Quasi-reversibility methods of optimal control for ill-posed final value diffusion equations

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Formulation of our problem

History of the quasi-reversibility method

Existence results of the state equations
- The ill-posed state equation
- The approximated state equation

Convergence of solutions

The optimal control problems
- The ill-posed optimal control problem
- The approximated optimal control problem
- Convergence of the approximated optimal controls
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1. Formulation of our problem
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Our control problem

Of concern is the following optimization problem:

$$\min_{(f, \rho) \in \mathcal{A}} \frac{1}{2} \left( \| \rho(\cdot, 0) - \rho^d \|_{L^2(\Omega)}^2 + \xi \| f \|_{L^2(Q)}^2 \right)$$  \hspace{1cm} (1.1)

subject to the constraints that $\rho$ solves the ill-posed state equation

$$\begin{cases} 
\rho_t + A\rho = f & \text{in } Q := \Omega \times (0, T), \\
\rho(\cdot, T) = \rho^T & \text{in } \Omega,
\end{cases}$$  \hspace{1cm} (1.2)

where $T > 0$, $f$ is the control, and $\rho^T \in L^2(\Omega)$ is a given function.

- $\rho^d$ is a given fixed target and $\xi \geq 0$ is a real parameter.

- Here,

$$\mathcal{A} = \{ (f, \rho) : f \in \mathbb{U}_{ad} \text{ and } \rho \text{ is the strong solution of (1.2)} \}$$
Assumption on the operator $A$

We assume the following.

1. The operator $A : D(A) \to L^2(\Omega)$ is positive, selfadjoint, invertible, and has a compact resolvent.

2. $\Omega \subset \mathbb{R}^N$ is an arbitrary open set. The regularity needed for $\Omega$ will depend on the boundary or exterior conditions associated with $A$.

3. The operator $-A$ generates a strongly continuous, analytic, and compact semigroup $S = (S(t))_{t \geq 0}$ on $L^2(\Omega)$ which is also submarkovian in the sense that each operator $S(t)$, $t \geq 0$, is positive, and also $L^\infty$-contractive, that is, for all $t \geq 0$,

$$\|S(t)u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}, \quad \forall \; u \in L^2(\Omega) \cap L^\infty(\Omega).$$
Observation 1

It follows from our assumptions on $A$ that the following hold.

1. The operator $A$ is given by a bilinear, symmetric, closed, and coercive form $\mathcal{E}$ with domain $\mathcal{V} := D(\mathcal{E}) = D(\sqrt{A})$ and norm

   $$\|u\|_{\mathcal{V}} := (\mathcal{E}(u, u))^{1/2}. \tag{1.3}$$

2. There is a constant $C > 0$ such that for every $u \in D(\mathcal{E})$,

   $$\|u\|_{\mathcal{V}} \geq C\|u\|_{L^2(\Omega)}. \tag{1.4}$$

3. For every $u \in D(A)$ and $v \in D(\mathcal{E})$ we have

   $$(Au, v)_{L^2(\Omega)} = \mathcal{E}(u, v). \tag{1.5}$$
Observation 2

1. The spectrum of $A$ is formed with eigenvalues $\lambda_n$ ($n \in \mathbb{N}$) satisfying
   \[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq \cdots \quad \text{and} \quad \lim_{n \to \infty} \lambda_n = +\infty. \]

2. The eigenvalues of $S(t)$ are $e^{-t\lambda_n}$, and possible zero. For each $\varepsilon > 0$ and $t \geq 0$, the operator $\varepsilon I + S(t)$ is invertible with inverse
   \[ \| (\varepsilon I + S(t))^{-1} \| \leq \frac{1}{\varepsilon}. \quad (1.6) \]

3. If $u \in L^2(\Omega)$ has the expansion $u = \sum_{k=1}^{\infty} a_k \phi_k$, then
   \[ S(t)u = \sum_{k=1}^{\infty} e^{-t\lambda_k} a_k \phi_k \quad \text{and} \quad (S(t)u, u)_{L^2(\Omega)} = \sum_{k=1}^{\infty} e^{-t\lambda_k} a_k^2 \geq 0. \]
Observation 3

