Optimal Control Design for Fluid Mixing: from Open-Loop to Closed-Loop

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Mixing phenomena

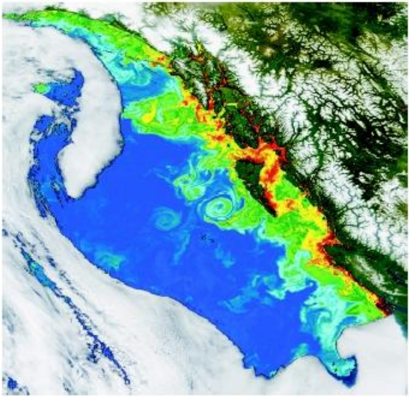
Mixing is to disperse one material or field in another medium. It occurs in many natural phenomena and industrial applications.

Mixing in painting

Mixing in baking

Mixing phenomena

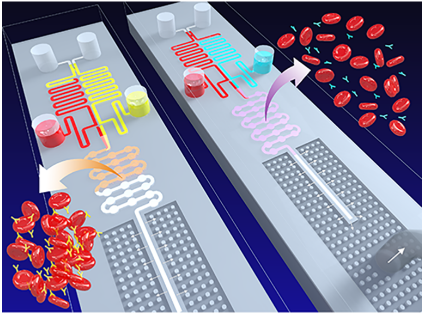
Spreading of a pollutant in the atmosphere



Mixing of temperature, salt, and nutrient in ocean*.

*http://www.waterencyclopedia.com/Mi-Oc/Ocean-Mixing.html

Mixing phenomena



Microfluidic mixing: controllable and fast mixing is critical for practical development of microfluidic and lab-on-chip devices*.

Optimal mixing?

*https://www.elveflow.com/microfluidic-reviews/microfluidic-flow-control/microfluidic-mixers-a-short-review/

Outline

- Optimal boundary control for fluid mixing via flow advection
- Feedback control (sub-optimal) for fluid mixing
  - instantaneous control design
  - approximation of the optimal open-loop control
- Asymptotic behavior of the nonlinear closed-loop system
- Numerical Implementation

Mixing modeled by transport equation

Consider the transport equation in an open bounded and connected domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, with a regular boundary Γ

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad \theta(0) = \theta_0, \quad x \in \Omega.$$

- θ: mass concentration or density distribution
- v: incompressible velocity field with no-penetration BC, that is,

$$\nabla \cdot v = 0, \quad v \cdot n|_{\Gamma} = 0.$$

- $\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}$, $p \in [1, \infty]$, $t > 0$.

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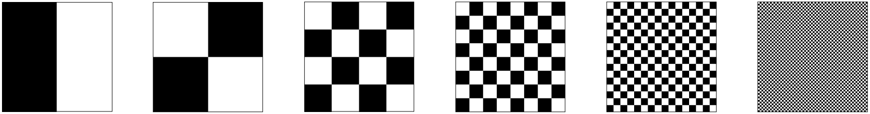

Mix-norm: negative Soblev norm

- **Mix-norm**: consider the 1D periodic interval $[0, L]$. Define

$$d(\theta, x, w) = \frac{1}{w} \int_{x-w/2}^{x+w/2} \theta(y)\, dy$$

for all $x, w \in [0, L]$. The mix-norm $M(\theta)$ is then obtained by averaging d^2 over x and w:

$$M^2(\theta) = \frac{1}{L^2} \int_0^L \int_0^L d^2(\theta, x, w)\, dx\, dw$$

*Mathew-Mezic-Petzold '05, Lin-Thiffeault-Doering '10, Thiffeault '11, etc

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$$M^2(\theta) = \frac{1}{L^2}\int_0^L \int_0^L d^2(\theta, x, w)\, dx\, dw \sim \|\theta\|^2_{H^{-1/2}}$$

In fact, any $H^{-\alpha}$-norm for $\alpha > 0$, which quantifies the weak convergence, can be used as a mix-norm.

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Some known results and open questions

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad \nabla \cdot v = 0, \quad x \in \Omega,$$
$$\theta(0) = \theta_0.$$

- Alberti-Crippa-Mazzucato '16: For $\theta_0 \in L^\infty(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} \theta_0 = 0$ and self-similar structure, there exists $v \in W^{s,p}$ uniformly in time, for some $s \geq 0$ and $1 \leq p \leq \infty$, such that
  1. if $s < 1$: perfect mixing in finite time, i.e., there is a time t^* such that $\lim_{t \to t^*} \|\theta(t, \cdot)\|_{H^{-1}} = 0$;
  2. if $s = 1$: exponential decay;
  3. if $s > 1$: polynomial decay. *However, it is unknown whether* $\|\theta(t, \cdot)\|_{H^{-1}}$ *decays exponentially in time for some* $s > 1$.

