REACHABLE SET FOR HAMILTON-JACOBI EQUATIONS WITH NON-SMOOTH HAMILTONIAN AND SCALAR CONSERVATION LAWS

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ABSTRACT. We give a full characterization of the range of the operator which associates, to any initial condition, the viscosity solution at time T of a Hamilton-Jacobi equation with convex Hamiltonian. Our main motivation is to be able to treat the case of convex Hamiltonians with no further regularity assumptions. We give special attention to the case $H(p) = |p|$, for which we provide a rather geometrical description of the range of the viscosity operator by means of an interior ball condition on the sublevel sets. From our characterization of the reachable set, we are able to deduce further results concerning, for instance, sharp regularity estimates for the reachable functions, as well as structural properties of the reachable set. The results are finally adapted to the case of scalar conservation laws in dimension one.

1. INTRODUCTION

We consider first-order Hamilton-Jacobi equations of the form

$$
(1) \qquad \begin{cases} \partial_t u + H(\nabla_x u) = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}
$$

where $N \geq 1$, $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, and the Hamiltonian $H : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a given convex function, with no further regularity assumptions. It is well-known that the initial-value problem (1) is well-posed in the sense of viscosity solutions [7, 17]. For any given positive time $T > 0$, the main goal of this work is to give a full characterization of the range of the operator

$$
(2) \qquad \begin{array}{rccc} S_T^+ : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_0 & \longmapsto & u(T, \cdot), \end{array}
$$

which associates, to any initial condition, the viscosity solution at time T of the equation (1).

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In what follows, the range of the operator S_T^+ will be referred to as *the reachable set*, and will be denoted by

$$(3) \qquad \mathcal{R}_T := \{u_T \in \mathrm{Lip}(\mathbb{R}^N) \, : \, \exists u_0 \in \mathrm{Lip}(\mathbb{R}^N) \text{ such that } S_T^+ u_0 = u_T\} \subset \mathrm{Lip}(\mathbb{R}^N).$$

The problem of characterizing $\mathcal{R}_T$ can be seen as a controllability problem in which the dynamics are governed by the PDE in (1), and the control is the corresponding initial condition. The characterization of the reachable set for evolutionary equations such as (1) is important when addressing the inverse problem of reconstructing the initial condition from an observation of the solution at some positive time $T > 0$. This inverse problem is well-known to be highly ill-posed due to the lack of regularity of the solutions, which gives raise to the loss of backward uniqueness [6, 11, 16] (multiple initial conditions result in the same solution after some time). Moreover, in real-life applications, the measurements of the solution are usually noisy, and it is often the case that no initial condition is compatible with the given observation. Hence, when addressing this inverse-design problem, the first step is to construct a reachable function which is as close as possible to the given noisy observation. This problem can be formulated as a minimum squares problem problem of the form

$$\underset{\varphi_T \in \mathcal{R}_T}{\mathrm{minimize}} \|\varphi_T(\cdot) - u_T(\cdot)\|_{L^2}^2,$$

and is studied in [12] for convex smooth Hamiltonians. Having a good characterization of $\mathcal{R}_T$ is obviously of great interest in order to determine whether existence and uniqueness of minimizers may hold or not, as well as to design optimization algorithms to find a good approximation of the minimizer $\varphi_T^* \in \mathcal{R}_T$.

When H is smooth and uniformly convex, i.e.

$$(4) \qquad H \in C^2(\mathbb{R}^N) \quad \text{and} \quad D^2 H(p) \geq c \, I_N \qquad \text{for some } c > 0,$$

the reachable set $\mathcal{R}_T$ is well-studied, and its characterization can be addressed by utilizing semi-concavity[1] estimates. More precisely, it is well-known that a necessary condition for $u_T \in \mathcal{R}_T$ is given by the following inequality[2] (see [11, 18])

$$(5) \qquad D^2 u_T \leq \frac{(D^2 H(\nabla u_T))^{-1}}{T} \qquad \text{in} \quad \mathbb{R}^N,$$

which is understood in the sense of viscosity solutions. Moreover, for the one-dimensional case in space, and for quadratic Hamiltonians in any space dimension, it is proven in [11, Theorem 2.2] that the semiconcavity inequality (5) is actually optimal, in the sense that (5) is equivalent to $u_T \in \mathcal{R}_T$.

In this work, we aim to give similar results for the case when $H : \mathbb{R}^N \longrightarrow \mathbb{R}$ does not fulfill the hypotheses (4), and is merely assumed to be a convex function. In this general context, where the Hamiltonian is neither smooth nor strictly convex, the viscosity solutions cannot be ensured to be semiconcave, and the (one-sided) regularizing effect of the equation (1) can no longer be expressed by means of differential inequalities such as (5). Nonetheless, we are still able to give a full characterization of the reachable set $\mathcal{R}_T$ by introducing a global condition, which is based on a family of test functions constructed by means of the Legendre-Fenchel transform of the Hamiltonian. As we will see in Theorem 2, for the level set equation ($H(p) = |p|$), this reachability condition

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[1] We recall that a function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be semiconcave if there exists a constant $c \in \mathbb{R}$ such that the function $x \mapsto f(x) - c|x|^2$ is concave.

[2] Here, D^2 stands for the Hessian matrix operator, and the inequality is understood in the usual partial order of symmetric matrices, i.e. $A \leq B$ if and only if $B - A$ is semidefinite positive.

can still be interpreted as a one-sided regularity condition, or semiconcavity condition, not for the solution itself, but for its level sets (see Remark 2).

1.1. **Characterization of the reachable set.** Let us state our first result, which gives a full characterization of the reachable set for the equation (1) when the Hamiltonian is merely assumed to be a convex function. This characterization identifies the functions u_T in $\mathcal{R}_T$ with those functions such that, for any $x \in \mathbb{R}^N$, there exists a function of the form

$$z \longmapsto T H^* \left(\frac{z-x}{T} \right) + c,$$

touching u_T from above at x, where H^* is the Legendre-Fenchel transform of the H. Let us recall that the Legendre-Fenchel transform of H is the function $H^* : \mathbb{R}^N \longrightarrow (-\infty, +\infty]$ defined by

$$(6) \qquad\qquad H^*(q) = \sup_{p \in \mathbb{R}^N} \{p \cdot q - H(p)\}, \qquad \forall q \in \mathbb{R}^N.$$

Note that the function H^* is convex and lower semicontiuous since it is the supremum of convex continuous functions. Note also that $H^*(q)$ may take infinite values whenever H is not superlinear. Indeed, this is the case for $H(p) = |p|$, whose Legendre-Fenchel transform satisfies $H^*(q) = +\infty$ for any $|q| > 1$.

Theorem 1. *Let $H : \mathbb{R}^N \to \mathbb{R}$ be a convex function, $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and $T > 0$. Set the family of functions*

$$\mathcal{F}_T(u_T) := \left\{ \varphi : z \mapsto T H^* \left(\frac{z-x_0}{T} \right) + c \; : \;\; x_0 \in \mathbb{R}^N, \; c \in \mathbb{R} \; s.t. \; \varphi(z) \geq u_T(z) \; \forall z \in \mathbb{R}^N \right\},$$

where H^* is the Legendre-Fenchel transform of H as defined in (6).

Then $u_T \in \mathcal{R}_T$ if and only if for all $x \in \mathbb{R}^N$, there exists $\varphi \in \mathcal{F}_T(u_T)$ such that $\varphi(x) = u_T(x)$.

