that, as pointed out in [19], the asymptotic behavior might be extremely complex.

Let us consider again the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^N \quad t > 0 \\ u(x,0) = \varphi(x) & \mathbb{R}^N. \end{cases}$$

We know that

$$\varphi \in L^1(\mathbb{R}^N) \Rightarrow \|u(t)\|_p \leqslant C_p \, t^{-\frac{N}{2}(1-\frac{1}{p})} \|\varphi\|_{L^1(\mathbb{R}^N)}.$$

We now use the scaling argument. For, we introduce the rescaled family

$$u_\lambda(x,t) = \lambda^N u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

It is easy to see that, for each $\lambda > 0$, u_λ solves the heat equation as well:

$$\begin{cases} u_{\lambda,t} - \Delta u_\lambda = 0 & \mathbb{R}^N, \, 0 < t < 1 \\ u_\lambda(0) = \varphi_\lambda = \lambda^N \varphi(\lambda x). \end{cases}$$

At this point it is important to observe that the initial data of the rescaled family of solutions satisfy

$$\varphi_\lambda \longrightarrow \int_{\mathbb{R}^N} \varphi(x) \, dx \, \delta_0, \quad \lambda \to \infty.$$

On the other hand, the decay rate of the solution u in L^p yields uniform bounds on the rescaled family of solutions u_λ:

$$\|u_\lambda(t)\|_p \leqslant C_p \, t^{-\frac{N}{2}(1-\frac{1}{p})}, \, 0 < t < 1.$$

This inequality holds uniformly on λ with a constant C_p independent of this parameter.

This estimate, together with the regularizing effect of the heat equation allows showing that, u_λ is relatively compact in $L^p(\mathbb{R}^N)$ for all $1 \leq p \leq \infty$, for each $t > 0$. In fact compactness holds in $C([\tau, 1]; L^p(\mathbb{R}^N))$, for all $0 < \tau < 1$. Let v be the limit of u_λ, after extraction of a suitable subsequence:

$$u_\lambda \to v, \quad \lambda \to \infty.$$

One can show that the limit v is the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^N \times (0,1) \\ v(0) = \int_{\mathbb{R}^N} \varphi \, dx \, \delta_0. \end{cases}$$

By uniqueness of the fundamental solution $v = \int \varphi \, dx \, G$. Consequently the limit holds along the whole familiy $\lambda \to \infty$.

Then

$$u_\lambda\big|_{t=1} \overset{\lambda\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} \varphi dx \, G\big|_{t=1} \, in \quad L^p(\mathbb{R}^N),$$

for all $1 \leq p \leq \infty$. This turns to be equivalent to the fact that

$$||u(t)|| - \int_{\mathbb{R}^N} \varphi dx \, G||_{L^p(\mathbb{R}^N)} t^{\frac{N}{2}(1-\frac{1}{p})} \underset{t\to\infty}{\longrightarrow} 0 \, .$$

This is precisely the first term in the asymptotic expansion we got through the explicit representation formula of solutions.

Summarizing, we see that sharp L^p decay estimates, together with the regularizing effect of parabolic equations (to gain compactness), scaling arguments and the uniqueness of the limit characterized as the solution of the limit system, allow obtaining the first term of the asymptotic expansion of the solution as $t \to \infty$.

This method works for much more general problems (variable coefficients, nonlinearities, etc.) but fails to give a complete asymptotic expansion.

In fact, as we show by means of the explicit representation formula, when the initial datum φ satisfies the further property that $|x|\varphi(x) \in L^1(\mathbb{R}^N)$, then we gain an extra decay rate of the order of $t^{-1/2}$. The scaling argument above in itself does not yield this extra information.

We have also mentioned above that this method allows exhibiting, in some particular cases a much more complex asymtotic behavior of solutions. We shall briefly discuss this matter in the following subsection.

