Quasi-reversibility methods of optimal control for ill-posed final value diffusion equations

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1. Formulation of our problem
2. History of the quasi-reversibility method
3. Existence results of the state equations
   - The ill-posed state equation
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4. Convergence of solutions
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Our control problem

- Of concern is the following optimization problem:

$$\min_{(f,\rho)\in\mathcal{A}} \frac{1}{2}\left(\|\rho(\cdot,0)-\rho^d\|^2_{L^2(\Omega)}+\xi\|f\|^2_{L^2(Q)}\right) \tag{1.1}$$

  subject to the constraints that ρ solves the ill-posed state equation

$$\begin{cases}\rho_t + A\rho = f & \text{in } Q := \Omega\times(0,T),\\ \rho(\cdot,T)=\rho^T & \text{in } \Omega,\end{cases} \tag{1.2}$$

  where $T>0$, f is the control, and $\rho^T\in L^2(\Omega)$ is a given function.
- ρ^d is a given fixed target and $\xi\geq 0$ is a real parameter.
- Here,

$$\mathcal{A}=\{(f,\rho): f\in\mathbb{U}_{ad} \text{ and } \rho \text{ is the strong solution of (1.2)}\}$$

Assumption on the operator A

We assume the following.

1. The operator $A : D(A) \to L^2(\Omega)$ is positive, selfadjoint, invertible, and has a compact resolvent.
2. $\Omega \subset \mathbb{R}^N$ is an arbitrary open set. The regularity needed for Ω will depend on the boundary or exterior conditions associated with A.
3. The operator $-A$ generates a strongly continuous, analytic, and compact semigroup $S = (S(t)_{t\geq 0}$ on $L^2(\Omega)$ which is also submarkovian in the sense that each operator $S(t)$, $t \geq 0$, is positive, and also L^∞-contractive, that is, for all $t \geq 0$,

$$\|S(t)u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}, \quad \forall\, u \in L^2(\Omega) \cap L^\infty(\Omega).$$

Observation 1

It follows from our assumptions on A that the following hold.

1. The operator A is given by a bilinear, symmetric, closed, and coercive form $\mathcal{E}$ with domain $\mathbb{V} := D(\mathcal{E}) = D(\sqrt{A})$ and norm

$$\|u\|_{\mathbb{V}} := (\mathcal{E}(u,u))^{1/2}. \tag{1.3}$$

2. There is a constant $C > 0$ such that for every $u \in D(\mathcal{E})$,

$$\|u\|_{\mathbb{V}} \geq C\|u\|_{L^2(\Omega)}. \tag{1.4}$$

3. For every $u \in D(A)$ and $v \in D(\mathcal{E})$ we have

$$(Au, v)_{L^2(\Omega)} = \mathcal{E}(u, v). \tag{1.5}$$

Observation 2

1. The spectrum of A is formed with eigenvalues λ_n $(n \in \mathbb{N})$ satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq \cdots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = +\infty.$$

2. The eigenvalues of $S(t)$ are $e^{-t\lambda_n}$, and possible zero. For each $\varepsilon > 0$ and $t \geq 0$, the operator $\varepsilon I + S(t)$ is invertible with inverse

$$\|(\varepsilon I + S(t))^{-1}\| \leq \frac{1}{\varepsilon}. \tag{1.6}$$

3. If $u \in L^2(\Omega)$ has the expansion $u = \sum_{k=1}^{\infty} a_k \phi_k$, then

$$S(t)u = \sum_{k=1}^{\infty} e^{-t\lambda_k} a_k \phi_k \text{ and } (S(t)u, u)_{L^2(\Omega)} = \sum_{k=1}^{\infty} e^{-t\lambda_k} a_k^2 \geq 0.$$