This characterization is somehow reminiscent of the definition of viscosity subsolution, and can actually be seen as a weaker notion of semiconcavity. Interesting cases are the power-like Hamiltonians of the form

$$(7) \qquad\qquad H(p) = \frac{|p|^\alpha}{\alpha}, \qquad \forall p \in \mathbb{R}^N, \qquad \text{for some } \alpha \in [1, \infty).$$

Note that, except for the quadratic case, $\alpha = 2$, Hamiltonians of the form (7) do not fulfil the hypotheses (4). If we consider $\alpha > 1$, then Theorem 1 implies that for any $T > 0$, $u_T \in \mathcal{R}_T$ if and only if, for any $x \in \mathbb{R}^N$, there exists a function of the form

$$z \longmapsto T \frac{\alpha - 1}{\alpha} \left| \frac{z-x}{T} \right|^{\frac{\alpha}{\alpha-1}} + c$$

touching u_T from above at x. From this observation, one can deduce the following regularity estimate for the functions in $\mathcal{R}_T$. The proof of this corollary is given in subsection 2.4.

Corollary 1. *Let H be of the form (7) for $\alpha > 1$ and $T > 0$. Then, for any $u_T \in \mathcal{R}_T$, the superdifferential of $u_T(x)$ is nonempty for all $x \in \mathbb{R}^N$, i.e. for all $x \in \mathbb{R}^N$ we have that*

$$D^+ u_T(x) := \{q \in \mathbb{R}^N \; : \; \exists \varphi \in C^1(\mathbb{R}^N) \;\; u_T - \varphi \leq 0, \; u_T(x) - \varphi(x) = 0 \; \nabla\varphi(x) = q\} \neq \emptyset.$$

Moreover, the following inequalities hold true:

(i) *If $1 < \alpha < 2$, then the superdifferential*

$$D^2 u_T(x) \leq \frac{\text{Lip}(u_T)^{2-\alpha}}{(\alpha - 1)T} I_N \qquad \forall x \in \mathbb{R}^N,$$

where $\text{Lip}(u_T)$ *stands for the Lipschitz constant of* u_T.

(ii) *If $\alpha > 2$, then*

$$D^2 u_T(x) \leq \frac{1}{(\alpha - 1)T \delta_x^{\alpha-2}} I_N \qquad \forall x \in \mathbb{R}^N \quad s.t. \quad \delta_x := \inf_{q \in D^+ u_T(x)} |q| > 0.$$

Remark 1.   (i) *From the statement (i) in the previous Corollary, we deduce that for $\alpha \in (1, 2)$ a necessary condition for $u_T \in \mathcal{R}_T$ is that u_T has to be semiconcave with a constant depending on T and the Lipschitz constant of u_T.*

(ii) *From the statement (ii) we can only deduce a weaker semiconcavity estimate for the regime $\alpha > 2$. More precisely, a semiconcavity estimate only holds at points x which are not critical points of u_T.*

(iii) *In addition, we observe that if x is a local maximum of u_T, then it holds that $D^2 u_T(x) \leq 0$. Hence, for the case $\alpha > 2$, we can slightly improve the result by saying that if $u_T \in \mathcal{R}_T$, then u_T is semiconcave at all points $x \in \mathbb{R}^N$ except for the critical points which are not local maxima (i.e. local minima and saddle points).*

Let us now look at the limit case $\alpha = 1$, i.e. when H is given by

$$(8) \qquad\qquad\qquad\qquad H(p) = |p|, \qquad \forall p \in \mathbb{R}^N,$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^N$. Note that, in this case, H is neither differentiable nor strictly convex, and this brings us to a quite different situation as compared to the regular strictly convex case $\alpha > 1$. The equation (1) with H given by (8) is also known as the level-set equation [20, 21] and is often used to describe the propagation of fronts, evolving in time, as the level sets of the viscosity solution to (1).

In our following result, we will see that, when H is given by (8), the reachable target $\mathcal{R}_T$ can be characterized by means of the following interior ball condition on the sublevel sets of u_T.

Definition 1. *Let $\Omega \subset \mathbb{R}^N$ be a closed set. We say that Ω satisfies the interior ball condition with radius $r > 0$ if for all $x \in \Omega$, there exists $y \in \Omega$ such that*

$$\overline{B(y, r)} \subset \Omega \qquad and \qquad x \in \overline{B(y, r)}.$$

We can now state the following theorem.

Theorem 2. *Let $u_T \in \text{Lip}(\mathbb{R}^N)$, $H(p) = |p|$ and $T > 0$. Then $u_T \in \mathcal{R}_T$ if and only if for all $\alpha \in \mathbb{R}$, the α-sublevel set defined as*

$$\Omega_\alpha(u_T) := \{x \in \mathbb{R}^N ; \quad u_T(x) \leq \alpha\}$$

satisfies the interior ball condition of Definition 1 with radius $r = T$.

Remark 2.   (i) *Recall that the convexity (resp. concavity) of a set can be characterized by the non-negativity (resp. non-positivity) of the curvature of its boundary. Taking this into account, we see that the interior ball condition of Theorem 2 implies that the curvature of the boundary of any sub-level set of u_T is bounded from above. Hence, the characterization of the reachable set $\mathcal{R}_T$ given in Theorem 2 can be seen as a semiconcavity condition on*
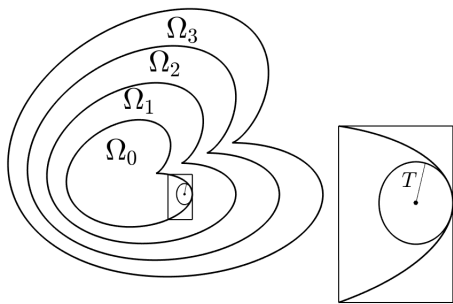
FIGURE 1. The sub-level sets of a function $u_T \in \mathrm{Lip}(\mathbb{R}^2)$ satisfying the interior ball condition from Theorem 2. The region of greatest curvature on the boundary of the sublevel set 0 is zoomed down in the box at the right.

the sublevel sets of u_T. In this case, the regularizing effect of the Hamilton-Jacobi equation is not observed on the solution, but rather on its sub-level sets.

(ii) We point out that the condition of Theorem 2 is indeed a one-sided regularity estimate for the boundary of the sub-level sets. As a matter of fact, the boundary needs not be smooth in general, and might contain corners, which, in view of the interior ball condition, will always be pointing towards the interior of the sub-level set. See Figure 1 for an illustration.

In the one-dimensional case in space, it is sufficient to check the interior ball condition on the local minima of u_T, and then, the above result can be formulated simply as follows:

Corollary 2. *Consider the one-dimensional case $N = 1$, and let $u_T \in \mathrm{Lip}(\mathbb{R})$, $H(p) = |p|$ and $T > 0$. Then, $u_T \in \mathcal{R}_T$ if and only if for any local minimum $x \in \mathbb{R}$ of u_T, there exists $x_0 \in \mathbb{R}$ such that $x \in [x_0 - T, \, x_0 + T]$ and $u_T(y) \leq u_T(x)$ for all $y \in [x_0 - T, \, x_0 + T]$.*

See Figure 2 for an illustration of this characterization.