1. If $u \in L^2(\Omega)$ has the expansion $u = \sum_{k=1}^{\infty} a_k \phi_k$, then for $\varepsilon > 0$

$$\left(\varepsilon I + S(T)\right) u = \sum_{k=1}^{\infty} \left(\varepsilon + e^{-T\lambda_k}\right) a_k \phi_k$$

and

$$\left(\varepsilon I + S(T)\right)^{-1} u = \sum_{k=1}^{\infty} \frac{a_k}{\varepsilon + e^{-T\lambda_k}} \phi_k.$$

2. Let $\mathbb{V}^*$ denote the dual of $\mathbb{V}$. Then $A$ can be also viewed as a bounded operator from $\mathbb{V}$ into $\mathbb{V}^*$ given by

$$\langle Au, v \rangle_{\mathbb{V}^*, \mathbb{V}} = \mathcal{E}(u, v). \quad (1.7)$$
Remark

1. One comes across such a model (1.2) while dealing of physical phenomena with missing data.

2. Actually in such a situation, one may not know when the phenomenon began or have information on the boundary.

3. Such models with missing data are therefore ill-posed in the sense of Hadamard. In particular, the uniqueness of solutions as well as the continuous dependence of solutions on the given data $f$ and $\rho^T$ are not always satisfied.

4. Even if we can prove using minimizing sequences that the optimization problem (1.1)-(1.2) has a unique solution $(\bar{f}, \bar{\rho})$, it will be difficult to characterize this solution since the increase of the state and the control will be linked as we shall see.
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Lattès and Lions 1969

Let $\varepsilon > 0$. Lattès and Lions used the following well-posed problem

\[
\begin{align*}
\frac{d}{dt}v_{\varepsilon}(t) + Av_{\varepsilon}(t) - \varepsilon A^2 v_{\varepsilon}(t) &= 0, \quad 0 < t < T \\
v_{\varepsilon}(T) &= g
\end{align*}
\]  
(2.1)

to approximate the ill-posed problem

\[
\begin{align*}
u'(t) + Au(t) &= 0, \quad 0 < t < T \\
u(T) &= g
\end{align*}
\]  
(2.2)

They used the initial value $u_{\varepsilon}(0) = v_{\varepsilon}(0)$ in the system

\[
\begin{align*}
\frac{d}{dt}u_{\varepsilon}(t) + Au_{\varepsilon}(t) &= 0, \quad 0 < t < T \\
u_{\varepsilon}(0) &= v_{\varepsilon}(0)
\end{align*}
\]  
(2.3)
They assumed that \( A \) is a selfadjoint operator on a Hilbert space \( H \) satisfying suitable conditions.

They have shown that if \( u_\varepsilon \) is a solutions of (2.3), then \( u_\varepsilon(T) \) converges to \( g \) in \( H \), as \( \varepsilon \downarrow 0 \).

They did not get a convergence of \( u_\varepsilon(t), 0 \leq t < T \), as \( \varepsilon \downarrow 0 \).

Also the operator carrying \( g \) into \( v_\varepsilon(0) \) in the system (2.1) has a large norm for small \( \varepsilon \) (of the order of \( e^{\varepsilon} \)).
Showalter 1974

- Showalter has approximated the ill-posed problem (2.2) with

\[
\begin{aligned}
  v_\varepsilon'(t) + \varepsilon A v_\varepsilon'(t) + A v_\varepsilon(t) &= 0, \quad 0 < t < T \\
  v_\varepsilon(T) &= g.
\end{aligned}
\]  

(2.4)

- The author used the initial condition \( u_\varepsilon(0) = v_\varepsilon(0) \) in the system

\[
\begin{aligned}
  u_\varepsilon'(t) + A u_\varepsilon(t) &= 0, \quad 0 < t < T \\
  u_\varepsilon(0) &= v_\varepsilon(0).
\end{aligned}
\]  

(2.5)

- He has proved that \( u_\varepsilon(T) \) converges to \( g \) in \( H \), as \( \varepsilon \downarrow 0 \), and that \( u_\varepsilon(t) \) converges to the solution \( u(t) \) of (2.2) in \( H \), uniformly in \( t \in [0, T] \), as \( \varepsilon \downarrow 0 \), if and only if such a solution exists.