2.5 How Universal is the Law of Asymptotic Simplification ?

Let us consider now the Cauchy problem for a linear parabolic equation with variable coefficients:

$$\begin{cases} u_t - \operatorname{div}(a(x)\nabla u) = 0 & \mathbb{R}^N,\, t > 0 \\ u(0) = \varphi, \end{cases}$$

where

$$0 < \alpha \leq a(x) \leq \beta \leq \infty \qquad \text{a.e. } \in \mathbb{R}^N.$$

For this equation, the argument above, based on mutiplying the equation by powers of the solution, and integration by parts, yields the same decay rate as for the constant coefficient heat equation. Namely, it follows that

$$||u(t)||_p \leq C_p t^{-\frac{N}{2}(1-\frac{1}{p})}||\varphi||_1.$$

Let us now perform the scaling argument. For, introduce the rescaled family of solutions:

$$u_\lambda = \lambda^N u(\lambda x, \lambda^2 t).$$

In this case they satisfy

$$\begin{cases} u_{\lambda,t} - \operatorname{div}(a(\lambda x)\nabla u_\lambda) = 0 \\ u_\lambda(0) = \varphi_\lambda = \lambda^N \varphi(\lambda x) \longrightarrow \int \varphi dx\, \delta_0. \end{cases}$$

Note that, in this case, due to the presence of the variable coefficient, the equation associated with u_λ depends on λ. In other words, the equation is not invariant under the scaling transformation. It is then important to analyze how does $a(\lambda x)$ behave as $\lambda \to \infty$. There are several cases to be distinguished:

- *Periodic case*: Assume that $a = a(x)$ is $[0,1]^N$-periodic i.e. it is periodic of period one in each of the N space variables. Then, $a_\lambda = a(\lambda x)$ is $\left[0, \frac{1}{\lambda}\right]^N$-periodic and

  $$a_\lambda \overset{\lambda\to\infty}{\longrightarrow} \bar{a} = \int_{[0,1]^N} a\, dy\,, \quad \lambda \to \infty.$$

  weakly-* in $L^\infty(\mathbb{R}^N)$.

  But, contradicting the first intuition, in this case the limit v of the rescaled solutions u_λ does not satisfy the equation

  $$v_t - \bar{a}\Delta v = 0.$$

  Indeed, the theory of Homogenization guarantees that the limit function solves the *homogenized* equation

  $$v_t - \operatorname{div}(A^* \nabla v) = 0,$$

where A^* is a constant coefficient symmetric and elliptic matrix that does not coincide with $\bar{a}I$ except in the trivial case where the coefficient $a = a(x)$ is constant.

For instance, in one space dimension ($N = 1$), the homogenized coefficient A^* is

$$A^* = \left(\int_0^1 \frac{1}{a} \, dy \right)^{-1}$$

which does indeed differ from the average $\bar{a}$ of a except when a is constant.

This can be easily seen in the context of the elliptic equation

$$-(\, a(\lambda x) u'_\lambda \,)' = f.$$

Indeed, explicit computations yield

$$a(\lambda x) u'_\lambda = - \int_{-\infty}^{x} f$$

and

$$u'_\lambda = - \frac{1}{a(\lambda x)} \int f \longrightarrow - \fint \frac{1}{a} \int_{-\infty}^{x} f.$$

Consequently,

$$u_\lambda \longrightarrow v \; \mathcal{D}'$$

where v solves

$$-a^* v'' = f$$

with

$$a^* = \left(\fint \frac{1}{a} \right)^{-1}.$$

But although the analysis of this case is not straightforward, the periodic coefficient case is an easy one. In fact, using Block waves one can get a complete asymptotic expansions of solutions as $t \to \infty$ (see [8] and [17]).

- *Asymptotically Constant Difussion:* Assume now that

$$a(x) = 1 + \epsilon(x), \quad \epsilon(x) \to 0, \, |x| \to \infty.$$

Then,

$$a(\lambda x) = 1 + \epsilon(\lambda x) \to 1, \quad \lambda \to \infty, \text{ a.e. } x \in \mathbb{R}^N.$$

In this case the limit v of the rescaled solutions u_λ does indeed satisfy

$$\begin{cases} v_t - \Delta v = 0 \\ v(0) = \int_{\mathbb{R}^N} \varphi(x) \, dx \, \varphi_0, \end{cases}$$

and therefore

$$v = \int_{\mathbb{R}^N} \varphi(x)\,dx\,G\,.$$

We refer to [9] for a further analysis of this case.