Remark 3. *In section 2, we shall prove in Corollary 3 that, as a consequence of Theorem 1, the concave functions satisfy the property of being reachable for all positive times $T > 0$. However, from Corollary 2, we can deduce that for the Hamiltonian $H(p) = |p|$, the concave functions are not the only ones satisfying this property. Indeed, if $u_T : \mathbb{R} \to \mathbb{R}$ is monotonically increasing or decreasing, the reachability condition from Corollary 2 is trivially satisfied, and then $u_T \in \mathcal{R}_T$ for all $T > 0$. Hence, monotone functions are in $\mathcal{R}_T$ for all T no matter they are concave or not. We recall that in the smooth strictly convex case, it follows from the necessary condition (5), that $u_T \in \mathcal{R}_T$ for all $T > 0$ if and only if u_T is concave.*

1.2. **Structural properties of the reachable set.** As a by-product of the characterization of $\mathcal{R}_T$ given in Theorem 1, we can also prove some results concerning the structural properties of the set of reachable functions $\mathcal{R}_T$ for the Hamilton-Jacobi equation (1). The precise statements of these results are given in subsection 2.1, and their proofs in subsection (2.4).

(i) The reachable set is decreasing in time, i.e. $\mathcal{R}_T \subset \mathcal{R}_{T'}$ for all $0 < T' < T$, and concave functions are reachable for all $T > 0$. See Corollary 3.
(ii) The minimum of two reachable functions is reachable. See Corollary 4.
(iii) If $H(p) = |p|^\alpha$, with $\alpha \geq 1$, then the reachable set $\mathcal{R}_T$ is star-shaped with center at the origin. See Corollary 5.
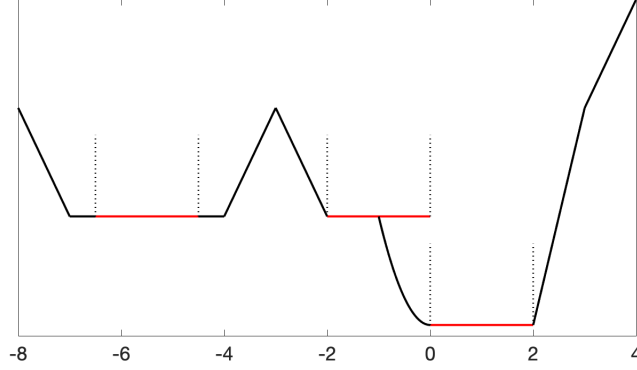
FIGURE 2. An example of function satisfying $u_T \in \mathcal{R}_T$ for the Hamiltonian $H(p) = |p|$ and $T = 1$. In view of Corollary 2, we only need to check the interior ball condition at the local minima of u_T. For each local minimum x of u_T in this plot, we have coloured in red a ball of radius 1 around x, in which u_T is smaller or equal than $u_T(x)$.

(iv) If $H(p) = |p|^2$, then $\mathcal{R}_T$ is convex, and if $H(p) = |p|$, then $\mathcal{R}_T$ is a non-convex cone with vertex at the origin. See Corollary 5.

1.3. **Reachable set for scalar conservation laws.** In the one-dimensional case in space, it is well-known that Hamilton-Jacobi equations and scalar conservation laws of the form

$$(9) \qquad \partial_t v + \partial_x[H(v)] = 0 \qquad \text{in } (0,T) \times \mathbb{R}$$

are intimately related. Indeed, if $u \in \mathrm{Lip}([0,T] \times \mathbb{R})$ is a viscosity solution to (1) with initial condition $u_0 \in \mathrm{Lip}(\mathbb{R})$, then the function $v \in L^\infty((0,T) \times \mathbb{R})$ given by

$$v(t,x) = \partial_x u(t,x) \qquad \text{for a.e. } (t,x) \in [0,T] \times \mathbb{R}$$

is the unique entropy solution to (9) with initial condition $v_0 = \partial_x u_0$ (see for instance [14, Theorem 1.1] and also [5, 6]).

In this section, we adapt the previous results to give a full characterization of the range of the operator

$$(10) \qquad \begin{array}{rccc} S_T^{SCL}: & L^\infty(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}) \\ & v_0 & \longmapsto & v(T,\cdot), \end{array}$$

which associates, to any initial condition v_0, the unique entropy solution [9, 15, 22] to the equation (9) at time T. We also define, for any $T > 0$, the reachable set for (9) as

$$(11) \qquad \mathcal{R}_T^{SCL} := \{v_T \in L^\infty(\mathbb{R}) \ : \ \exists v_0 \in L^\infty(\mathbb{R}^N) \text{ such that } S_T^{SCL} v_0 = v_T\} \subset L^\infty(\mathbb{R}),$$

For the scalar conservation law (9) with a flux H satisfying (4), it is well-known [6, 8, 10, 13] that for any $T > 0$ and $v_T \in L^\infty(\mathbb{R})$, the property $v_T \in \mathcal{R}_T^{SCL}$ is equivalent to the one-sided-Lipschitz condition

$$(12) \qquad \partial_p H(v(t,y)) - \partial_p H(v(t,x)) \leq \frac{y-x}{t} \qquad \text{for a.e. } x \leq y.$$

In the general convex case, in which H is not necessarily differentiable nor strictly convex, the one-sided-Lipschitz inequality (12) does not hold in general. Nonetheless, we can adapt Theorem 1 in the following way to give a full characterization of the functions in $\mathcal{R}_T^{SCL}$.

Theorem 3. *Let $H : \mathbb{R} \to \mathbb{R}$ be a convex function, $v_T \in L^\infty(\mathbb{R})$ and $T > 0$. Then $v_T \in \mathcal{R}_T^{SCL}$ if and only if*

(13)
$$
\begin{array}{c}
\text{for all } x \in \mathbb{R}, \text{ there exists } x_0 \in \mathbb{R} \text{ such that the function} \\
z \longmapsto \int_0^z v_T(y)dy - T\, H^*\left(\frac{z - x_0}{T}\right) \text{ has a global maximum at } x.
\end{array}
$$

Sharp one-sided regularity estimates are for power-like fluxes of the form $|p|^\alpha$ with $\alpha > 1$ are given in [10]. The limit case $\alpha = 1$ is again different since H is no longer differentiable. The following theorem provides a full characterization of the functions in $\mathcal{R}_T^{SCL}$, when the flux is the absolute value.

Theorem 4. *Let $v_T \in L^\infty(\mathbb{R})$, $H(p) = |p|$ and $T > 0$. Then, $v_T \in \mathcal{R}_T^{SCL}$ if and only if*

(14)
$$
\operatorname{sgn}(v_T(y)) - \operatorname{sgn}(v_T(x)) \leq \frac{y - x}{T} \qquad \text{for a.e. } x, y \in \operatorname{supp}(v_T) \text{ satisfying } x \leq y.
$$

Here, the sign function $\operatorname{sgn} : \mathbb{R} \setminus \{0\} \to \{-1, 1\}$ is defined as

$$
\operatorname{sgn}(z) := \left\{ \begin{array}{ll} -1 & \text{if } z < 0 \\ 1 & \text{if } z > 0. \end{array} \right.
$$

The above result must be interpreted as follows: in order for v_T to be reachable, any sign change, from negative to positive, must be separated by an interval of length $2T$ where v_T vanishes. More precisely, if we define the supports of the positive and negative parts of v_T

$$
A_+ = \{x \in \mathbb{R};\ v_T(x) > 0\} \qquad \text{and} \qquad A_- = \{x \in \mathbb{R};\ v_T(x) < 0\}.
$$

then it must hold that

$$
y - x \geq 2T, \qquad \text{for a.e. } x \in A_- \text{ and for a.e. } y \in A_+ \text{ with } x \leq y.
$$

See Figure 3 for an illustration of a function v_T satisfying this property.