- In these convergences, the norm of the function carrying \( g \) to \( v_\varepsilon(0) \) in (2.4) is also quite large for small values of \( \varepsilon \).
Miller 1973

- Miller has addressed the problem of large norm by finding an optimal perturbation of the operator $A$.

- The author stated that it should be possible to make the norm in the order of $c/\varepsilon$ rather than $e^{c/\varepsilon}$ and derive conditions on the perturbation to achieve the best possible results.

- As above the author approximated (2.2) with

\[
\begin{cases}
  v'(t) + g(A)v(t) = 0, & 0 < t < T \\
  v(T) = g,
\end{cases}
\]

and again solved the problem forward using $v(0)$ as an initial datum.

- Miller called this method, stabilized quasi-reversibility.
Clark and Oppenheimer 1994

They have approximated the ill-posed problem (2.2) with the well-posed problem

\[
\begin{align*}
    u'(t) + Au(t) &= 0, \quad 0 < t < T \\
    \varepsilon u(0) + u(T) &= g.
\end{align*}
\]  

(2.6)

Let \( u_\varepsilon \) be a solution of (2.6). They have shown that \( u_\varepsilon(T) \) converges to \( g \) in \( H \), as \( \varepsilon \downarrow 0 \), and that \( u_\varepsilon(t) \) converges to the solution \( u(t) \) in \( H \) (if such a solution exists), uniformly in \( t \in [0, T] \), as \( \varepsilon \downarrow 0 \).

They have also obtained a better polynomial convergence rate.

We mention that the approximation (2.6) is known in the literature as the "quasi-boundary value method".
Finally, Denche and Bessila have approximated the ill-posed (2.2) with the following well-posed problem:

\[
\begin{aligned}
    & u'(t) + Au(t) = 0, \quad 0 < t < T \\
    & u(T) - \varepsilon u'(0) = g.
\end{aligned}
\]  

They have obtained nice convergence results as the ones proved by Clark and Oppenheimer but here with a logarithmic convergence rate.
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Definition (Strong solutions)

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$. A function $\rho \in C([0, T]; L^2(\Omega))$ is said to be a strong solution of the ill-posed problem (1.2) if the following assertions hold:

- **Regularity:**
  
  $$\rho_t(\cdot, t) \in L^2(\Omega) \text{ and } \rho(\cdot, t) \in D(A) \text{ for a.e. } t \in (0, T)$$

  and the first equation in (1.2) is satisfied for a.e. $t \in (0, T)$.

- **Final condition:**
  
  $$\rho(\cdot, T) = \rho^T \text{ a.e. in } \Omega.$$
Lemma (Necessary and sufficient conditions for existence of solutions)

Let \( f \in L^2(Q) \) and \( \rho^T \in L^2(\Omega) \) have the expansions

\[
f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t) \phi_k \quad \text{and} \quad \rho^T = \sum_{k=1}^{\infty} b_k \phi_k.
\]

Then the following assertions are equivalent.

1. The ill-posed system (1.2) has a strong solution \( \rho \).
2. The following two series converge

\[
\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \left( \int_0^T e^{t\lambda_k} |f_k(t)| \, dt \right)^2 < \infty. \tag{3.1}
\]

(3.1) is similar to the so called Gevrey condition.
Proof

Let \( f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t) \phi_k \) and \( \rho^T = \sum_{k=1}^{\infty} b_k \phi_k \).