Observation 3

1. If $u \in L^2(\Omega)$ has the expansion $u = \sum_{k=1}^{\infty} a_k \phi_k$, then for $\varepsilon > 0$

$$\big(\varepsilon I + S(T)\big)u = \sum_{k=1}^{\infty} \big(\varepsilon + e^{-T\lambda_k}\big) a_k \phi_k \text{ and}$$

$$\big(\varepsilon I + S(T)\big)^{-1} u = \sum_{k=1}^{\infty} \frac{a_k}{\varepsilon + e^{-T\lambda_k}} \phi_k.$$

2. Let $\mathbb{V}^\star$ denote the dual of $\mathbb{V}$. Then A can be also viewed as a bounded operator from $\mathbb{V}$ into $\mathbb{V}^\star$ given by

$$\langle Au, v\rangle_{\mathbb{V}^\star,\mathbb{V}} = \mathcal{E}(u, v). \tag{1.7}$$

Remark

1. One comes across such a model (1.2) while dealing of physical phenomena with missing data.
2. Actually in such a situation, one may not know when the phenomenon began or have information on the boundary.
3. Such models with missing data are therefore ill-posed in the sense of Hadamard. In particular, the uniqueness of solutions as well as the continuous dependence of solutions on the given data f and ρ^T are not always satisfied.
4. Even if we can prove using minimizing sequences that the optimization problem (1.1)-(1.2) has a unique solution $(\bar{f}, \bar{\rho})$, it will be difficult to characterize this solution since the increase of the state and the control will be linked as we shall see.

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Lattès and Lions 1969

- Let $\varepsilon > 0$. Lattès and Lions used the following well-posed problem

$$\begin{cases} v_\varepsilon'(t) + Av_\varepsilon(t) - \varepsilon A^2 v_\varepsilon(t) = 0, & 0 < t < T \\ v_\varepsilon(T) = g \end{cases} \tag{2.1}$$

to approximate the ill-posed problem

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T \\ u(T) = g. \end{cases} \tag{2.2}$$

- They used the initial value $u_\varepsilon(0) = v_\varepsilon(0)$ in the system

$$\begin{cases} u_\varepsilon'(t) + Au_\varepsilon(t) = 0, & 0 < t < T \\ u_\varepsilon(0) = v_\varepsilon(0). \end{cases} \tag{2.3}$$

Lattès and Lions 1969

- They assumed that A is a selfadjoint operator on a Hilbert space H satisfying suitable conditions.
- They have shown that if u_ε is a solutions of (2.3), then $u_\varepsilon(T)$ converges to g in H, as $\varepsilon \downarrow 0$.
- They did not get a convergence of $u_\varepsilon(t)$, $0 \leq t < T$, as $\varepsilon \downarrow 0$.
- Also the operator carrying g into $v_\varepsilon(0)$ in the system (2.1) has a large norm for small ε (of the order of $e^{\frac{c}{\varepsilon}}$).

Showalter 1974

- Showalter has approximated the ill-posed problem (2.2) with

$$\begin{cases} v'_\varepsilon(t) + \varepsilon A v'_\varepsilon(t) + A v_\varepsilon(t) &= 0, \quad 0 < t < T \\ v_\varepsilon(T) &= g. \end{cases} \tag{2.4}$$

- The author used the initial condition $u_\varepsilon(0) = v_\varepsilon(0)$ in the system

$$\begin{cases} u'_\varepsilon(t) + A u_\varepsilon(t) &= 0, \quad 0 < t < T \\ u_\varepsilon(0) &= v_\varepsilon(0). \end{cases} \tag{2.5}$$

- He has proved that $u_\varepsilon(T)$ converges to g in H, as $\varepsilon \downarrow 0$, and that $u_\varepsilon(t)$ converges to the solution $u(t)$ of (2.2) in H, uniformly in $t \in [0, T]$, as $\varepsilon \downarrow 0$, if and only if such a solution exists.
- In these convergences, the norm of the function carrying g to $v_\varepsilon(0)$ in (2.4) is also quite large for small values of ε.