So far, in all the examples, asymptotic simplification occurs and solutions behave as a gaussian process in a suitable homogeneous medium. But this is not true for all coefficients $0 < \alpha \le a(x) \le \beta < \infty$ since complexity arises as well. The following holds:

Lemma. ([19])
Given any sequence $\{f_j\}_{j\in\mathbb{N}}$, bounded in $L^\infty(\mathbb{R}^N)$, we can construct a function $g \in L^\infty(\mathbb{R}^N)$ such that the weak- accumulation points of the rescaled family*

$$g_\lambda(x) = g(\lambda x) \quad \text{as } \lambda \to \infty$$

contain the closure of the sequence $\{f_j\}_{j\in\mathbb{N}}$ on the weak- topology.*

In particular, for a suitable g, theset of accumulation points of the family $\{g_\lambda\}$ may cover a ball in $L^\infty(\mathbb{R}^N)$.

Proof. The Lemma can be easily proved by the "zooming method" that we describe briefly.

We first cut each function f_i in the annulus $2^{-j} < |x| < 2^j$ and then zoom it with a zooming factor $\lambda_{ij} > 0$. In this way we get the new function

$$f_{ij}(x) = f_i\left(\frac{x}{\lambda_{ij}}\right) \qquad 2^{-j}\lambda_{ij} < |x| < 2^j\lambda_{ij}.$$

We arrange all these annulae so that they become disjoint by choosing appropriate values of λ_{ij} and then define the function g as being

$$g(x) = f_i\left(\frac{x}{\lambda_{ij}}\right) \quad \text{in each of the annulae.}$$

We then consider the rescaled family $g_\lambda = g(\lambda x)$. It is important to observe that, along the sequence $\lambda = \lambda_{ij}$, $g_\lambda \equiv f_i$. Therefore, the function f_i is included in the set of accumulation points of g_λ and this for all indexes i.

This concludes the proof of the Lemma.

■

This has important consequences in the asymptotic behavior of solutions of parabolic equations with coefficients having the structure of the function g in the Lemma. Indeed, consider for instance the parabolic equation with variable density

$$\rho(x)\, u_t - \Delta u = 0.$$

The density ρ, according to the previous Lemma, can be chosen such that the set of accumulation points of the rescaled family ρ_λ contains any sequence $\{f_j\}$. The accumulation points of the rescaled family of solutions u_λ then solve the limiting equations

$$\begin{cases} f_j(x)\, v_t - \Delta v = 0 \\ v(0) = \int_{\mathbb{R}^N} \varphi\, \delta_0, \end{cases}$$

for $j \geq 1$.

Consequently, the set of accumulation points of the rescaled family u_λ contains the whole family of fundamental solutions G_j of these equations, i.e.

$$\lim_{\lambda\to\infty} u_\lambda \supset \left\{ \int \varphi\, dx\, G_1,\ \int \varphi\, dx\, G_2,\ \int \varphi\, dx\, G_3 \ldots \right\}.$$

This shows that, although for some important cases asymptotic simplication occurs, the large time asymptotic behavior may be quite complex in general. The problem of obtaining a complete asymptotic expansion to any order for general variable coefficient parabolic equations is still open. We refer to [19] for a deeper discussion of this issue.

3 The Wave Equation

3.1 The Dirichlet problem

Let us consider now the wave equation in a bounded domain Ω with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - \Delta u = 0 & \Omega \times (0,\infty) \\ u|_{\partial\Omega} = 0 & \\ u(0) = u_0,\ u_t(0) = u_1 & \text{in } \Omega. \end{cases}$$

It is a model for the vibrations of strings and membranes and also for the propagation of acoustic waves.