(b) \Rightarrow (a): Assume that the two series in (5.1) converge. Define

\[
\rho(\cdot, t) := \sum_{k=1}^{\infty} e^{(T-t)\lambda_k} b_k \phi_k - \int_t^T \sum_{k=1}^{\infty} e^{(\tau-t)\lambda_k} f_k(\tau) \phi_k \, d\tau, \quad t \in [0, T].
\]

In that case, it is easy to verify that \( \rho \) is a strong solution of (1.2).
(a) $\Rightarrow$ (b):

Let $\rho$ be a solution of (1.2) with $f = 0$. Since $\rho(\cdot, 0) \in L^2(\Omega)$, it follows that $\rho(\cdot, 0) = \sum_{k=1}^{\infty} a_k \phi_k$. Calculating we get that

$$S(T)\rho(\cdot, 0) = \sum_{k=1}^{\infty} e^{-T\lambda_k} a_k \phi_k = \rho^T = \sum_{k=1}^{\infty} b_k \phi_k. \quad (3.2)$$

(3.2) implies that $e^{-T\lambda_k} a_k = b_k$. Thus, $a_k = b_k e^{T\lambda_k}$. Since $\rho(\cdot, 0) \in L^2(\Omega)$, then $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Thus, $\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty$.

Let $\rho$ be a solution of (1.2) with $\rho^T = 0$. Then $\rho(\cdot, 0) = \sum_{k=1}^{\infty} a_k \phi_k$. Therefore,

$$S(T)\rho(\cdot, 0) = \sum_{k=1}^{\infty} e^{-T\lambda_k} a_k \phi_k = -\sum_{k=1}^{\infty} \int_0^T e^{-(T-\tau)\lambda_k} f_k(\tau) \phi_k \, d\tau.$$
Proof Cont

This implies that

\[ e^{-T \lambda_k} a_k = - \int_0^T e^{-(T-\tau) \lambda_k} f_k(\tau) \, d\tau. \]

Thus, \( a_k = - \int_0^T e^{t \lambda_k} f_k(t) \, dt. \)

Since \( \sum_{k=1}^{\infty} |a_k|^2 < \infty \), we can deduce that the series

\[ \sum_{k=1}^{\infty} \left( \int_0^T e^{t \lambda_k} f_k(t) \, dt \right)^2 < \infty. \]
The approximated state equation

For any $\varepsilon > 0$, we approximate (1.2) with the well-posed problem:

$$
\begin{aligned}
\rho_t^\varepsilon + A\rho^\varepsilon &= f \quad \text{in } Q, \\
\varepsilon\rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) &= \rho^T \quad \text{in } \Omega.
\end{aligned}
$$

(3.3)

Definition (Let $\varepsilon > 0$, $f \in L^2(Q)$, and $\rho^T \in L^2(\Omega)$)

A $\rho^\varepsilon \in C([0, T]; L^2(\Omega))$ is said to be a strong solution of (3.3) if

- Regularity:

  $$
  \rho_t^\varepsilon(\cdot, t) \in L^2(\Omega) \text{ and } \rho^\varepsilon(\cdot, t) \in D(A) \text{ for a.e. } t \in (0, T).
  $$

- Initial condition:

  $$
  \varepsilon\rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = \rho^T \text{ a.e. in } \Omega.
  $$

(3.4)
A good candidate for a solution

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$.