The rest of the paper is structured as follows. Section 2 is devoted to Hamilton-Jacobi equations. In subsection 2.1, we present some corollaries concerning the structural properties of $\mathcal{R}_T$, that can be deduced from Theorem 1. Then, in subsection 2.2 we give some prelimiaries about Hamilton-Jacobi equations and the Hopf-Lax formula which are then used in subsections 2.3 and 2.4 to prove our results. In section 3, we prove the characterization of the reachable set given in Theorem 3 for scalar conservation laws (9) with general convex flux, and we also prove Theorem 4 for the case when the flux is the absolute value. Finally, we conclude the paper with a section describing our conclusions and presenting a couple of open questions.
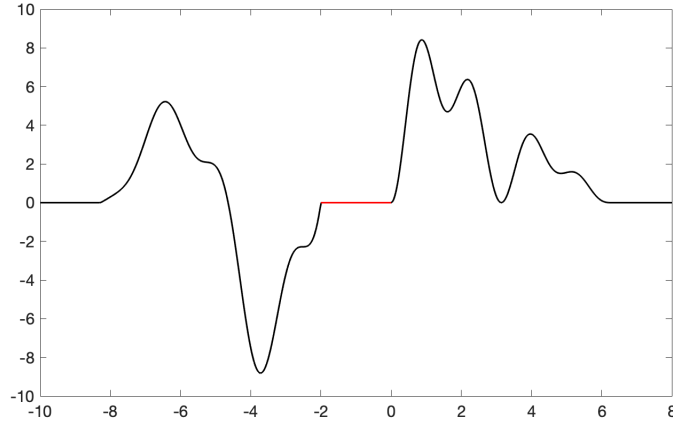
FIGURE 3. An example of a function $v_T \in \mathcal{R}_T^{SCL}$ for the flux $H(p) = |p|$ and $T = 1$.

2. HAMILTON-JACOBI EQUATIONS

In this section, we deal with Hamilton-Jacobi equations of the form (1) with a convex Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$, and without making any further regularity assumptions. As announced in the introduction, for a given $T > 0$, our main goal is to prove the full characterization (necessary and sufficient condition) given in Theorem 1 for the reachable set $\mathcal{R}_T$, defined as in (3), and also prove its main properties. Before addressing the proofs of our results, let us state in the following subsection the results concerning the structural properties of $\mathcal{R}_T$, that can be deduced from Theorem 1.

2.1. **Reachable set: main properties.** Theorem 1 has some interesting consequences, revealing information about the structure of the reachable set $\mathcal{R}_T$, and the way it evolves as we increase the time horizon T. The following result ensures that the reachable set decreases as T increases, and that concave functions have the property of being reachable for all $T > 0$. The corollary is proved in subsection 2.4.

Corollary 3. *Let $H : \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then,*

$$\text{for any} \quad 0 < T' < T, \quad \text{we have} \quad \mathcal{R}_T \subset \mathcal{R}_{T'}.$$

Moreover, if $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ is a concave function, then, $u_T \in \mathcal{R}_T$ for all $T > 0$.

Remark 4. *Corollary 3 states that concavity is a sufficient condition for a function to be reachable for all $T > 0$. However, it is not necessary in general. Indeed, it can be proved (see Remark 3) that, if one considers the one-dimensional case with the Hamiltonian given by $H(p) = |p|$, any globally Lipschitz monotone (increasing or decreasing) function $u_T : \mathbb{R} \to \mathbb{R}$ is reachable for all $T > 0$, even if it is not concave. It differs from the smooth uniformly convex case (4), where, due to the necessary condition (5), a function is reachable for all $T > 0$ if and only if it is a concave function.*

Another interesting consequence of Theorem 1 is the following corollary, which roughly says that the minimum of two reachable targets is reachable. The proof of this corollary is omitted as it is a straightforward consequence of Theorem 1.
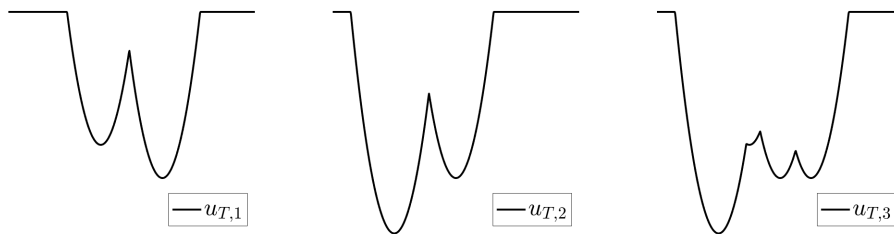
FIGURE 4. Three examples of reachable targets in time $T = 1$, with compact support, for the Hamiltonian $H(p) = |p|^2/2$. The first two examples were constructed using the statement (ii) in Corollary 4, whereas the third one was constructed using the statement (i) as the minimum of the two first examples.

Corollary 4. *Let $T > 0$, let $H : \mathbb{R}^N \to \mathbb{R}$ be a convex function, and let H^* be its Legendre-Fenchel transform as defined in* (6)*. Then, the following statements hold true.*

(i) *For any $u_T, v_T \in \mathcal{R}_T$, the function $w_T(x) := \min\{u_T(x), v_T(x)\}$ satisfies $w_T \in \mathcal{R}_T$.*

(ii) *If in addition H^* is locally Lipschitz, then for any $u_T \in \mathcal{R}_T$, $x_0 \in \mathbb{R}^N$ and $c \in \mathbb{R}$, the function*

$$w_T(x) := \min\left\{u_T(x),\ T\,H^*\left(\frac{x - x_0}{T}\right) + c\right\}$$

satisfies $w_T \in \mathcal{R}_T$.

Note that in (ii), the assumption of H^* being a locally Lipschitz continuous function is needed to guarantee that $w_T \in \mathrm{Lip}(\mathbb{R}^N)$. Corollary 4 provides, in particular, a simple method to construct reachable functions with compact support when H^* is locally Lipschitz. Note that the zero function is reachable for any $T > 0$. Then, for any given finite set $\{(x_i, c_i)\}_{i=1}^k \subset \mathbb{R}^N \times \mathbb{R}$, we can define the function

$$u_T(x) := \min\left\{0,\ T\,H^*\left(\frac{x - x_1}{T}\right) + c_1,\ \ldots,\ T\,H^*\left(\frac{x - x_k}{T}\right) + c_k\right\}$$

which, in view of Corollary 4, satisfies $u_T \in \mathcal{R}_T$. Of course, the method can readily be applied to larger collections of points $\{(x_i, c_i)\}_{\mathcal{I}} \subset \mathbb{R}^N \times \mathbb{R}$, under the assumption of c_i being uniformly bounded from below. See Figure 4 for an illustration of this result.

The last property about the reachable set $\mathcal{R}_T$ that we are going to present as a consequence of Theorem 1 applies to power-like Hamiltonians of the form (7). The following corollary ensures that the reachable set $\mathcal{R}_T$ is *star-shaped* with center the origin, i.e.