Miller 1973

- Miller has addressed the problem of large norm by finding an optimal perturbation of the operator A.
- The author stated that it should be possible to make the norm in the order of c/ε rather than $e^{c/\varepsilon}$ and derive conditions on the perturbation to achieve the best possible results.
- As above the author approximated (2.2) with

$$\begin{cases} v'(t) + g(A)v(t) = 0, \quad 0 < t < T \\ v(T) = g, \end{cases}$$

  and again solved the problem forward using $v(0)$ as an initial datum.
- Miller called this method, stabilized quasi-reversibility.

Clark and Oppenheimer 1994

- They have approximated the ill-posed problem (2.2) with the well-posed problem

$$\begin{cases} u'(t) + Au(t) = 0, & 0 < t < T \\ \varepsilon u(0) + u(T) = g. \end{cases} \tag{2.6}$$

- Let u_ε be a solution of (2.6). They have shown that $u_\varepsilon(T)$ converges to g in H, as $\varepsilon \downarrow 0$, and that $u_\varepsilon(t)$ converges to the solution $u(t)$ in H (if such a solution exists), uniformly in $t \in [0, T]$, as $\varepsilon \downarrow 0$.
- They have also obtained a better polynomial convergence rate.
- We mention that the approximation (2.6) is known in the literature as the "quasi-boundary value method".

Denche and Bessila 2005

- Finally, Denche and Bessila have approximated the ill-posed (2.2) with the following well-posed problem:

$$\begin{cases} u'(t) + Au(t) = 0, \quad 0 < t < T \\ u(T) - \varepsilon u'(0) = g. \end{cases} \tag{2.7}$$

- They have obtained nice convergence results as the ones proved by Clark and Oppenheimer but here with a logarithmic convergence rate.

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Definition (Strong solutions)

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$. A function $\rho \in C([0,T]; L^2(\Omega))$ is said to be a strong solution of the ill-posed problem (1.2) if the following assertions hold:

- Regularity:

$$\rho_t(\cdot, t) \in L^2(\Omega) \text{ and } \rho(\cdot, t) \in D(A) \text{ for a.e. } t \in (0, T)$$

  and the first equation in (1.2) is satisfied for a.e. $t \in (0, T)$.

- Final condition:

$$\rho(\cdot, T) = \rho^T \quad \text{a.e. in } \Omega.$$

Lemma (Necessary and sufficient conditions for existenceof solutions)

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$ have the expansions

$$f(\cdot,t) = \sum_{k=1}^{\infty} f_k(t)\phi_k \quad \text{and} \quad \rho^T = \sum_{k=1}^{\infty} b_k\phi_k.$$

Then the following assertions are equivalent.

1. *The ill-posed system (1.2) has a strong solution ρ.*
2. *The following two series converge*

$$\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \left(\int_0^T e^{t\lambda_k} |f_k(t)| \, dt \right)^2 < \infty. \qquad (3.1)$$

(3.1) *is similar to the so called Gevrey condition.*

Proof

Let $f(\cdot,t) = \sum_{k=1}^{\infty} f_k(t)\phi_k$ and $\rho^T = \sum_{k=1}^{\infty} b_k\phi_k$.
(b) $\Rightarrow$ (a): Assume that the two series in (5.1) converge. Define

$$\rho(\cdot,t) := \sum_{k=1}^{\infty} e^{(T-t)\lambda_k} b_k\phi_k - \int_t^T \sum_{k=1}^{\infty} e^{(\tau-t)\lambda_k} f_k(\tau)\phi_k \, d\tau, \quad t \in [0,T].$$

In that case, it is easy to verify that ρ is a strong solution of (1.2).