1. Recall that $S = (S(t))_{t \geq 0}$ is the semigroup generated by $-A$.
2. For $\varepsilon > 0$ and $t \in [0, T]$, we let

$$
\rho^\varepsilon(\cdot, t) := S(t)(\varepsilon I + S(T))^{-1}\rho^T
- \int_0^T S(T + t - \tau)(\varepsilon I + S(T))^{-1}f(\cdot, \tau) \, d\tau
+ \int_0^t S(t - \tau)f(\cdot, \tau) \, d\tau.
$$

(3.5)
A good candidate for a solution

Using the semigroup property we easily get the following identity:

\[
\rho^\varepsilon(\cdot, t) = S(t)(\varepsilon I + S(T))^{-1}\rho^T
- \int_0^t S(t - \tau)\left[ I - S(T)(\varepsilon I + S(T))^{-1}\right] f(\cdot, \tau) \, d\tau
- \int_t^T S(T + t - \tau)(\varepsilon I + S(T))^{-1}f(\cdot, \tau) \, d\tau. \tag{3.6}
\]

Theorem (Existence of solutions of the approximated equation)

Let \( f \in L^2(Q) \) and \( \rho^T \in L^2(\Omega) \). Then, the function \( \rho^\varepsilon \) given in (3.5) or (3.6) is the unique strong solution of the approximated system (3.3) and it depends continuously on the given data \( \rho^T \) and \( f \).
Since the semigroup $S$ is analytic, we have that $S(t)u \in D(A)$ for all $u \in L^2(\Omega)$ and $t > 0$. In addition, $\frac{d}{dt} (S(t)u) = -AS(t)u$ for all $t > 0$, and $(\varepsilon l + S(T))^{-1} \rho^T \in D(A)$ for every $\varepsilon > 0$.

A simple computation shows that for a.e. $t \in (0, T)$,

$$\rho^\varepsilon(\cdot, t) = -A\rho^\varepsilon(\cdot, t) + f(\cdot, t).$$

It follows from (3.5) that

$$\varepsilon \rho^\varepsilon(\cdot, 0) = \varepsilon (\varepsilon l + S(T))^{-1} \rho^T$$

$$- \varepsilon \int_0^T S(T - \tau)(\varepsilon l + S(T))^{-1} f(\cdot, \tau) \, d\tau. \quad (3.7)$$
Ideas of the proof

Using (3.5) and the semigroup property again, we also get

$$\rho^\varepsilon(\cdot, T) = S(T) (\varepsilon I + S(T))^{-1} \rho^T$$

$$+ \varepsilon \int_0^T S(T - \tau) (\varepsilon I + S(T))^{-1} f(\cdot, \tau) \, d\tau. \tag{3.8}$$

Combining (3.7)-(3.8), we get

$$\varepsilon \rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = (\varepsilon I + S(T)) (\varepsilon I + S(T))^{-1} \rho^T = \rho^T.$$

Using that $S$ is contractive we get that for every $t \in [0, T]$,

$$\|\rho^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon} \|\rho^T\|_{L^2(\Omega)} + \frac{1}{\varepsilon} \|f\|_{L^2(Q)} + \|f\|_{L^2(Q)}. \tag{3.9}$$

Uniqueness is easy to prove.
The approximated adjoint equation

The approximated adjoint equation is given by

\[
\begin{cases}
-p_t^\epsilon + Ap^\epsilon = 0 & \text{in } Q, \\
\epsilon p^\epsilon(\cdot, T) + p^\epsilon(\cdot, 0) = p^T & \text{in } \Omega.
\end{cases}
\]

(3.10)

Definition (Let \(p^T \in L^2(\Omega)\))

A function \(p^\epsilon \in C([0, T]; L^2(\Omega))\) is a strong solution of (3.10) if

- Regularity:

\(p^\epsilon \in C([0, T]; L^2(\Omega)) \cap C^1((0, T); L^2(\Omega)) \cap C((0, T); D(A)).\)

- Initial condition:

\(\epsilon p^\epsilon(\cdot, T) + p^\epsilon(\cdot, 0) = p^T \text{ a.e. in } \Omega.\)
Theorem (Existence of solutions of approximated adjoint equation)

Let $p^T \in L^2(\Omega)$. Then, there exists a unique strong solution $p^\varepsilon$ to (3.10). In addition, $p^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)$ and there is a constant $C > 0$ independent of $\varepsilon$ such that

$$
\|p^\varepsilon\|_{C([0, T]; L^2(\Omega))}^2 + \|p^\varepsilon\|_{L^2((0, T); \mathbb{V})}^2 \leq \frac{C}{\varepsilon^2} \|p^T\|_{L^2(\Omega)}^2.
$$