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- Elgindi-Zlatoš '18: The answer is affirmative for

$$1 < s < \frac{1 + \sqrt{5}}{2} < 2 \quad \text{and} \quad p \in \left[1, \ \frac{2}{2s + 1 - \sqrt{5}}\right).$$

Objective: optimal mixing via flow advection

- Optimal mixing via Stokes flows

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0.$$

- Optimal mixing via Navier-Stokes flows
    - **Passive scalar**: one-way coupling. The transported scalar does not influence the velocity field

$$\frac{\partial v}{\partial t} - \nu \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \nabla \cdot v = 0.$$

    - **Active scalar**: two-way coupling. The transported scalar influences the velocity field through local forces (such as buoyancy)

$$\frac{\partial v}{\partial t} - \nu \Delta v + v \cdot \nabla v + \nabla p = \theta \mathrm{e}, \quad \nabla \cdot v = 0,$$

    where e is a unit vector in the direction of buoyancy*.

*Hou-Li '05, Chae '06, H.-Kukavica-Zane '13, '15, '16, H. *et. al* '18, etc.

Mixing in Stokes flows

Consider

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad x \in \Omega,$$

where the velocity field is govern by

$$\frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0, \quad v \cdot n|_{\Gamma} = 0.$$

- p: pressure; ν: viscosity

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Motivated by the observation that moving walls accelerate mixing compared to fixed walls with no-slip boundary condition*, we consider the Navier slip boundary control for mixing

$$v \cdot n|_{\Gamma} = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + kv \cdot \tau)|_{\Gamma} = g \cdot \tau.$$

- n and τ are the outward unit normal and tangential vectors to the boundary Γ
- $\mathbb{D}(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$: deformation tensor
- $k > 0$: coefficient of friction
- g: control input function

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Forward Model Simulations ($\nu = 1$ and $k = 0.5$)

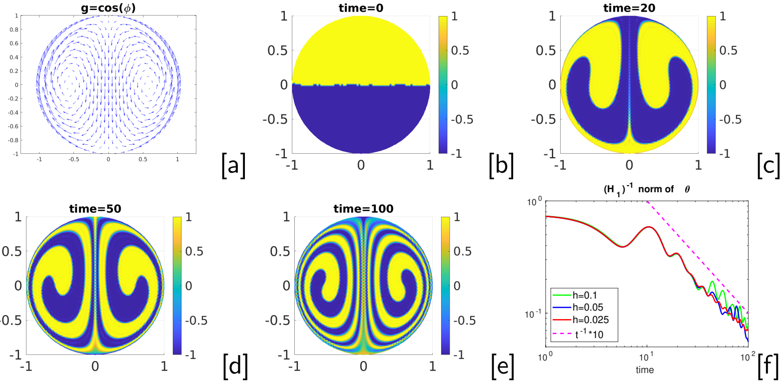

Fig. 1. $\Omega = \{(x,y): x^2+y^2<1\}$ and $g = \cos(\phi)\tau$. [a]: sample of velocity field. The maximum magnitude is roughly 0.4. [b, c, d, e]: θ at t=0, 20, 50, 100. [f]: $(H^1(\Omega))'$ norms of θ in time. All the contour figures of θ are using the data when the mesh size $h = 0.025$.

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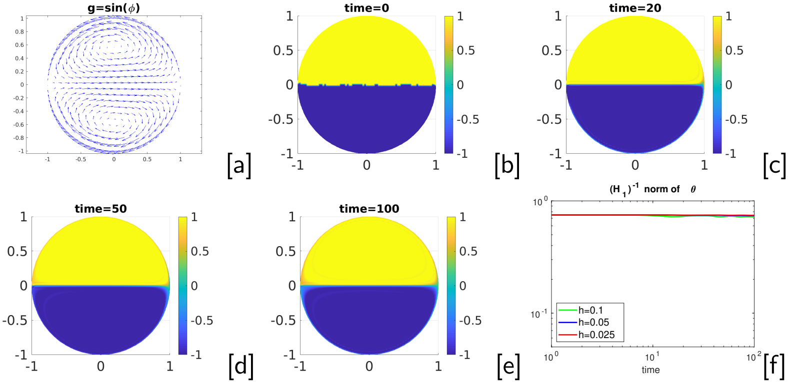

Fig. 2: $g = \sin(\phi)\tau$. [a]: sample of velocity field. The maximum magnitude is roughly 0.4. [b,c,d,e]: θ at t=0, 20, 50, 100. [f]: $(H^1(\Omega))'$ norms of θ in time. All the contour figures of θ are using the data when the mesh size $h = 0.025$.