(15) $$\forall u_T \in \mathcal{R}_T \quad \text{and} \quad \forall \lambda \in [0,1], \quad \text{we have} \quad \lambda u_T \in \mathcal{R}_T.$$

For the particular case $\alpha = 2$, the set $\mathcal{R}_T$ is additionally convex, and if $\alpha = 1$, then $\mathcal{R}_T$ is actually a non-convex cone with vertex at the origin, i.e.

$$(16) \qquad\qquad \forall u_T \in \mathcal{R}_T \quad \text{and} \quad \forall \lambda \in [0, \infty), \quad \text{we have} \quad \lambda u_T \in \mathcal{R}_T.$$

Corollary 5. *Let $T > 0$ and let $H : \mathbb{R}^N \to \mathbb{R}$ be given by (7) for some $\alpha \in [1, \infty)$. Then the reachable set $\mathcal{R}_T$ is star-shaped with center the origin, i.e. (15) holds. Moreover, if $\alpha = 2$, then $\mathcal{R}_T$ is convex, and if $\alpha = 1$, then $\mathcal{R}_T$ is a cone with vertex at the origin.*

The proof of the corollary is given in subsection 2.4.

2.2. **Preliminaries.** Let us recall some elementary facts about viscosity solutions to Hamilton-Jacobi equations of the form (1) that are well-known in the literature and will be used throughout our proofs. Let us recall from (2) in the introduction that, for any $T > 0$, the (forward) viscosity operator S_T^+ associates, to any initial condition u_0, the viscosity solution to (1) at time $t = T$. It is well-known that the viscosity solution to (1) can be given by the so-called Hopf-Lax formula (see for instance [1, 2, 3]). Then, for any $T > 0$, the operator S_T^+ can be explicitly defined as

$$(17) \qquad\qquad S_T^+ u_0(x) = \min_{y \in \mathbb{R}^N} \left\{ u_0(y) + T H^* \left(\frac{x - y}{T} \right) \right\},$$

where H^* is defined as in (6).

The simplest way to characterize the reachable set $\mathcal{R}_T$, which actually applies to more general Hamiltonians of the form $H(x, p)$, is to perform a backward-forward resolution of (1), by means of the so-called *backward viscosity operator* (see [4])

$$\begin{aligned} S_T^- : \quad & \mathrm{Lip}(\mathbb{R}^N) \quad \longrightarrow \quad & \mathrm{Lip}(\mathbb{R}^N) \\ & u_T \quad \longmapsto \quad & S_T^- u_T = w(0, \cdot), \end{aligned}$$

where $w \in \mathrm{Lip}([0, T] \times \mathbb{R}^N)$ is the unique *backward viscosity solution* to (1) with terminal condition u_T. We recall that $w \in \mathrm{Lip}([0, T] \times \mathbb{R}^N)$ is a backward viscosity solution to (1) if and only if the function $v(t, c) := w(T - t, x)$ is a forward viscosity solution to

$$\partial_t v - H(D_x v) = 0 \qquad \text{in } [0, T] \times \mathbb{R}^N.$$

As well as for the forward viscosity solutions, existence, uniqueness and stability of backward viscosity solutions for the terminal value problem associated to the Hamilton-Jacobi equation (1) can be proved by means of the vanishing viscosity method, i.e. the backward viscosity solution can be obtained as the limit when $\varepsilon \to 0^+$ of the solution u_ε to the terminal value problem

$$\begin{cases} \partial_t u_\varepsilon + \varepsilon \Delta u_\varepsilon + H(D_x u_\varepsilon) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u_\varepsilon(T, \cdot) = u_T & \text{in } \mathbb{R}^N. \end{cases}$$

Let us now recall the reachability condition for the initial-value problem (1) which, for any $T > 0$, identifies the reachable targets in time T with the fixed points of the composition operator $S_T^+ \circ S_T^-$. Under the assumption of $H : \mathbb{R}^N \to \mathbb{R}$ being a convex function and $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, we have that

$$(18) \qquad\qquad u_T \in \mathcal{R}_T \quad \text{if and only if} \quad S_T^+ \circ S_T^- u_T = u_T$$

The proof of (18) is exactly the same as the one of [11, Theorem 2.1], which is a direct consequence of [11, Proposition 4.7] (see also [4, 19]), and we omit the proof here.

As well as for the forward viscosity solutions, there is a Hopf-Lax formula for the backward viscosity solutions to (1) with terminal condition $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, which reads as

$$(19) \qquad S_T^- u_T(x) = \max_{y \in \mathbb{R}^N} \left\{ u_T(y) - T\, H^* \left(\frac{y-x}{T} \right) \right\}.$$

Let us finish the subsection with the proof of the following elementary property of H^*, which will be used in the sequel.

Lemma 1. *Let $H : \mathbb{R}^N \to \mathbb{R}$ be a convex function and let H^* be its Legendre-Fenchel transform. Then, for any constant $C > 0$, we have*

$$H^*(q) \geq C\,|q| - \max_{p \in \overline{B(0,C)}} H(p) \qquad \forall q \in \mathbb{R}^N,$$

where $\overline{B(0,C)}$ is the closure of the ball of radius C centered at the origin.

Proof. Let $C > 0$ be any positive constant. Since H is convex and takes values in $\mathbb{R}$, we deduce that H is continuous, and then we have

$$\max_{p \in \overline{B(0,C)}} H(p) < \infty.$$

Now, using the definition of H^* in (6), for any $q \in \mathbb{R}^N \setminus \{0\}$, we can take $p = C\frac{q}{|q|} \in \overline{B(0,C)}$ and then deduce that

$$H^*(q) \geq C|q| - H\left(C\frac{q}{|q|} \right) \geq C|q| - \max_{p \in \overline{B(0,C)}} H(p).$$

Similarly, for $q = 0$, by taking $p = 0$ we obtain

$$H^*(q) \geq -H(0) \geq C|q| - \max_{p \in \overline{B(0,C)}} H(p).$$

$\square$

2.3. **Proof of Theorems 1 and 2.** We start with the proof of Theorem 1.

Proof of Theorem 1. Let $u_T \in \mathcal{R}_T$ be a reachable target. By (18), we have that, for all $x \in \mathbb{R}^N$, there exists $x_0 \in \mathbb{R}^N$ such that

$$u_T(x) = S_T^- u_T(x_0) + T\, H^* \left(\frac{x - x_0}{T} \right).$$

Using the definition of S_T^- in (19), we deduce that

$$S_T^- u_T(x_0) \geq u_T(z) - T\, H^* \left(\frac{z - x_0}{T} \right) \qquad \forall z \in \mathbb{R}^N.$$

Hence, by setting $c = S_T^- u_T(x_0)$, we obtain that the function

$$\varphi(z) = c + T\, H^* \left(\frac{z - x_0}{T} \right)$$

satisfies $\varphi \in \mathcal{F}_T(u_T)$ and $\varphi(x) = u_T(x)$.