Proof Cont

(a) $\Rightarrow$ (b):

- Let ρ be a solution of (1.2) with $f = 0$. Since $\rho(\cdot,0) \in L^2(\Omega)$, it follows that $\rho(\cdot,0) = \sum_{k=1}^{\infty} a_k\phi_k$. Calculating we get that

$$S(T)\rho(\cdot,0) = \sum_{k=1}^{\infty} e^{-T\lambda_k} a_k\phi_k = \rho^T = \sum_{k=1}^{\infty} b_k\phi_k. \qquad (3.2)$$

  (3.2) implies that $e^{-T\lambda_k}a_k = b_k$. Thus, $a_k = b_k e^{T\lambda_k}$. Since $\rho(\cdot,0) \in L^2(\Omega)$, then $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Thus, $\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty$.
- Let ρ be a solution of (1.2) with $\rho^T = 0$. Then $\rho(\cdot,0) = \sum_{k=1}^{\infty} a_k\phi_k$. Therefore,

$$S(T)\rho(\cdot,0) = \sum_{k=1}^{\infty} e^{-T\lambda_k} a_k\phi_k = -\sum_{k=1}^{\infty} \int_0^T e^{-(T-\tau)\lambda_k} f_k(\tau)\phi_k \, d\tau.$$

Proof Cont

- This implies that

$$e^{-T\lambda_k} a_k = -\int_0^T e^{-(T-\tau)\lambda_k} f_k(\tau)\, d\tau.$$

- Thus, $a_k = -\int_0^T e^{t\lambda_k} f_k(t)\, dt.$
- Since $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, we can deduce that the series

$$\sum_{k=1}^{\infty} \left(\int_0^T e^{t\lambda_k} f_k(t)\, dt \right)^2 < \infty.$$

□

The approximated state equation

For any $\varepsilon > 0$, we approximate (1.2) with the well-posed problem:

$$\begin{cases} \rho^{\varepsilon}_t + A\rho^{\varepsilon} = f & \text{in } Q, \\ \varepsilon\rho^{\varepsilon}(\cdot, 0) + \rho^{\varepsilon}(\cdot, T) = \rho^T & \text{in } \Omega. \end{cases} \tag{3.3}$$

Definition (Let $\varepsilon > 0$, $f \in L^2(Q)$, and $\rho^T \in L^2(\Omega)$)

A $\rho^{\varepsilon} \in C([0, T]; L^2(\Omega))$ is said to be a strong solution of (3.3) if

- Regularity:

$$\rho^{\varepsilon}_t(\cdot, t) \in L^2(\Omega) \text{ and } \rho^{\varepsilon}(\cdot, t) \in D(A) \text{ for a.e. } t \in (0, T).$$

- Initial condition:

$$\varepsilon\rho^{\varepsilon}(\cdot, 0) + \rho^{\varepsilon}(\cdot, T) = \rho^T \text{ a.e. in } \Omega. \tag{3.4}$$

A good candidate for a solution

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$.

1. Recall that $S = (S(t))_{t\geq 0}$ is the semigroup generated by $-A$.
2. For $\varepsilon > 0$ and $t \in [0, T]$, we let

$$\begin{aligned}\rho^\varepsilon(\cdot, t) :=& S(t)\big(\varepsilon I + S(T)\big)^{-1}\rho^T \\ & - \int_0^T S(T + t - \tau)\big(\varepsilon I + S(T)\big)^{-1} f(\cdot, \tau)\, d\tau \\ & + \int_0^t S(t - \tau) f(\cdot, \tau)\, d\tau. \end{aligned} \tag{3.5}$$

A good candidate for a solution

- Using the semigroup property we easily get the following identity:

$$\begin{aligned}\rho^{\varepsilon}(\cdot,t) =& S(t)\big(\varepsilon I + S(T)\big)^{-1}\rho^{T} \\ & - \int_0^t S(t-\tau)\Big[I - S(T)(\varepsilon I + S(T))^{-1}\Big] f(\cdot,\tau)\, d\tau \\ & - \int_t^T S(T+t-\tau)(\varepsilon I + S(T))^{-1} f(\cdot,\tau)\, d\tau. \qquad (3.6)\end{aligned}$$

Theorem (Existence of solutions of the approximated equation)

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$. Then, the function ρ^{ε} given in (3.5) *or* (3.6) *is the unique strong solution of the approximated system* (3.3) *and it depends continuously on the given data ρ^T and f.*