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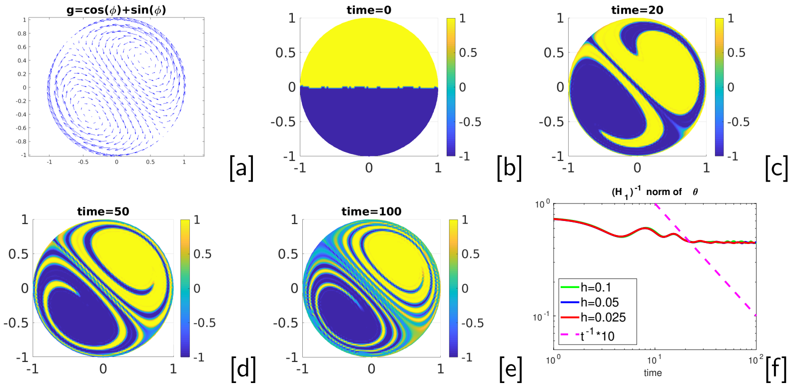

Fig. 3: $g = \cos(\phi)\tau + \sin(\phi)\tau$. [a]: sample of velocity field. The maximum magnitude is roughly 0.4. [b,c,d,e]: θ at t=0, 20, 50, 100. [f]: $(H^1(\Omega))'$ norms of θ in time. All the contour figures of θ are using the data when the mesh size $h = 0.025$.

Problem formulation: optimal bilinear control

- **Minimize**

$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g\|^2_{U_{ad}}, \quad \gamma > 0, \quad (P),$$

for a given $T > 0$, subject to

$$\begin{cases} \frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \\ \frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = 0, \\ \nabla \cdot v = 0, \end{cases}$$

with Navier slip boundary control:

$$v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + k v \cdot \tau)|_\Gamma = g \cdot \tau,$$

and initial conditions $\theta(0) = \theta_0$ and $v(0) = v_0$. Here $\gamma > 0$ is the control weight and U_{ad} stands for the set of admissible controls.

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- **Procedures**: (1) Prove the well-posedness of problem (P); (2) Identify the set of admissible controls; (3) Prove the existence of an optimal control and establish the optimality conditions.

Challenges in analysis and computation

- *Nonlinearity:* The nonlinear coupling due to advection essentially leads to a nonlinear control and non-convex optimization problem.
- *Zero diffusivity:* Differentiability leads to a high-order regularity required for the velocity field.
- *Boundary Control:*
  1. Creation of vorticity on the domain boundary;
  2. Compatibility conditions may come into play even in the case of non-smooth solutions.

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- *Boundary Control:*
  1. Creation of vorticity on the domain boundary;
  2. Compatibility conditions may come into play even in the case of non-smooth solutions.
- *Computation:*
  1. Mass conservation of scalar transport in incompressible flows;
  2. Small-scale structures and large gradients of the scalar field will develop in the mixing process.

Well-posedness of problem (P)

- Cost functional:

$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g\|^2_{U_{ad}}, \quad \gamma > 0. \quad (P)$$

Consider

$$(-\Delta + I)\eta = \theta, \quad \frac{\partial \eta}{\partial n}|_\Gamma = 0.$$

Let $\Lambda = (-\Delta + I)^{1/2}$. Then

$$\|\theta\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta\|_{L^2(\Omega)} = \|\Lambda\eta\|_{L^2(\Omega)} = \|\eta\|_{H^1(\Omega)}.$$

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Let $\Lambda = (-\Delta + I)^{1/2}$. Then
$$\|\theta\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta\|_{L^2(\Omega)} = \|\Lambda\eta\|_{L^2(\Omega)} = \|\eta\|_{H^1(\Omega)}.$$
- Define
$$V^s_n(\Omega) = \{v \in H^s(\Omega) : \text{div } v = 0, \ v \cdot n|_\Gamma = 0\}, \quad s \geq 0,$$
$$V^s_n(\Gamma) = \{g \in H^s(\Gamma) : g \cdot n|_\Gamma = 0\}, \quad s \geq 0.$$
- For $(\theta_0, v_0) \in L^\infty(\Omega) \times V^0_n(\Omega)$, there exists $g \in L^2(0, T; V^0_n(\Gamma))$ such that J is finite.