For the reverse implication, let us first prove that, for any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, it holds that

(20) $$u_T(x) \le S_T^+ \circ S_T^- u_T(x) \qquad \forall x \in \mathbb{R}^N.$$

In view of (19), we have

$$S_T^- u_T(y) \ge u_T(x) - T\,H^*\left(\frac{x-y}{T}\right) \qquad \forall x, y \in \mathbb{R}^N,$$

which implies that

$$u_T(x) \le \min_{y \in \mathbb{R}^N} \left\{ S_T^- u_T(y) + T\,H^*\left(\frac{x-y}{T}\right) \right\} = S_T^+ \circ S_T^- u_T(x) \qquad \forall x \in \mathbb{R}^N.$$

Now, let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ be such that, for all $x \in \mathbb{R}^N$, there exists $\varphi \in \mathcal{F}_T(u_T)$ satisfying $\varphi(x) = u_T(x)$. This means that there exist $x_0 \in \mathbb{R}^N$ and $c \in \mathbb{R}$ such that

$$u_T(x) = c + T\,H^*\left(\frac{x-x_0}{T}\right)$$

and

$$u_T(z) \le c + T\,H^*\left(\frac{z-x_0}{T}\right) \qquad \forall z \in \mathbb{R}^N.$$

This in particular implies, as a consequence of (19), that $c = S_T^- u_T(x_0)$. Hence, using (17) we deduce that

$$
\begin{aligned}
S_T^+ \circ S_T^- u_T(x) &= \min_{y \in \mathbb{R}^N} \left\{ S_T^- u_T(y) + T\,H^*\left(\frac{y-x}{T}\right) \right\} \\
&\le S_T^- u_T(x_0) + T\,H^*\left(\frac{x_0-x}{T}\right) = u_T(x).
\end{aligned}
$$

Combining this inequality with (20) we deduce that $S_T^+ \circ S_T^- u_T(x) = u_T(x)$ for all $x \in \mathbb{R}^N$, and then we can use the general reachability criterion (18) to deduce that $u_T \in \mathcal{R}_T$. □

Let us now prove Theorem 2 using the conclusion of Theorem 1.

Proof of Theorem 2. Note first of all that the Legendre-Fenchel transform of $H(p) = |p|$ is given by

(21) $$H^*(q) = \begin{cases} 0 & \text{if } |q| \le 1 \\ +\infty & \text{if } |q| > 1. \end{cases}$$

In view of the form of H^*, the functions in $\mathcal{F}_T$ defined in the statement of Theorem 1 are simply functions which are constant in a ball of radius T and infinity elsewhere. Therefore, the reachability condition from Theorem 1, in this case, reads as follows:

(22) $\quad \forall x \in \mathbb{R}^N, \quad \exists x_0 \in \mathbb{R}^N \quad \text{such that} \quad x \in \overline{B(x_0, T)} \quad \text{and} \quad u_T(y) \le u_T(x) \quad \forall y \in \overline{B(x_0, T)}.$

It is easy to prove that this property is equivalent to the interior ball condition from Definition 1 with $r = T$. Let us first assume that (22) holds. Then, for any $\alpha \in \mathbb{R}$ and $x \in \Omega_\alpha(u_T)$, we have that there exists a ball $\overline{B(x_0, T)}$ containing x such that

$$u_T(y) \le u_T(x) \le \alpha, \qquad \forall y \in \overline{B(x_0, T)},$$

which implies that $\overline{B(x_0, T)} \subset \Omega_\alpha(u_T)$.

On the other hand, if the interior ball condition holds with $r = T$, then for any $x \in \mathbb{R}^N$ we have that $x \in \Omega_\alpha(u_T)$ with $\alpha = u_T(x)$. Hence, by the interior ball condition, there exists $x_0 \in \Omega_\alpha(u_T)$ such that $x \in \overline{B(x_0, T)} \subset \Omega_\alpha(u_T)$, which then implies that

$$u_T(y) \leq \alpha = u_T(x) \qquad \forall y \in \overline{B(x_0, T)}.$$

$\square$

2.4. **Proof of Corollaries 1, 3 and 5.** We start by proving the regularity result given in Corollary 1 for power-like Hamiltonians.

Proof of Corollary 1. We start by noticing that, since $H(p) = |p|^\alpha$ for some $\alpha > 1$, the its Legendre–Fenchel transform is given by

$$H^*(q) = \frac{\alpha - 1}{\alpha} |q|^{\frac{\alpha}{\alpha - 1}}.$$

Then, a straightforward computation gives the following:

$$(23) \qquad \nabla H^*(q) = |q|^{\frac{2-\alpha}{\alpha-1}} q \qquad \forall q \in \mathbb{R}^N,$$

and

$$(24) \qquad D^2 H^*(q) \leq \frac{1}{\alpha - 1} |q|^{\frac{2-\alpha}{\alpha-1}} I_N, \qquad \forall q \in \mathbb{R}^N.$$

Now, from Theorem 1, we have that if $u_T \in \mathcal{R}_T$, then for any $x \in \mathbb{R}^N$ there exists a function $\varphi : \mathbb{R}^N \to \mathbb{R}$ of the form

$$\varphi(z) := u_T(z) - T H^* \left(\frac{z - x_0}{T} \right) - c$$

for some $x_0 \in \mathbb{R}^N$ and $c \in \mathbb{R}$ such that $\varphi(\cdot)$ attains its maximum at x.

This implies that $0 \in D^+ \varphi(x)$, which then implies, using (23), that

$$(25) \qquad \nabla H^* \left(\frac{x - x_0}{T} \right) = \left| \frac{x - x_0}{T} \right|^{\frac{2-\alpha}{\alpha-1}} \frac{x - x_0}{T} \in D^+ u_T(x).$$

It then follows that $\overline{D^+ u_T(x)} \neq \emptyset$ for all $x \in \mathbb{R}^N$.

Let us now prove the semiconcavity inequalities. Since $\varphi(\cdot)$ attains its maximum at x, we have that its Hessian matrix at x is semidefinite negative, i.e. $D^2 \varphi(x)$. Then, using (24) we obtain that

$$(26) \qquad \begin{aligned} D^2 u_T(x) &\leq \frac{1}{T} D^2 H^* \left(\frac{x - x_0}{T} \right) \\ &\leq \frac{1}{\alpha - 1} \frac{|x - x_0|^{\frac{2-\alpha}{\alpha-1}}}{T^{\frac{1}{\alpha-1}}} I_N. \end{aligned}$$

We now need to use an estimate for the quantity $|x - x_0|$, taking into account that the exponent $\frac{2-\alpha}{\alpha-1}$ has different sign depending whether $\alpha \in (1, 2)$ or $\alpha > 2$.

If $\alpha \in (1,2)$, we can use (25), and the Lipschitz constant of u_T, that we denote by $\text{Lip}(u_T)$, to deduce that

$$|x - x_0| \leq T \, \text{Lip}(u_T)^{\alpha-1}.$$

Note that $u_T \in \text{Lip}(\mathbb{R}^N)$ implies that $|q| \leq \text{Lip}(u_T)$ for all $q \in D^+ u_T(x)$, and for all $x \in \mathbb{R}^N$.

Hence, combining the above inequality with (26), along with the fact that the exponent $\frac{2-\alpha}{\alpha-1}$ is positive, we deduce that

$$D^2 u_T(x) \leq \frac{\text{Lip}(u_T)^{2-\alpha}}{(\alpha-1)T} I_N.$$

Let us now assume that $\alpha > 2$. If we have

$$\delta_x := \inf_{q \in D^+ u_T(x)} |q| > 0,$$

we can deduce from (25) that

$$|x - x_0| \geq T \, \delta_x^{\alpha-1}.$$

Hence, combining this with (26), and the fact that the exponent $\frac{2-\alpha}{\alpha-1}$ is negative, we deduce that

$$D^2 u_T(x) \leq \frac{1}{(\alpha-1)\delta_x^{\alpha-2} T} I_N.$$

$\square$

Let us now prove Corollary 3.