Ideas of the proof

- Since the semigroup S is analytic, we have that $S(t)u \in D(A)$ for all $u \in L^2(\Omega)$ and $t > 0$. In addition, $\frac{d}{dt}(S(t)u) = -AS(t)u$ for all $t > 0$, and $\left(\varepsilon I + S(T)\right)^{-1}\rho^T \in D(A)$ for every $\varepsilon > 0$.
- A simple computation shows that for a.e. $t \in (0, T)$,

$$\rho_t^\varepsilon(\cdot, t) = -A\rho^\varepsilon(\cdot, t) + f(\cdot, t).$$

- It follows from (3.5) that

$$\begin{aligned}\varepsilon\rho^\varepsilon(\cdot, 0) =& \varepsilon\left(\varepsilon I + S(T)\right)^{-1}\rho^T \\ &- \varepsilon\int_0^T S(T-\tau)\left(\varepsilon I + S(T)\right)^{-1} f(\cdot, \tau)\, d\tau. \end{aligned} \qquad (3.7)$$

Ideas of the proof

- Using (3.5) and the semigroup property again, we also get

$$\begin{aligned}\rho^{\varepsilon}(\cdot, T) =& S(T)\big(\varepsilon I + S(T)\big)^{-1}\rho^T \\ &+ \varepsilon \int_0^T S(T-\tau)\big(\varepsilon I + S(T)\big)^{-1} f(\cdot,\tau)\, d\tau. \end{aligned} \tag{3.8}$$

  Combining (3.7)-(3.8), we get

$$\varepsilon\rho^{\varepsilon}(\cdot,0) + \rho^{\varepsilon}(\cdot, T) = \big(\varepsilon I + S(T)\big)\big(\varepsilon I + S(T)\big)^{-1}\rho^T = \rho^T.$$

- Using that S is contractive we get that for every $t \in [0, T]$,

$$\|\rho^{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \leq \frac{1}{\varepsilon}\|\rho^T\|_{L^2(\Omega)} + \frac{1}{\varepsilon}\|f\|_{L^2(Q)} + \|f\|_{L^2(Q)}. \tag{3.9}$$

- Uniqueness is easy to prove. □

The approximated adjoint equation

The approximated adjoint equation is given by

$$\begin{cases} -p_t^\varepsilon + Ap^\varepsilon = 0 & \text{in } Q, \\ \varepsilon p^\varepsilon(\cdot, T) + p^\varepsilon(\cdot, 0) = p^T & \text{in } \Omega. \end{cases} \tag{3.10}$$

Definition (Let $p^T \in L^2(\Omega)$)

A function $p^\varepsilon \in C([0, T]; L^2(\Omega))$ is a strong solution of (3.10) if

- Regularity:

$$p^\varepsilon \in C([0, T]; L^2(\Omega)) \cap C^1((0, T); L^2(\Omega)) \cap C((0, T); D(A)).$$

- Initial condition:

$$\varepsilon p^\varepsilon(\cdot, T) + p^\varepsilon(\cdot, 0) = p^T \text{a.e. in } \Omega.$$

Theorem (Existence of solutions of approximated adjoint equation)

Let $p^T \in L^2(\Omega)$*. Then, there exists a unique strong solution* p^ε *to* (3.10)*. In addition,* $p^\varepsilon \in L^2((0,T);\mathbb{V}) \cap H^1((0,T);\mathbb{V}^\star)$ *and there is a constant* $C > 0$ *independent of* ε *such that*

$$\|p^\varepsilon\|^2_{C([0,T];L^2(\Omega))} + \|p^\varepsilon\|^2_{L^2((0,T);\mathbb{V})} \le \frac{C}{\varepsilon^2}\|p^T\|^2_{L^2(\Omega)}. \tag{3.11}$$

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Convergence of $\rho^\varepsilon(\cdot, T)$

$$\lim_{\varepsilon \downarrow 0} \|\rho^\varepsilon(\cdot, T) - \rho^T\|_{L^2(\Omega)} = 0.$$