This system is well-posed in the energy space $H_0^1(\Omega) \times L^2(\Omega)$. Then, given $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ there exists a unique solution $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$. Moreover, the energy

$$E(t) = \frac{1}{2} \int_\Omega \left[|\nabla u(x,t)|^2 + u_t^2(x,t) \right] dx$$

is conserved in time, i.e.

$$E(t) \,=\, E(0), \;\forall t > 0\,.$$

This system exhibits the following features that are in opposition with those characterizing the heat equation:

- The system is purely conservative.
- The system lacks of smoothing properties.
- The system is time reversible.

The property of conservation of energy can be easily proved by the energy method. In this case, multiplying the equation by u_t and integrating by parts we get

$$\frac{dE}{dt} = \int_\Omega u_{tt}\, u_t \,+\, \nabla u \,\cdot\, \nabla u_t dx = \int_\Omega (u_{tt} \,-\, \Delta u)\; u_t\, dx \,=\, 0.$$

Solutions of the wave equation can be easily developed in Fourier series. Consider again the eigenvalue problem

$$\begin{cases} -\Delta\phi_j(x) \,=\, \lambda_i \phi_j(x) & \Omega \\ \phi_j\big|_{\partial\Omega} \,=\, 0 & j = 1, 2, \ldots \end{cases}$$

Then

$$u(x,t) \,=\, \sum_{j=1}^{\infty} \left(a_j \cos(\sqrt{\lambda_j}\, t) \,+\, \frac{bj}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) \right) \phi_j(x),$$

where $\{a_j\}$ and $\{b_j\}$ are the Fourier coefficients of the initial data

$$u_0(x) \,=\, \sum_{j=1}^{\infty} a_j \phi_j(x), \quad u_1(x) \,=\, \sum_{j=1}^{\infty} b_j \phi_j(x).$$

In other words:

$$u \,=\, \sum_{k=1}^{\infty} u_k(t)\, \phi_k(x)$$

where

$$\begin{cases} u_k'' \,+\, \lambda_k\, u_k \,=\, 0, & t > 0 \\ u_k(0) \,=\, a_k, \; u_k'(0) \,=\, b_k, & k = 1, \ldots, \infty \end{cases}$$

Consequently, the wave equation is equivalent to a system of infinitely many uncoupled harmonic oscillators.

3.2 The Damped Wave Equation

The system above is purely conservative. But in most physical systems friction is present. Friction or damping is also a very efficient way for stabilizing engineering systems. This was already observed by L. Maxwell in his work "Governors" on the dynamical properties of the steam-engine in the end of XIX-th century.

The damped wave equation reads as follows

$$\begin{cases} u_{tt} - \Delta u + d u_t = 0 & \Omega \times (0,\infty) \\ u = 0 & \partial\Omega \times (0,\infty) \\ u(0) = u_0,\ u_t(0) = u_1 & \Omega \end{cases}$$

where $d > 0$ is the damping coefficients.

The energy method yields the following *energy dissipation law*

$$\frac{dE}{dt} = -d \int_\Omega u_t^2 \leq 0 \ .$$

The following questions arise naturally: *Does* $E(t) \to 0,\ t \to \infty$ *?* and, if yes, *How Fast ?*

In this case the Fourier method yields the following representation formula

$$u = \sum_{k=1}^{\infty} u_k(t)\, \phi_k(x)$$

where the coefficients u_k obey the ODE

$$u_k'' + \lambda_k\, u_k + d u_k' = 0.$$

The two roots of the characteristic polynomial are

$$r_k = \frac{-d \pm \sqrt{d^2 - 4\lambda_k}}{2}.$$

Accordingly, two cases have to be distinguished:

- $d \leq 2\sqrt{\lambda_1} \Rightarrow \mathrm{Re}(r_k^{\pm}) = -\frac{d}{2}, \quad \forall k$
- $d > 2\sqrt{\lambda_1} \Rightarrow \exists$ finite number of real eigenvalues.