Theorem (Existence, H., AMO (2018))

Assume that $(\theta_0, v_0) \in L^\infty(\Omega) \times V^0_n(\Omega)$. There exists an optimal solution $g \in L^2(0, T; V^0_n(\Gamma))$ to problem (P).

Identification of the set of admissible controls

- To establish the first-order necessary optimality conditions, one needs $\theta \in H^1(\Omega)$, which requires $\theta_0 \in H^1(\Omega)$ and

$$\boxed{\int_0^T \|\nabla v\|_{L^\infty} \, d\tau < \infty.}$$

This causes the challenges!

In fact, by Agmon's inequality

$$\|\nabla v\|_{L^\infty} \le c\|v\|_{H^{1+d/2+\epsilon}}, d = 2, 3,$$

for some $\epsilon > 0$.

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for some $\epsilon > 0$.

- As a result, the time derivative of g needs to be penalized

$$U_{ad} = L^2(0, T; V_n^{d/2-1/2+\varepsilon}(\Gamma)) \cap H^{d/4-1/4+\varepsilon/2}(0, T; V_n^0(\Gamma)), \qquad d = 2, 3,$$

for any $\varepsilon > 0$.

An approximating control approach: passive scalar

Consider

$$\frac{\partial \theta_\epsilon}{\partial t} + v_\epsilon \cdot \nabla \theta_\epsilon = \epsilon \Delta \theta_\epsilon, \quad v_\epsilon = v,$$

with Neumann BC

$$\epsilon \frac{\partial \theta_\epsilon}{\partial n}|_\Gamma = 0.$$

For a given $T > 0$, find a control

$$\boxed{g_\epsilon \in U_{\epsilon_{ad}} = L^2(0, T; V_n^0(\Gamma))}$$

minimizing the cost functional

$$J_\epsilon(g_\epsilon) = \frac{1}{2}\|\theta_\epsilon(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g_\epsilon\|^2_{U_{\epsilon_{ad}}}. \qquad (\mathrm{P}_\epsilon)$$

Convergence Issues:

- Does $\theta_\epsilon \to \theta$ as $\epsilon \to 0$? Yes! (with the condition that $\int_0^T \|\nabla v\|_{L^\infty} \, d\tau < \infty$).
- Let g_ϵ^{opt} be the optimal solution to (P_ϵ).
  - Does $g_\epsilon^{opt} \to g^{opt}$ as $\epsilon \to 0$? Yes!
  - Is g^{opt} a solution to the original problem (P)? Yes!

First-order optimality system via an approximating control approach

- To summarize, if $(g^{opt}, v^{opt}, \theta^{opt})$ is the optimal solution, then it satisfies

State Equations
$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta = 0, \quad \theta(0) = \theta_0, \\ \partial_t v - \nu \Delta v + \nabla p = 0, \quad \nabla \cdot v = 0, \quad v(0) = v_0, \\ v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + kv \cdot \tau)|_\Gamma = g \cdot \tau, \end{cases}$$

Adjoint Equations
$$\begin{cases} -\partial_t \rho - v \cdot \nabla \rho = 0, \quad \rho(T) = \Lambda^{-2}\theta(T), \\ -\partial_t w - \nu \Delta w + \nabla q = \theta \nabla \rho, \quad \nabla \cdot w = 0, \quad w(T) = 0, \\ w \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(w) \cdot \tau + kw \cdot \tau)|_\Gamma = 0, \end{cases}$$

Optimality Condition: $\boxed{g = \frac{1}{\gamma} w|_\Gamma.}$

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Adjoint Equations $\begin{cases} -\partial_t \rho - v \cdot \nabla \rho = 0, \quad \rho(T) = \Lambda^{-2}\theta(T), \\ -\partial_t w - \nu \Delta w + \nabla q = \theta \nabla \rho, \quad \nabla \cdot w = 0, \quad w(T) = 0, \\ w \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(w) \cdot \tau + kw \cdot \tau)|_\Gamma = 0, \end{cases}$

Optimality Condition: $\boxed{g = \frac{1}{\gamma} w|_\Gamma.}$

Theorem (Uniqueness, H. (AMO, 2018))

For $d = 2$ and $\gamma > 0$ sufficiently large, there exists at most one optimal controller to problem (P).