(3.11)
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**Convergence of $\rho^\varepsilon(\cdot, T)$**

\[
\lim_{\varepsilon \downarrow 0} \| \rho^\varepsilon(\cdot, T) - \rho^T \|_{L^2(\Omega)} = 0.
\]

**Proof**

Let $\rho^T = \sum_{k=1}^{\infty} b_k \phi_k \in L^2(\Omega)$ and $f \in L^2(Q)$. Using (3.5) we get

\[
\| \rho^\varepsilon(\cdot, T) - \rho^T \|_{L^2(\Omega)}^2 \leq \sum_{k=1}^{\infty} \varepsilon^2 b_k^2 (\varepsilon + e^{-T \lambda_k})^{-2} \nonumber \\
+ \left\| I - S(T)(\varepsilon I + S(T))^{-1} \right\|^2 \| f \|^2_{L^2(Q)}. \tag{4.1}
\]

Using the dominated convergence theorem, it is easily seen that

\[
\lim_{\varepsilon \downarrow 0} \left\| I - S(T)(\varepsilon I + S(T))^{-1} \right\| = 0. \tag{4.2}
\]
Next, fix $\delta > 0$. Choose $N \in \mathbb{N}$ large such that $\sum_{k=N+1}^{\infty} b_k^2 < \frac{\delta}{2}$. Then,

$$\sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left( \varepsilon + e^{-T\lambda_k} \right)^{-2} \leq \varepsilon^2 \sum_{k=1}^{N} b_k^2 e^{2T\lambda_k} + \frac{\delta}{2}.$$ 

Now, let $\varepsilon > 0$ be such that $\varepsilon^2 < \frac{\delta}{4} \left( \sum_{k=1}^{N} b_k^2 e^{2T\lambda_k} \right)^{-1}$.

This implies that $\sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left( \varepsilon + e^{-T\lambda_k} \right)^{-2} < \delta$. We have shown that

$$\lim_{\varepsilon \downarrow 0} \sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left( \varepsilon + e^{-T\lambda_k} \right)^{-2} = 0. \quad (4.3)$$

Combining (4.2)-(4.3) and using (4.1), we get the result.
General convergence results

Let $f \in L^2(Q)$, $\rho^T \in L^2(\Omega)$, and $\rho^\varepsilon$ be given by (3.5).

- The system (1.2) has a strong solution $\rho$ if and only if the sequence $\{\rho^\varepsilon(\cdot, 0)\}$ converges in $L^2(\Omega)$, as $\varepsilon \downarrow 0$.

- In addition, as $\varepsilon \downarrow 0$, we have that
  
  $\rho^\varepsilon(\cdot, t)$ converges strongly to $\rho$ in $L^2(\Omega)$, uniformly in $t \in [0, T]$,

  $\rho^\varepsilon$ converges strongly to $\rho$ in $L^2((0, T), V) \cap H^1((0, T); V^*)$. 

Let \( f \in L^2(Q) \) and \( \rho^T \in L^2(\Omega) \) have the expansions

\[
f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t) \phi_k \quad \text{and} \quad \rho^T = \sum_{k=1}^{\infty} b_k \phi_k. \tag{4.4}
\]

If there exists \( \gamma > 0 \) such that the series

\[
\sum_{k=1}^{\infty} b_k^2 e^{\gamma T \lambda_k} \quad \text{and} \quad \sum_{k=1}^{\infty} \int_0^T e^{\gamma t \lambda_k} |f_k(t)|^2 \, dt \tag{4.5}
\]

converge, then

\[
\lim_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)} = 0
\]

with rate \( \varepsilon^{\gamma - 2} \).
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Our ill-posed control problem