But, always, independently of the value of the damping constant $d > 0$ it follows that:

$$\max_k \mathrm{Re}(r_k) \geq \max_k \mathrm{Re}(r_k)\Big|_{d=2\sqrt{\lambda_1}} = -\sqrt{\lambda_1}\,.$$

This means that, among the class of constant dampers, the one that produces the best decay rate for the energy is $d = 2\sqrt{\lambda_1}$ in which case the energy decays exponentially with a rate $-2\sqrt{\lambda_1}$.

In particular it is not true, as one could expect in a first approach, that increasing the damping constant necessarily yields a better decay rate. This occurs for $0 \leq d \leq d = 2\sqrt{\lambda_1}$ but it is not longer true for $d > 2\sqrt{\lambda_1}$. This phenomenon is refered to as "overdamping" in the engineering literature.

One can do much better by means of non-constant damping coefficients. But the corresponding spectral problem is not easy to deal with, since it is not self-adjoint.

3.3 Boundary damping

An interesting example is the one related with the wave equation with boundary damping which leads to a new phenomenon: the *vanishing waves.*

Consider the $1 - d$ wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ t > 0 \\ u(0,t) = 0, & t > 0 \\ u_x(1,t) + u_t(1,t) = 0, & t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & 0 < x < 1. \end{cases}$$

The energy dissipation law now reads as follows:

$$\frac{dE}{dt}(t) = -u_t^2(1,t)$$

In this particular case not only solutions decay exponentially as $t \to \infty$. But, in fact, solutions vanish in finite time ($T = 2$). This is closely related with the fact that the spectrum of the system is empty and indicates that the evolution process under consideration is highly irreversible.

To prove this property of vanishing of waves it is convenient to decompose the d'Alembert operator in two first order transport equations:

$$u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$$

A simple analysis of the behavior of solutions along characteristic lines allows showing that the vanbishing of waves property is indeed true in time $T = 2$. Of course, at this level, the fact that in the dissipative boundary condition $u_x(1,t) + u_t(1,t) = 0$ one of these transport operators appears plays a key role.

But this example is impossible to reproduce for general equations or higher dimensional situations and it is very specific to the $1 - d$ wave equation with constant coefficients. In fact it does not even hold for the $1 - d$ wave equation with variable coefficients (see [5]).

The boundary conditions for which solutions vanish in final time are often refered to as transparent boundary conditions since, as the analysis of this $1-d$ models shows, waves, when reaching the dissipative boundary condition are not reflected at all. Transparent boundary conditions are very useful in numerics since they allow reproducing in a bounded domain the properties of the Cauchy problem in the whole space. The fact of being able to reduce the problem to solving a system in a bounded domain, of course, reduces significantly the computational cost of the numerical method.

3.4 Internal damping

There is an intermediate damping mechanism: the one in which the damping term is localized in a subdomain ω of the domain Ω where the equation holds:

$$\begin{cases} u_t \, - \, \Delta u \, + \, 1_\omega u_t = 0 \\ u\big|_{\partial\Omega} \, = \, 0 \\ u(0) = u_0, \qquad u_t(0) = u_1. \end{cases}$$

Here and in the sequel 1_ω denotes the characteristic function of the set ω.

In this case the energy dissipation law reads

$$\frac{dE}{dt} = -\int_\omega u_t^2 \, dx.$$

Theorem. *The energy decays exponentially provided ω is a neighborhood of a subset of the boundary of the form*

$$\Gamma(x_0) = \{x \, \in \, \partial\Omega \, : \, (x \, - \, x_0) \cdot n(x) \, > \, 0\} \, ,$$

for some $x_0 \in \mathbb{R}^N$.

More precisely, there exist positive constants $C_{\Omega,\omega}$ and $\alpha_{\Omega,\omega}$, depending on Ω and ω such that

$$E(t) \leqslant C_{\Omega,\omega} e^{-\alpha_{\Omega,\omega} t} E(0),$$

for every solution of the system.

Remark.