Mixing in Navier-Stokes flows (d=2)

- **Case I:** Passive scalar field

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0$$

advected by the Navier-Stokes flow

$$\frac{\partial v}{\partial t} - \nu \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \nabla \cdot v = 0.$$

- **Case II:** Active scalar field advected by the buoyancy-driven flow,

$$\frac{\partial v}{\partial t} - \nu \Delta v + v \cdot \nabla v + \nabla p = \theta \mathrm{e}, \quad \nabla \cdot v = 0,$$

where $\mathrm{e} = (0,1)^T$ is a unit vector in the direction of the gravitational acceleration.

x

- **Minimize**

$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g\|^2_{U_{ad}} - \frac{\zeta}{2}\int_0^T \|\nabla \times v\|^2_{L^2}\, dt, \quad \gamma, \zeta > 0, \quad (P_{NS})$$

for given $T > 0$, where $\nabla \times v = \partial_1 v_2 - \partial_2 v_1$ (H.-Wu, SICON '18, JDE '19).

Numerical implementation for mixing via Stokes flows

- We restrict our interest to the control input function of the form

$$g(x,t) = \sum_{i=1}^{M} \alpha_i(t) b_i(x) \in L^2(0,T;V_n^0(\Gamma)),$$

  where $\{b_1, \cdots, b_M\} \subset V_n^0(\Gamma)$ are prescribed and $\alpha_i(t) \in L^2(0,T)$ are the control functions in time.

- Numerical schemes (H.-Zheng, 2020)
  - Taylor-Hood finite element algorithm together with projection method and BDF2 time discretization for solving the Stokes equations
  - Runge-Kutta DG method for solving the transport equation
  - Gradient decent method with backtracking; conjugate gradient with Polak-Ribiére (CG-PR) and Flether-Reeves (CG-FR) methods

Ex. 1: $g = \alpha \cos(\phi)\tau$ and $\theta_0 = \tanh(y/0.1)$

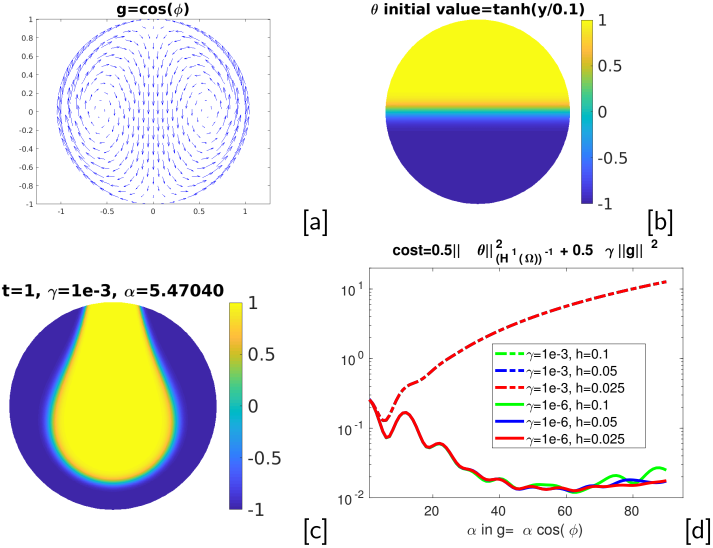

Figure 4. $\Omega = \{(x, y) \colon x^2 + y^2 < 1\}$ with a uniform triangulation and $h = 0.025$.