Proof of Corollary 3. The fact that the reachable set $\mathcal{R}_T$ decreases in time is a direct consequence of the semigroup property of S_T^+. Indeed, if $u_T \in \mathcal{R}_T$, then there exists $u_0 \in \text{Lip}(\mathbb{R}^N)$ such that $S_T^+ u_0 = u_T$. Then, for any $T' \in (0, T)$, consider the initial condition

$$\tilde{u}_0(x) = S_{T-T'}^{HJ} u_0(x).$$

By the semigroup property we have that

$$S_{T'}^{HJ} \tilde{u}_0(x) = S_{T'}^{HJ}(S_{T-T'}^{HJ} u_0(x)) = S_T^+ u_0(x) = u_T(x),$$

implying that $u_T \in \mathcal{R}_{T'}$. This proves that $\mathcal{R}_T \subset \mathcal{R}_{T'}$.

Let us now prove the second part of the Corollary. Let $u_T \in \text{Lip}(\mathbb{R}^N)$ be concave and fix any $T > 0$. In view of Theorem 1, it suffices to prove that, for all $x \in \mathbb{R}^N$, there exists $x_0 \in \mathbb{R}^N$ and $c \in \mathbb{R}$ such that

(27) $\qquad u_T(x) = T \, H^* \left(\frac{x - x_0}{T} \right) + c \quad \text{and} \quad u_T(z) \leq T \, H^* \left(\frac{z - x_0}{T} \right) + c \quad \forall z \in \mathbb{R}^N.$

Since u_T is concave, for any $x \in \mathbb{R}^N$, there exists $p_0 \in \mathbb{R}^N$ such that

(28) $\qquad\qquad\qquad u_T(z) \leq p_0 \cdot (z - x) + u_T(x) \qquad \forall z \in \mathbb{R}^N.$

On the other hand, it is well-known that the convex conjugate of a convex function is convex and lower semi-continuous. This, combined with the superlinearity of H^* proved in Lemma 1, implies the existence of $q_0 \in \mathbb{R}^N$ satisfying

$$H^*(q_0) - p_0 \cdot q_0 = \min_{q \in \mathbb{R}^N} \{H^*(q) - p_0 \cdot q\} > -\infty,$$

and then we have

(29) $$H^*(q) \geq p_0 \cdot (q - q_0) + H^*(q_0) \qquad \forall q \in \mathbb{R}^N.$$

Set $x_0 := x - Tq_0$. For any $z \in \mathbb{R}^N$, we can plug $q = \frac{z - x_0}{T}$ into (29) and multiply by T to obtain

$$T\,H^*\left(\frac{z - x_0}{T}\right) \geq p_0 \cdot (z - x) + T\,H^*(q_0).$$

Finally, combining this inequality with (28), we deduce that

$$u_T(z) \leq T\,H^*\left(\frac{z - x_0}{T}\right) - T\,H^*(q_0) + u_T(x), \quad \forall z \in \mathbb{R}^N.$$

By the choice of x_0, we observe that the above inequality is actually an equality for $z = x$. Then (27) follows with $c = u_T(x) - T\,H^*(q_0)$, and the corollary is proved. $\qquad\square$

We end the section with the proof of Corollary 5.

Proof of Corollary 5. We start with the cases $\alpha = 2$ and $\alpha = 1$. The case $\alpha = 2$ follows directly from the characterization of the $\mathcal{R}_T$ given by the semiconcavity condition (5), which in this case reads as

$$D^2 u_T \leq \frac{I_N}{T}, \qquad \text{in the viscosity sense.}$$

Note that if u_T and v_T both satisfy this inequality, then so does $\lambda u_T + (1 - \lambda)v_T$ for all $\lambda \in [0, 1]$.

The case $\alpha = 1$ follows from Theorem 2. Indeed, for any $u_T \in \mathcal{R}_T$ and $\lambda > 0$ we have that, for all $\gamma \in \mathbb{R}$ the sublevel set $\Omega_\gamma(\lambda u_T)$ is given by

$$\Omega_\gamma(\lambda u_T) = \{x \in \mathbb{R}^N ; \quad \lambda u_T(x) \leq \gamma\} = \Omega_{\gamma/\lambda}(u_T),$$

which satisfies the interior ball condition with radius $r = T$ since u_T is reachable.

Let us now consider $\alpha \in (1, \infty)$. For any $u_T \in \mathcal{R}_T$ and $\lambda \in (0, 1)$, we shall check that λu_T satisfies the reachability condition from Theorem 1. First of all note that, since H is of the form (7), its Legendre-Fenchel H^* transform is given by

$$H^*(q) = \frac{\alpha - 1}{\alpha} |q|^{\frac{\alpha}{\alpha - 1}}$$

Since $u_T \in \mathcal{R}_T$, for any $x \in \mathbb{R}^N$, there exists $x_0 \in \mathbb{R}^N$ and $c \in \mathbb{R}$ such that the function

$$\phi(z) = \frac{\alpha - 1}{\alpha} T^{-\frac{1}{\alpha - 1}} |z - x_0|^{\frac{\alpha}{\alpha - 1}} + c$$

satisfies

$$\phi(x) = u_T(x) \qquad \text{and} \qquad \phi(z) \geq u_T(z) \qquad \forall \in \mathbb{R}^N.$$

Hence, the function $\psi(z) = \lambda\phi(z)$ satisfies

$$\psi(x) = \lambda u_T(x) \qquad \text{and} \qquad \psi(z) \geq \lambda u_T(z) \qquad \forall \in \mathbb{R}^N.$$

Now, since $\psi(z)$ can be written as

$$\psi(z) = \frac{\alpha - 1}{\alpha}\left(\frac{T}{\lambda^{\alpha-1}}\right)^{-\frac{1}{\alpha-1}}|z - x_0|^{\frac{\alpha}{\alpha-1}} + \lambda c,$$

we deduce, from Theorem 1, that $\lambda u_T \in \mathcal{R}_{T'}$ with $T' = \dfrac{T}{\lambda^{\alpha-1}}$. Note that $\alpha > 1$ and $\lambda \in (0,1)$ imply that $T' > T$. Finally, since the reachable set is decreasing in time (see Corollary 3), we conclude that $\lambda u_T \in \mathcal{R}_T$. $\qquad\square$

3. SCALAR CONSERVATION LAWS

In this section we prove the results given in Theorem 3 and 4 concerning the characterization of the reachable set $\mathcal{R}_T^{SCL}$ for the scalar conservation law

$$(30) \qquad \begin{cases} \partial_t v + \partial_x H(v) = 0 & \text{in } (0, T) \times \mathbb{R} \\ v(0, \cdot) = v_0 & \text{in } \mathbb{R}. \end{cases}$$

Proof of Theorem 3. The proof consists in checking that condition (13) is equivalent to the condition of Theorem 1 for the function

$$u_T(x) := \int_0^x v_T(y)dy \qquad \forall x \in \mathbb{R}.$$

Then, since $\partial_x u_T(x) = v_T(x)$ for a.e. $x \in \mathbb{R}$, we have that v_T is reachable for the equation (30) if and only if u_T is reachable for the equation (1). But, in view of Theorem 1, u_T is reachable for (1) if and only if, for all $x \in \mathbb{R}$, there exists x_0 such that the function

$$z \longmapsto \int_0^z v_T(y)dy - T\,H^*\left(\frac{z - x_0}{T}\right)$$

has a global maximum at x, and the proof is concluded. $\qquad\square$

We end the section with the proof of Theorem 4 stated in the introduction, which corresponds to the application of Theorem 3 to the case $H(p) = |p|$.