- Here $n(x)$ denotes the unit normal vector to the domain Ω at the boundary point $x \in \partial\Omega$.
- The subset $\Gamma(x_0)$ of the boundary is constituted by the points in which the ray going from the reference point x_0 to x exits the domain Ω.
- In the context of the linear wave equation under consideration a sharp necessary condition for the exponential decay was given in [1] in Geometric Optics terms. This is the so called Geometric Control Condition and reads as follows: The exponential

stability property holds if all rays of Geometric Optics, after possibly bouncing on the boundary of the domain Ω, reach the subdomain ω where the damping term is effective.

- Similar results hold for semilinear wave equations as well, see [6] and [22].

Sketch of the proof:

- Step 1: It suffices to check that there exist $C, T > 0$:

$$E(0) \leqslant C \int_0^T \int_\omega u_t^2 dxdt,$$

for every solution.

This inequality, refered often as *observability inequality*, indicates that the dissipated energy is proportional to the energy within the system.

This observability inequality, combined with the energy dissipation law yields

$$E(T) - E(0) = -\int_0^T \int_\omega u_t^2 dxdy \leqslant -\frac{1}{C} E(0),$$

and consequently,

$$E(T) \leqslant (\frac{C-1}{C}) E(0).$$

Accordingly, the semigroup at time $t = T$, $S(T)$, is a contraction.

The semigroup property then yields the exponential decay. Indeed, when $t = kT$ it follows that

$$\|S(t)\| = \|S(kT)\| \leqslant \gamma^k = e^{k \log \gamma} = e^{-|\log \gamma| t/T}.$$

- Step 2: We now proceed to the prove the observability inequality.

The difference between the damped and the conservative wave equation being the damping term itself, it suffices to prove the inequality for the conservative wave equation.

$$\begin{cases} u_{tt} - \Delta u_\lambda = 0 & \Omega \times (0, T) \\ u\big|_{\partial\Omega} = 0 & \\ u(0) = u_0, u_t(0) = u_1 & \Omega. \end{cases}$$

The question is now whether

$$E(0) \leqslant C \int_0^T \int_\omega u_t^2 dxdt.$$

This inequality does indeed hold and can be proved using the so called multiplier method. It consists in multiplying the equation by u, u_t, $(x - x_0) \cdot \nabla u$,... and integrating by parts to later combine the identities one obtains (see [14] and [15] for a systematic description of this method).

This method is very much the same as that developed by Pohozaev in the context of elliptic equations. Pohozaev's identity allows getting connections between the total energy of solutions of elliptic problems and its energy concentrated on the boundary.

Let us recall how this identity may be established for the eigenvalue problem

$$-\Delta u = \lambda u \quad \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Multiplying the equation by $x \cdot \nabla u$ (we assume without loss of generality that $x_0 = 0$) we get

$$\lambda \int_\Omega u x \cdot \nabla u dx = -\frac{N\lambda}{2} \int_\Omega u^2 dx$$

and

$$\begin{aligned} -\int_\Omega \Delta u x \cdot \nabla u dx &= \int_\Omega \nabla u \cdot \nabla (x \cdot \nabla u) dx - \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 (x \cdot n) d\sigma \\ &= \int_\Omega \left[|\nabla u|^2 + x \cdot \nabla \frac{|\nabla u|^2}{2} \right] - \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 (x \cdot n) d\sigma \\ &= \left(1 - \frac{N}{2}\right) \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 (x \cdot n) d\sigma. \end{aligned}$$

Moreover, multiplying the equation by u it follows that

$$\int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2 = \lambda$$

provided we normalized the eigenfunctions by the condition $\left(\int_\Omega u^2 = 1\right)$. Combining these identities we get

$$\left(1 - \frac{N}{2}\right) \lambda - \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 (x \cdot n) d\sigma = -\frac{N}{2} \lambda.$$

Thus

$$\lambda = \frac{1}{2} \int_{\partial\Omega} (x \cdot n) \left| \frac{\partial u}{\partial n} \right|^2 d\sigma \leqslant \frac{1}{2} \int_{\Gamma(x_0)} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma.$$

Consequently, it follows that

$$\boxed{\int_\Omega |\nabla u|^2 dx \leqslant \frac{1}{2} \int_{\Gamma(x_0)} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma.}$$

It is important to note that this inequality holds independently of the frecuency λ, and it does indeed guarantee that the energy of the eigenfunctions is uniformly bounded by the energy concentrated on the subset of the boundary $\Gamma(x_0)$.