Ex. 1. $\gamma = $ 1e-6

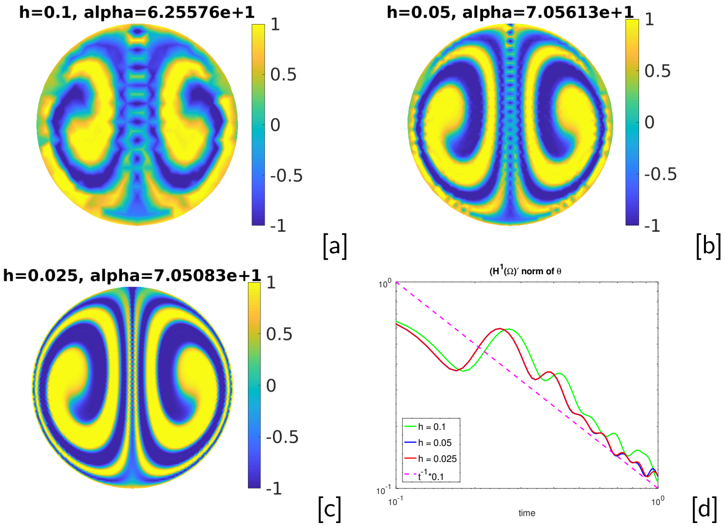

Figure 5. [a,b,c]: optimal numerical solutions of θ at $t = 1$ under meshes $h = 0.1$, $h = 0.05$, and $h = 0.025$, respectively. [d]: $(H^1(\Omega))'$-norms of θ in time.

Ex. 2. Cosine and Sine based controls

Consider $\theta_0 = \sin(2\pi y)$ and $g = \alpha_1(t)\cos(2\phi)\tau + \alpha_2(t)\sin(2\phi)\tau$

$$\alpha_1(t) = \sum_{i=1}^{N} \alpha_{1i}\chi_{((i-1)\Delta s,\, i\Delta s)}(t), \quad \alpha_2(t) = \sum_{i=1}^{N} \alpha_{2i}\chi_{((i-1)\Delta s,\, i\Delta s)}(t)$$

for $\alpha_{1i}, \alpha_{2i} \in \mathbb{R}$.

- N is a positive integer, $\Delta s = \frac{1}{N}$ is the segmentation interval in time.
- $\chi_{((i-1)\Delta s,\, i\Delta s)}(t) = 1$ is the characteristic function.
- For convenience, we let $\vec{\alpha} = (\alpha_{11}, \ldots, \alpha_{1N}, \alpha_{21}, \ldots, \alpha_{2N})^T \in \mathbb{R}^{2N}$.

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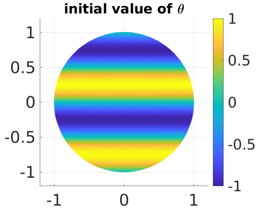
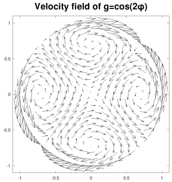
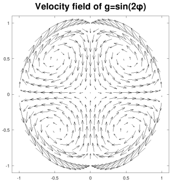


[a]



[b]



[c]

Figure 5: [a] $\theta_0 = \sin(2\pi y)$; velocity fields induced by $g = \cos(2\phi)\tau$ in [b] and $g = \sin(2\phi)\tau$ in [c].

Ex. 2. $\gamma = $ 1e-6

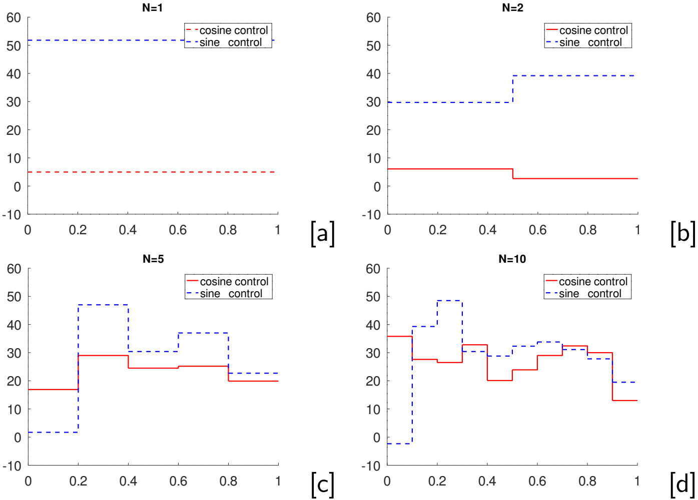


Figure 7: Optimal control weight $\vec{\alpha}$. [a]: $N = 1$, cost=9.69e-03. [b]: $N = 2$, cost=9.10e-03. [c]: $N = 5$, cost=5.91e-03. [d]: $N = 10$, cost=5.65e-03.