Proof

Let $\rho^T = \sum_{k=1}^{\infty} b_k \phi_k \in L^2(\Omega)$ and $f \in L^2(Q)$. Using (3.5) we get

$$\|\rho^\varepsilon(\cdot, T) - \rho^T\|^2_{L^2(\Omega)} \leq \sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left(\varepsilon + e^{-T\lambda_k}\right)^{-2} + \left\| I - S(T)\left(\varepsilon I + S(T)\right)^{-1} \right\|^2 \|f\|^2_{L^2(Q)}. \quad (4.1)$$

Using the dominated convergence theorem, it is easily seen that

$$\lim_{\varepsilon \downarrow 0} \left\| I - S(T)\left(\varepsilon I + S(T)\right)^{-1} \right\| = 0. \quad (4.2)$$

Proof Cont

Next, fix $\delta > 0$. Choose $N \in \mathbb{N}$ large such that $\sum_{k=N+1}^{\infty} b_k^2 < \frac{\delta}{2}$. Then,

$$\sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left(\varepsilon + e^{-T\lambda_k}\right)^{-2} \leq \varepsilon^2 \sum_{k=1}^{N} b_k^2 e^{2T\lambda_k} + \frac{\delta}{2}.$$

Now, let $\varepsilon > 0$ be such that $\varepsilon^2 < \frac{\delta}{4} \left(\sum_{k=1}^{N} b_k^2 e^{2T\lambda_k} \right)^{-1}$.

This implies that $\sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left(\varepsilon + e^{-T\lambda_k}\right)^{-2} < \delta$. We have shown that

$$\lim_{\varepsilon \downarrow 0} \sum_{k=1}^{\infty} \varepsilon^2 b_k^2 \left(\varepsilon + e^{-T\lambda_k}\right)^{-2} = 0. \tag{4.3}$$

Combining (4.2)-(4.3) and using (4.1), we get the result. □

General convergence results

Let $f \in L^2(Q)$, $\rho^T \in L^2(\Omega)$, and ρ^ε be given by (3.5).

- The system (1.2) has a strong solution ρ if and only if the sequence $\{\rho^\varepsilon(\cdot, 0)\}$ converges in $L^2(\Omega)$, as $\varepsilon \downarrow 0$.
- In addition, as $\varepsilon \downarrow 0$, we have that

$$\rho^\varepsilon(\cdot, t) \text{ converges strongly to } \rho \text{ in } L^2(\Omega), \text{ uniformly in } t \in [0, T],$$

$$\rho^\varepsilon \text{ converges strongly to } \rho \text{ in } L^2((0, T), \mathbb{V}) \cap H^1((0, T); \mathbb{V}^\star).$$

Rate of convergence

Let $f \in L^2(Q)$ and $\rho^T \in L^2(\Omega)$ have the expansions

$$f(\cdot,t) = \sum_{k=1}^{\infty} f_k(t)\phi_k \text{ and } \rho^T = \sum_{k=1}^{\infty} b_k\phi_k. \quad (4.4)$$

If there exists $\gamma > 0$ such that the series

$$\sum_{k=1}^{\infty} b_k^2 e^{\gamma T\lambda_k} \text{ and } \sum_{k=1}^{\infty} \int_0^T e^{\gamma t\lambda_k}|f_k(t)|^2\, dt \quad (4.5)$$

converge, then

$$\lim_{\varepsilon\downarrow 0} \|\rho^\varepsilon(\cdot,T) - \rho^T\|_{L^2(\Omega)} = 0$$

with rate $\varepsilon^\gamma\gamma^{-2}$.