As a consequence of this, with some extra work, taking into account that the subdomain ω is a neighborhood of Γ_{x_0}, it can also be proved that

$$\boxed{\int_\Omega |\nabla u|^2 dx \leqslant C \int_\omega |\nabla u|^2\, dx,}$$

with a constant C which is independent of the eigenfunction.

The same occurs for the wave equation when T is large enough, namely when $T > \operatorname{diam}(\Omega \setminus \omega)$.

■

This Theorem shows that, when ω is a neighborhood of a large enough subset of the boundary, the exponential decay property holds. As a complement to this result it is important to observe that, in general, for any non-empty open subset ω of Ω the property of the exponential decay of the energy is no longer true but still the energy of every solution tends to zero without uniform decay rate. The fact that the energy of each solutions tends to zero can be proved as an application of La Salle's invariance principle.

3.5 Damping localized on narrow sets

We have discussed the case where the damping is localized in an open non-empty subset of the domain Ω. But the question makes also sense whatever ω is. It is for instance natural to analyze the case where ω is a measurable set of positive measure. In this case, a Lyapunov type argument allows showing that the energy of each individual trajectory tends to zero. The question of whether exponential decay holds under suitable additional assumptions is open in the context of damping sets of positive measure.

Applying La Salle's invariance argument and using the energy E of the system as Lyapunov function one can show that every solution tends to zero if and only the following uniqueness or unique continuation property holds: *The only solution of the damped equation*

$$\begin{cases} u_{tt} - \Delta u + 1_w u_t = 0 \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

such that $u_t = 0$ *in the damping region* $\omega \times (0, \infty)$ *is the trivial one* $u \equiv 0$.

However, taking into account that the damping term vanishes under the condition that $u_t = 0$ in the damping region $\omega \times (0, \infty)$, it is in fact equivalent to proving the same unique continuation property for the undamped equation

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

The question is then whether $u_t = 0$ in $\omega \times (0, \infty)$ for the solutions of the conservative wave equation implies that $u \equiv 0$.

To analyze this question it is convenient to develop solutions in Fourier series. In fact we set $v = u_t$. Obviously, v is also a solution of the Dirichlet problem for the wave equation and $v = 0$ in $\omega \times (0, \infty)$. The Fourier representation formula yields

$$v(x,t) = \sum a_k^{\pm} e^{\pm i\sqrt{\lambda_k} t} \phi_k(x),$$

where the sum runs along the whole sequence of eigenpairs of the Laplace operator.

We now use the orthogonality propertiy of complex exponentials in infinite time

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{i\lambda t} e^{-i\mu t}\, dt = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

Then, the fact that $v = 0$ in $\omega \times (0, \infty)$ implies

$$\int_0^\infty \int_\omega v^2(x,t)\, dx\, dt \equiv 0$$

and the following alternative holds: Either $v \equiv 0$ or $\phi_k^2 = 0$ in ω for some k.

However, the second possibility can inmediately be excluded since the eigenfunctions of the Laplacean are analytic and therefore they can not vanish in a set of positive measure without being trivially zero everywhere. Thus, we conclude that $u_t \equiv v \equiv 0$. Therefore, $u = u(x)$ and it solves the elliptic problem

$$\begin{cases} -\Delta u = 0 \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

The only solution of this elliptic equation being $u \equiv 0$, we deduce that the desired unique continuation property holds.

As a consequence of this analysis it follows that for damping sets ω of positive measure one can guarantee that all solutions tend to zero. The problem of getting conditions under which the decay rate is uniform and exponential is open, as we mentioned before.