Ex. 2 (cont'd)

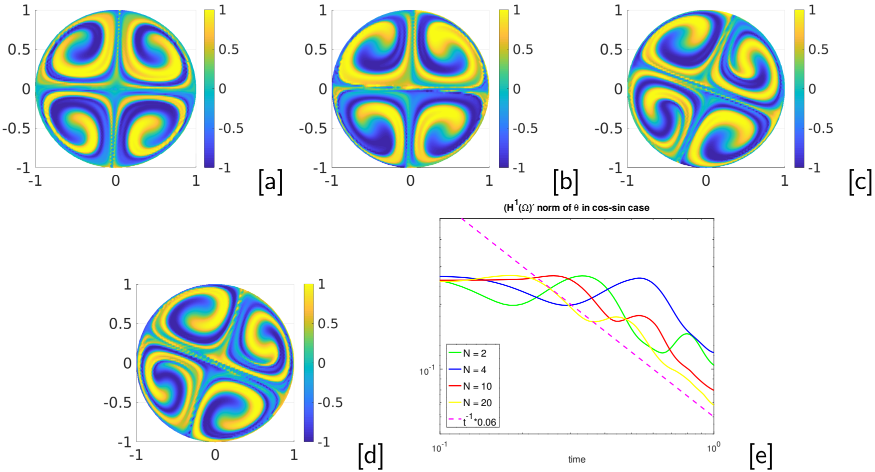


Figure 8: Optimal θ when $t = 1$ using the cos-sin based control design. [a]: $N = 1$, cost=9.69e-03. [b]: $N = 2$, cost=9.10e-03. [c]: $N = 5$, cost=5.91e-03. [d]: $N = 10$, cost=5.65e-03. [e]: $(H^1(\Omega))'$-norms in time.

Ex. 3. Cosine, Sine and 1 based controls

Consider 5 basis functions (1, $\cos(\theta)$, $\sin(\theta)$, $\cos(2\theta)$, $\sin(2\theta)$) and 10 time segments. The initial guesses are taken as alternating sequences of $\alpha^0 = \pm 80$ and ± 60, respectively. The mesh size $h = 0.0125$.

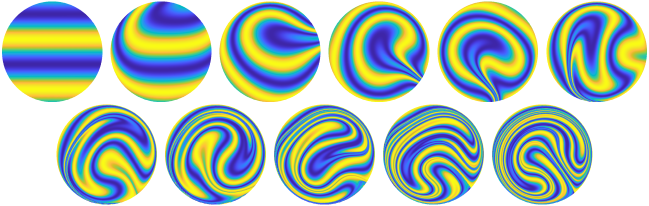


Figure 9: $N = 10$, $\alpha^0 = \pm 80$. CG-FR method. Time $t = 0, 0.1, \cdots, 1$.

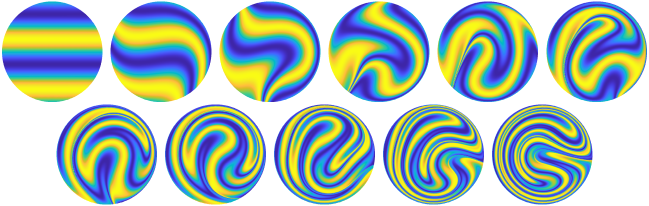


Figure 10: $N = 10$, $\alpha^0 = \pm 60$. CG-FR method. Time $t = 0, 0.1, \cdots, 1$.

Feedback law for transport equation

- Recall

$$\begin{cases} \frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad \theta(0) = \theta_0, \\ \nabla \cdot v = 0, \quad v \cdot n|_\Gamma = 0, \\ (-\Delta + I)\eta = \theta, \quad \frac{\partial \eta}{\partial n}|_\Gamma = 0. \end{cases}$$

- **Observation:** Taking the L^2-inner product of the transport equation with η and integrating by parts yield

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{(H^1(\Omega))'} = (\theta \nabla \eta, v) = (\mathbb{P}(\theta \nabla \eta), v),$$

where $\mathbb{P}\colon L^2(\Omega) \to V^0_n(\Omega)$. This inspires the choice of v:

$$v = -c(t)\mathbb{P}(\theta \nabla \eta), \quad c(t) > 0.$$

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- In particular, let $c(t) = ||\mathbb{P}(\theta\nabla\eta)||^{-2}_{L^2}$, then

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{(H^1(\Omega))'} = -1.$$

Feedback law for transport equation (cont'd)