Proof of Theorem 4. First of all, we recall that the Legendre-Fenchel transform of $H(p) = |p|$ is given by the function

$$(31) \qquad H^*(q) = \begin{cases} 0 & \text{if } |q| \leq 1 \\ +\infty & \text{else.} \end{cases}$$

We first prove that (14) implies (13), and then we will prove the reversed implication. Let v_T satisfy (14). For any $x \in \mathbb{R}$, define the points

$$x_1 := \sup\{y \in (-\infty, x] \text{ such that } v_T(z) \geq 0 \text{ for a.e. } z \in (y, x)\}$$

and

$$x_2 := \inf\{y \in [x, +\infty) \text{ such that } v_T(z) \leq 0 \text{ for a.e. } z \in (x, y)\}.$$

By the choice of x_1 and x_2, we have that for any $\varepsilon > 0$, the sets

$$[x_1 - \varepsilon, \, x_1] \cap \{v_T(y) < 0\} \qquad \text{and} \qquad [x_2, \, x_2 + \varepsilon] \cap \{v_T(y) > 0\}$$

have both positive measure, whence, by the assumption (14), and letting $\varepsilon \to 0^+$, we deduce that

$$(32) \qquad\qquad\qquad\qquad\qquad x_2 - x_1 \geq 2T.$$

Moreover, by the choice of x_1 and x_2, we have $v_T(z) \geq 0$ for a.e. $z \in (x_1, x)$ and $v_T(z) \leq 0$ for a.e. $z \in (x, x_2)$. This implies that the function $g : [x_1, x_2] \to \mathbb{R}$, defined by

$$(33) \qquad\qquad g(z) = \int_0^z v_T(y)dy, \qquad \forall z \in [x_1, \, x_2], \quad \text{has a global maximum at } z = x.$$

Then, by (32), along with the fact that $x \in (x_1, x_2)$, implies that there exists $x_0 \in (x_1, x_2)$ such that

$$[x_0 - T, \, x_0 + T] \subset (x_1, \, x_2) \qquad \text{and} \qquad x \in [x_0 - T, \, x_0 + T]$$

Finally, for this choice of x_0, and using (31), we obtain that

$$\int_0^z v_T(y)dy - T \, H^* \left(\frac{z - x_0}{T} \right) = \begin{cases} g(z) & \text{for } z \in [x_0 - T, \, x_0 + T] \\ -\infty & \text{else,} \end{cases}$$

and we can deduce from (33) that the function

$$z \longmapsto \int_0^z v_T(y)dy - T \, H^* \left(\frac{z - x_0}{T} \right)$$

has a global maximum at x.

Let us prove the reversed implication. Consider a function $v_T \in L^\infty(\mathbb{R})$ satisfying (13). For any $x \leq y$, it is obvious that, if $y - x \geq 2T$, then (14) trivially holds. It is therefore sufficient to prove that the property (14) holds in any interval of length $2T$. Let $(a, b) \subset \mathbb{R}$ be any interval with $b - a = 2T$, and set

$$x_1 := \sup\{y \in (a, b) \text{ such that } v_T(z) \geq 0 \text{ for a.e. } z \in (a, y)\}.$$

If $x_1 = b$, then $v_T(x) \geq 0$ for a.e. $x \in (a, b)$, which implies that $v_T(x) > 0$ for a.e. $x \in \operatorname{supp}(v_T) \cap (a, b)$, and hence, property (14) holds in (a, b).

If we have $x_1 \in [a, b)$, by the definition of x_1, it holds that, for any $\varepsilon > 0$, there exists $x_\varepsilon \in (x_1, x_1 + \varepsilon]$ such that

$$\int_{x_1}^{x_\varepsilon} v_T(y)dy < 0,$$

which implies that the function

$$(34) \qquad\qquad\qquad\qquad y \longmapsto g(y) = \int_0^y v_T(z)dz$$

satisfies $g(x_1) > g(x_\varepsilon)$. Using the assumption (13), together with the particular form of H^* in (31), we have that

$$(35) \quad \forall x \in \mathbb{R}, \quad \exists x_0 := x_0(x) \in \mathbb{R} \quad \text{s.t.} \quad |x - x_0| \leq T \text{ and } g(x) \geq g(y) \ \forall y \in [x_0 - T, x_0 + T].$$

In particular, applying this property to x_ε, and the fact that $b - a = 2T$, we have that $g(y) \leq g(x_1)$ for all $y \in [x_1, b]$.

We can now deduce that $v_T(y) \leq 0$ for a.e. $y \in (x_1, b)$. This is indeed equivalent to prove that the function $g(x)$ defined in (34) is nonincreasing in $[x_1, b]$. Assume for a contradiction that

$$\exists z_1, z_2 \in [x_0, b] \quad \text{with } z_1 < z_2 \text{ and } g(z_2) > g(z_1).$$

Then we have $g(z_1) < g(z_2) \leq g(x_0)$, which together with $z_1 \in (x_0, z_2) \subset (a, b)$ leads to a contradiction with the statement (35). We have then proved that $v_T(x) \geq 0$ for a.e. $x \in (a, x_1)$ and $v_T(x) \leq 0$ for a.e. $x \in (x_1, b)$, which implies that (14) holds in (a, b).

$\square$

4. CONCLUSIONS AND OPEN QUESTIONS

In this work we studied the range of the operator that associates, to any initial condition, the solution at time T of nonlinear first-order partial differential equations such as Hamilton-Jacobi equations and scalar conservation laws. In the case when the Hamiltonian (resp. the flux) is smooth and uniformly convex, the range of this operator is well-understood, and can be characterized by means of semiconcavity estimates for Hamilton-Jacobi equations, and by one-sided Lipschitz condition for scalar conservation laws. Our goal in this work was to extend this results to the more general case when the Hamiltonian is not necessarily smooth nor strictly convex, and is merely assumed to be a convex function. Note that in this case, semiconcavity estimates are not available.

Our characterization of the reachable set for Hamilton-Jacobi equations relies on the use of the Hopf-Lax formula for the viscosity solution. This result is then adapted to the case of scalar conservation laws in one space dimension by using the link between both equations.

In the particular case of Hamilton-Jacobi equations with $H(p) = |p|$, we give a rather geometrical description of the reachable set by means of an interior ball condition on the sublevel sets of the target, which yields a one-sided regularity estimate for the boundary of the sublevel sets.

Finally, we use our main results to deduce several structural properties of the reachable set. For instance, we can prove that for power-like Hamiltonians of the form $H(p) = |p|^\alpha$, with $\alpha \geq 1$, the reachable set is star-shaped with center at the origin. Moreover, if $\alpha = 2$, the reachable set is convex, and if $\alpha = 1$, then it consists of a (non-convex) cone.

Open questions. Let us conclude the paper with two questions that we were not able to answer, and might be addressed in forthcoming works.

(i) We proved that for the case of Hamilton-Jacobi equations with power-like Hamiltonian, the reachable set is star-shaped, with center at the origin. Although it seems reasonable that the same property should hold for the case of general convex Hamiltonians, we were not able to provide a rigorous proof.

(ii) Concerning the same star-shaped property for the reachable set, we proved that the origin is a center of the domain, however, we cannot confirm whether or not other function than zero could be centers of this star-shaped set, i.e. a function $u_T^* \in \mathcal{R}_T$ such that

$$\forall u_T \in \mathcal{R}_T, \text{ and } \forall \lambda \in [0, 1], \qquad \lambda u_T + (1 - \lambda) u_T^* \in \mathcal{R}_T.$$

For instance, the set of concave functions is a convex set contained in the reachable set, which makes it a good candidate to find other centers.

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