- Let $f \in L^2(Q)$ and $\rho_T \in L^2(\Omega)$ satisfy

$$\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \left( \int_0^T e^{t\lambda_k} |f_k(t)| \, dt \right)^2 < \infty.$$  \hspace{1cm} (5.1)

- We want to solve the optimization problem (1.1)-(1.2).
- Let $U_{ad}$ be the closed and convex subset of $L^2(Q)$ given by

$$U_{ad} := \left\{ f \in L^2(Q) \mid \sum_{k=1}^{\infty} \left( \int_0^T e^{t\lambda_k} |f_k(t)| \, dt \right)^2 < \infty \right\}.$$  

- Let

$$\mathcal{A} = \{(f, \rho^T) \in U_{ad} \times L^2(\Omega) : \rho^T \text{ satisfies (5.1)} \}.$$
Observations

1. We know that $\mathcal{A} \neq \emptyset$.

2. Since $\rho \in C([0, T]; L^2(\Omega))$, we know that $\rho(\cdot, 0) \in L^2(\Omega)$. We can thus define the cost function

$$J(f, \rho) = \frac{1}{2} \left( \|\rho(\cdot, 0) - \rho^d\|_{L^2(\Omega)}^2 + \xi \|f\|_{L^2(Q)}^2 \right),$$

where $\rho^d \in L^2(\Omega)$ and $\xi > 0$ is a real parameter.

3. Using minimizing sequences, the structure of $J$, we can prove that there exists a unique $(\bar{\rho}, \bar{\rho}) \in \mathcal{A}$ solution to (1.1)-(1.2). Moreover,

$$\int_{\Omega} (\rho(x, 0) - \bar{\rho}(x, 0))(\bar{\rho}(x, 0) - \rho^d(x))dx + \int_Q \xi \bar{f}(f - \bar{f}) \, dxdt \geq 0$$

for all $(f, \rho) \in \mathcal{A}$.
The approximated optimal control problem

Let $\mathcal{U}_{ad} \subset L^2(Q)$ be closed and convex. Minimize

$$
\min_{(f, \rho^\varepsilon) \in \mathcal{U}_{ad} \times L^2(Q)} J^\varepsilon(f) := \frac{1}{2} \left( \| \rho^\varepsilon(\cdot, 0) - \rho^d \|^2_{L^2(\Omega)} + \xi \| f \|^2_{L^2(Q)} \right)
$$

(5.3)

subject to the constraints that

$$
\begin{cases}
\rho^\varepsilon_t + A \rho^\varepsilon = f & \text{in } Q, \\
\varepsilon \rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) = \rho^T & \text{in } \Omega.
\end{cases}
$$

(5.4)

Existence and uniqueness of optimal controls

For every $\varepsilon > 0$, there exists a unique $f^\varepsilon \in \mathcal{U}_{ad}$ solution of the minimization problem (5.3)-(5.4). The associated state $\rho^\varepsilon$ is the unique strong solution of (3.3) with $f$ replaced by $f^\varepsilon$. 
Ideas of the proof

- Since $J^\varepsilon$ is bounded from below by zero, it is possible to construct a minimizing sequence $(f^{\varepsilon n})_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} J^\varepsilon(f^{\varepsilon n}) = \inf_{f \in \mathcal{U}_{ad}} J^\varepsilon(f). \quad (5.5)$$

- There exist $f^\varepsilon \in L^2(Q)$, $\rho_T^\varepsilon, \rho_0^\varepsilon \in L^2(\Omega)$, and $\rho^\varepsilon \in L^2((0, T); \mathbb{V})$ such that, as $n \to \infty$, we have the following convergences:

$$f^{\varepsilon n} \to f^\varepsilon \quad \text{weakly in } L^2(Q), \quad (5.6a)$$

$$\rho^{\varepsilon n}(\cdot, 0) \to \psi_0^\varepsilon \quad \text{weakly in } L^2(\Omega), \quad (5.6b)$$

$$\rho^{\varepsilon n}(\cdot, T) \to \psi_T^\varepsilon \quad \text{weakly in } L^2(\Omega), \quad (5.6c)$$

$$\rho^{\varepsilon n} \to \rho^\varepsilon \quad \text{weakly in } L^2((0, T); \mathbb{V}). \quad (5.6d)$$
Theorem (Optimality conditions)