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Our ill-posed control problem

- Let $f \in L^2(Q)$ and $\rho_T \in L^2(\Omega)$ satisfy

$$\sum_{k=1}^{\infty} b_k^2 e^{2T\lambda_k} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \left(\int_0^T e^{t\lambda_k} |f_k(t)| \, dt \right)^2 < \infty. \tag{5.1}$$

- We want to solve the optimization problem (1.1)-(1.2).
- Let $\mathbb{U}_{ad}$ be the closed and convex subset of $L^2(Q)$ given by

$$\mathbb{U}_{ad} := \left\{ f \in L^2(Q) \text{ such that } \sum_{k=1}^{\infty} \left(\int_0^T e^{t\lambda_k} |f_k(t)| \, dt \right)^2 < \infty \right\}.$$

- Let

$$\mathcal{A} = \{(f, \rho^T) \in \mathbb{U}_{ad} \times L^2(\Omega) : \ \rho^T \text{ satisfies(5.1)}\}.$$

Observations

1. We know that $\mathcal{A} \neq \emptyset$.
2. Since $\rho \in C([0,T];L^2(\Omega))$, we know that $\rho(\cdot,0) \in L^2(\Omega)$. We can thus define the cost function

$$J(f,\rho) = \frac{1}{2}\left(\|\rho(\cdot,0) - \rho^d\|^2_{L^2(\Omega)} + \xi\|f\|^2_{L^2(Q)}\right), \qquad (5.2)$$

where $\rho^d \in L^2(\Omega)$ and $\xi > 0$ is a real parameter.

3. Using minimizing sequences, the structure of J, we can prove that there exists a unique $(\bar{f},\bar{\rho}) \in \mathcal{A}$ solution to (1.1)-(1.2). Moreover,

$$\int_\Omega (\rho(x,0) - \bar{\rho}(x,0))(\bar{\rho}(x,0) - \rho^d(x))dx + \int_Q \xi \bar{f}(f - \bar{f})\; dxdt \geq 0$$

for all $(f,\rho) \in \mathcal{A}$.

The approximated optimal control problem

Let $\mathcal{U}_{ad} \subset L^2(Q)$ be closed and convex. Minimize

$$\min_{(f,\rho^\varepsilon)\in\mathcal{U}_{ad}\times L^2(Q)} J^\varepsilon(f) := \frac{1}{2}\left(\|\rho^\varepsilon(\cdot,0)-\rho^d\|^2_{L^2(\Omega)} + \xi\|f\|^2_{L^2(Q)}\right) \tag{5.3}$$

subject to the constraints that

$$\begin{cases} \rho^\varepsilon_t + A\rho^\varepsilon = f & \text{in } Q, \\ \varepsilon\rho^\varepsilon(\cdot,0) + \rho^\varepsilon(\cdot,T) = \rho^T & \text{in } \Omega. \end{cases} \tag{5.4}$$

Existence and uniqueness of optimal controls

For every $\varepsilon > 0$, there exists a unique $f^\varepsilon \in \mathcal{U}_{ad}$ solution of the minimization problem (5.3)-(5.4). The associated state ρ^ε is the unique strong solution of (3.3) with f replaced by f^ε.

Ideas of the proof

- Since J^ε is bounded from below by zero, it is possible to construct a minimizing sequence $(f^{\varepsilon n})_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} J^\varepsilon(f^{\varepsilon n}) = \inf_{f\in\mathcal{U}_{\text{ad}}} J^\varepsilon(f). \tag{5.5}$$

- There exist $f^\varepsilon \in L^2(Q)$, $\rho_T^\varepsilon, \rho_0^\varepsilon \in L^2(\Omega)$, and $\rho^\varepsilon \in L^2((0,T);\mathbb{V})$ such that, as $n\to\infty$, we have the following convergences:

$$f^{\varepsilon n} \to f^\varepsilon \quad \text{weakly in } L^2(Q), \tag{5.6a}$$

$$\rho^{\varepsilon n}(\cdot,0) \to \psi_0^\varepsilon \quad \text{weakly in } L^2(\Omega), \tag{5.6b}$$

$$\rho^{\varepsilon n}(\cdot,T) \to \psi_T^\varepsilon \quad \text{weakly in } L^2(\Omega), \tag{5.6c}$$

$$\rho^{\varepsilon n} \to \rho^\varepsilon \quad \text{weakly in } L^2((0,T);\mathbb{V}). \tag{5.6d}$$