Similar problems arise when the damper is localized in manifolds of lower dimension (see for instance [12] and [16]). One of the most classical examples is the case where the damper is localized in one single point x_0 of the domain.

Let us consider this problem in $1-d$ the domain Ω under consideration being $\Omega = (0,\pi)$:

$$\begin{cases} u_{tt} - u_{xx} + u_t(x_0,t)\delta_{x_0} = 0 \\ u(0,t) = u(\pi,t) = 0. \end{cases}$$

In this case, the Fourier series development of solution reads

$$u = \sum a_k^{\pm} e^{\pm ikt} \sin(kx).$$

The damping being localized at the point $x_0 \in \Omega$ the question we analyze is closely related to how much of the energy of vibrations do we estimate through the observation of $\displaystyle\int_0^T u_t^2(x_0,t)\,dt$?

The answer is easy in this case. Indeed, by taking $T = 2\pi$ and using the orthogonality properties of trigonometric polynomials it follows that

$$\int_0^{2\pi} u_t^2(x_0,t)\,dt = \int_0^{2\pi} \left| \sum \pm a_k^{\pm} ike^{\pm ikt} \sin(kx_0) \right|^2 dt = 2\pi \sum_k \left|a_k^{\pm}\right|^2 k^2 \sin^2(kx_0).$$

According to this we have to exclude the cases where x_0/π is rational or not. Indeed,

- $x_0/\pi \in \mathbb{Q} \Rightarrow \exists\, k\colon\ \sin(kx_0) = 0.$

In this case there are Fourier components that we do not see at all! Consequently, the observed quantity does not constitute a norm in the space of solutions. In what concerns the damped equation, this means that there are solutions that are not damped at all, whose energy remains constant in time.

- $x_0/\pi \in \mathbb{Q}^c \Rightarrow \gamma_k = \sin(kx_0) \neq 0$ for all k. In this case, for all $T \geq 2\pi$ we obtain the inequality

$$\sum_k \gamma_k^2 |a_k^{\pm}|^2 k^2 \leq C \int_0^T |u_t(x_0,t)|^2\,dt.$$

However, the asymptotic behavior of the weights γ_k depends on the class of irrationality in which $\frac{x_0}{\pi}$ lies. In particular,

- When $\frac{x_0}{\pi}$ is an algebraic number of degree 2,

$$|\gamma_k| \sim c/|k|$$

- By the contrary, if $\frac{x_0}{\pi}$ is a Liouville number

$$|\gamma_k| \to 0\,, \qquad \text{exponentially as } k \to 0\,.$$

According to this, in the best case, for an optimal choice of x_0, i.e. when x_0/π is an algebraic number of degree 2, we loose one derivative on the observability inquality in the sense that we get

$$\|(u_0,u_1)\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)} \leq C \int_0^{2\pi} u_t^2(x_0,t)\,dt,$$

instead of the sharp inequality

$$E(0) \leq C \int_0^{2\pi} u_t^2(x_0, t)\, dt,$$

one could expect by comparison with the case where the damping is effective in an open subset of the domain. The latter is never true, whatever the choice of x_0 is.

This weakened observability estimates lead to polynomial decay rates for smooth solutions of the damped wave equation (see [12]). In $1-d$ these results were explained in terms of ray properties in [16]. There it was shown that one could concentrate solutions of the wave equation along rays, cancelling each other, so that the observability inequelity above with a defect of one derivative becomens sharp.

In $1-d$ wave equations with variable coefficients behave very much the same as the constant coefficient one discussed above. However, in that case, one can not rely on the property of orthogonality of trigonometric polynomials. The classical *Ingham inequality* plays the same role atthis level (see [21]).

The problem of combining Geometric Optics tools with others like diophantine approximations or ergodicity theory to obtain poilynomial decay rates when the damping acts on narrow sets in several space dimensions is completely open.

The same can be said about the case where the damper is located in moving points. We refer to [3] for the analysis of the approximate control property for the heat equation. But nothing is known for in the context of stabilization of the wave equation.

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