Let \((\rho^\varepsilon, f^\varepsilon)\) be the solution of the minimization problem (5.3)-(5.4). Then, there exists \(q^\varepsilon \in L^2((0, T); \mathbb{V}) \cap H^1((0, T); \mathbb{V}^*)\) such that we have the following optimality systems:

\[
\begin{aligned}
\rho^\varepsilon_t + A\rho^\varepsilon &= f^\varepsilon & \text{in } Q, \\
\varepsilon\rho^\varepsilon(\cdot, 0) + \rho^\varepsilon(\cdot, T) &= \rho^T & \text{in } \Omega,
\end{aligned}
\]  

(5.7)

and

\[
\begin{aligned}
-q^\varepsilon_t + Aq^\varepsilon &= 0 & \text{in } Q, \\
\varepsilon q^\varepsilon(\cdot, T) + q^\varepsilon(\cdot, 0) &= \rho^\varepsilon(\cdot, 0) - \rho^d & \text{in } \Omega,
\end{aligned}
\]  

(5.8)

and

\[
\int_Q (\xi f^\varepsilon - q^\varepsilon)(f - f^\varepsilon) \, dx \, dt \geq 0, \quad \forall f \in \mathcal{U}_{ad}.
\]  

(5.9)
Theorem (Convergence of optimal solutions)

Let \((\bar{f}, \bar{\rho})\) be a solution of the minimization problem \((1.1)-(1.2)\), and let \((f^\varepsilon, \rho^\varepsilon, q^\varepsilon)\) be as above.

- Assume that \(f^\varepsilon \in U_{ad}\) and that \(\text{Int}(U_{ad}) \neq \emptyset\).
- Then, there exists \(\bar{q} \in L^2(Q)\) such that, as \(\varepsilon \downarrow 0\), we have the following convergences:
  
  \[
  \begin{align*}
  f^\varepsilon & \text{ converges strongly to } \bar{f} \in L^2(Q) \text{ and } \bar{f} \in U_{ad}, \\
  \rho^\varepsilon & \text{ converges weakly to } \bar{\rho} \in L^2((0, T); V) \cap H^1((0, T); V^*), \\
  \rho^\varepsilon(\cdot, T) & \text{ converges strongly to } \rho^T \in L^2(\Omega), \\
  \rho^\varepsilon(\cdot, 0) & \text{ converges strongly to } \bar{\rho}(\cdot, 0) \in L^2(\Omega), \\
  q^\varepsilon & \text{ converges weakly to } \bar{q} \in L^2(Q).
  \end{align*}
  \]
Assume that $\text{Int}(U_{ad}) \neq \emptyset$. Then, $(\bar{f}, \bar{\rho})$ is the solution of the minimization problem (1.1)-(1.2) if and only if there exists $\bar{q} \in L^2(Q)$ such that the triple $(\bar{f}, \bar{\rho}, \bar{q})$ satisfies the following singular optimality systems: $\bar{\rho}$ is a strong solution of

$$
\begin{cases}
\bar{\rho}_t + A\bar{\rho} = \bar{f} & \text{in } Q, \\
\bar{\rho}(\cdot, T) = \rho^T & \text{in } \Omega,
\end{cases}
$$

(5.15)

and

$$
\int_Q \bar{q}(\phi_t + A\phi) \, dx \, dt = 0
$$

(5.16)

for every $\phi \in D((0, T))$, $D(A)$, and finally

$$
\int_Q (\xi \bar{f} - \bar{q})(f - \bar{f}) \, dx \, dt \geq 0, \quad \forall f \in U_{ad}.
$$

(5.17)
THANK YOU!