Theorem (Optimatiy conditions)

Let $(\rho^\varepsilon, f^\varepsilon)$ *be the solution of the minimization problem* (5.3)-(5.4)*. Then, there exists* $q^\varepsilon \in L^2((0,T);\mathbb{V}) \cap H^1((0,T);\mathbb{V}^\star)$ *such that we have the following optimality systems:*

$$\begin{cases} \rho_t^\varepsilon + A\rho^\varepsilon = f^\varepsilon & \text{in } Q, \\ \varepsilon\rho^\varepsilon(\cdot,0) + \rho^\varepsilon(\cdot,T) = \rho^T & \text{in } \Omega, \end{cases} \tag{5.7}$$

and

$$\begin{cases} -q_t^\varepsilon + Aq^\varepsilon = 0 & \text{in } Q, \\ \varepsilon q^\varepsilon(\cdot,T) + q^\varepsilon(\cdot,0) = \rho^\varepsilon(\cdot,0) - \rho^d & \text{in } \Omega, \end{cases} \tag{5.8}$$

and

$$\int_Q (\xi f^\varepsilon - q^\varepsilon)(f - f^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall f \in \mathcal{U}_{ad}. \tag{5.9}$$

Theorem (Convergence of optimal solutions)

Let $(\bar{f}, \bar{\rho})$ *be a solution of the minimization problem* (1.1)-(1.2), *and let* $(f^\varepsilon, \rho^\varepsilon, q^\varepsilon)$ *be as above.*

- *Assume that* $f^\varepsilon \in \mathbb{U}_{ad}$ *and that* $Int(\mathbb{U}_{ad}) \neq \emptyset$.
- *Then, there exists* $\bar{q} \in L^2(Q)$ *such that, as* $\varepsilon \downarrow 0$, *we have the following convergences:*

$$f^\varepsilon \text{ converges strongly to } \bar{f} \in L^2(Q) \text{ and } \bar{f} \in \mathbb{U}_{ad}, \tag{5.10}$$

$$\rho^\varepsilon \text{ converges weakly to } \bar{\rho} \in L^2((0,T);\mathbb{V}) \cap H^1((0,T);\mathbb{V}^\star), \tag{5.11}$$

$$\rho^\varepsilon(\cdot, T) \text{ converges strongly to } \rho^T \in L^2(\Omega), \tag{5.12}$$

$$\rho^\varepsilon(\cdot, 0) \text{ converges strongly to } \bar{\rho}(\cdot, 0) \in L^2(\Omega), \tag{5.13}$$

$$q^\varepsilon \text{ converges weakly to } \bar{q} \in L^2(Q). \tag{5.14}$$

Optimality conditions for the ill-posed control problem

Assume that $\mathrm{Int}(\mathbb{U}_{ad}) \neq \emptyset$. Then, $(\bar{f}, \bar{\rho})$ is the solution of the minimization problem (1.1)-(1.2) if and only if there exists $\bar{q} \in L^2(Q)$ such that the triple $(\bar{f}, \bar{\rho}, \bar{q})$ satisfies the following singular optimality systems: $\bar{\rho}$ is a strong solution of

$$\begin{cases} \bar{\rho}_t + A\bar{\rho} = \bar{f} & \text{in } Q, \\ \bar{\rho}(\cdot, T) = \rho^T & \text{in } \Omega, \end{cases} \tag{5.15}$$

and

$$\int_Q \bar{q}\Big(\phi_t + A\phi\Big)\, dxdt = 0 \tag{5.16}$$

for every $\phi \in \mathcal{D}((0, T)), D(A))$, and finally

$$\int_Q (\xi\bar{f} - \bar{q})(f - \bar{f})\,\mathrm{d}x\,\mathrm{d}t \geq 0, \quad \forall f \in \mathbb{U}_{ad}. \tag{5.17}$$

THANK YOU!