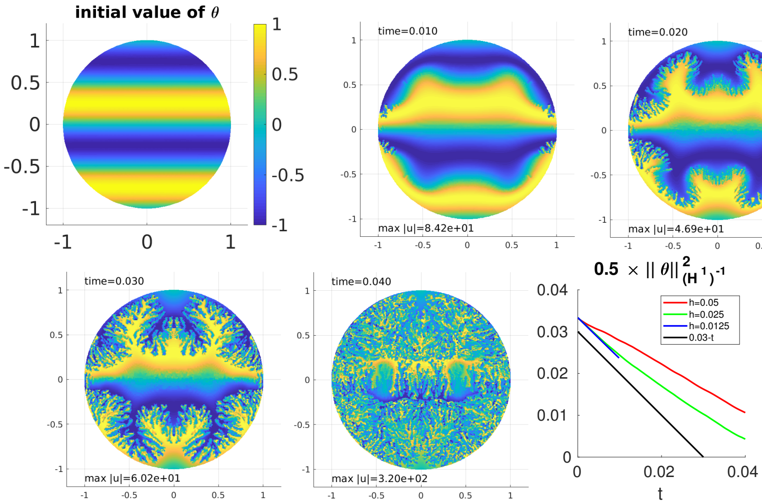

Fig. 4. Initial value: $\theta_0 = \sin(2\pi y)$. Snapshots of density evolution at $t = 0.01, 0.02, 0.03, 0.04$, with $v = -\frac{\mathbb{P}(\theta\nabla\eta)}{||\mathbb{P}(\theta\nabla\eta)||^2_{L^2}}$, $h = 0.0125$.

Construction of feedback laws

- **Instantaneous control design vs. Approximation of the optimality system**
- Recall that

$$\textbf{State equations} \qquad \begin{cases} \partial_t \theta = -v \cdot \nabla \theta, & \theta(0) = \theta_0, \\ \partial_t v = Av + Bg, & v(0) = v_0, \end{cases}$$

where $A = \mathbb{P}\Delta$, $\mathbb{P}\colon L^2(\Omega) \to V_n^0(\Omega)$, and B is the control input operator.

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$$\begin{cases} \theta^{i+1} = \theta^i - \delta v^{i+1} \cdot \nabla \theta^i, \\ (-\Delta + I)\eta^{i+1} = \theta^{i+1}, \quad \frac{\partial \eta^{i+1}}{\partial n}|_\Gamma = 0, \\ v^{i+1} = v^i + \delta A v^{i+1} + B g^{i+1}. \end{cases} \tag{1}$$

- Consider now the cost functional for one time step

$$J(g^{i+1}) = \frac{1}{2}\|\Lambda^{-1}\theta^{i+1}\|_{L^2}^2 + \frac{\gamma}{2}\|g^{i+1}\|_{U_{ad}}^2.$$

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- This method is closely tied to receding horizon control (RHC) or model predictive control (MPC) with finite time horizon (cf. Hinze-Kunisch '97, Hinze-Volkwein '02).

Construction of feedback laws (cont'd)

- Let (ρ^{i+1}, w^{i+1}) be the adjoint state of (θ^{i+1}, v^{i+1}). Applying the Euler-Lagrange method leads to

$$\rho^{i+1} = \eta^{i+1}, \quad (I - \delta A)w^{i+1} = \delta\mathbb{P}(\theta^i \nabla \rho^{i+1}), \tag{2}$$

and the optimality condition

$$\gamma g^{i+1} + B^* w^{i+1} = 0. \tag{3}$$

The optimality system (1)–(3) admits a unique solution due to the quadratic cost functional and the uniqueness of (1).

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- Compute $(g^{i+1}, v^{i+1}, \theta^{i+1})$ recursively by setting $g_0^i = 0$, which turns out to be the semi-implict time discretization of the closed-loop system

$$\begin{cases} \partial_t \theta = -v \cdot \nabla \theta, \quad \theta(0) = \theta_0, \\ \partial_t v = Av + Bg, \quad v(0) = v_0, \\ g = -\gamma^{-1} \delta B^* (I - hA)^{-1} \mathbb{P}(\theta \nabla \eta) \quad \text{(sub-optimal)}. \end{cases}$$

Optimize-then-approximate approach (OTA)

Approximation of the optimality system:

- Discretize the optimality system over one step time sub-interval, and then carry the information for the next time sub-interval;
- The state and the adjoint equations will be formulated forward and backward in time, but just for one step, so that the adjoint state can be related to the state explicitly;
- A recursive procedure leads to a sup-optimal continuous feedback law.

State Equations $\begin{cases} \theta^{i+1} = \theta^i - \delta v^{i+1} \cdot \nabla \theta^i, \\ (-\Delta + I)\eta^{i+1} = \theta^{i+1}, \quad \frac{\partial \eta^{i+1}}{\partial n}|_\Gamma = 0, \\ v^{i+1} = v^i + \delta A v^{i+1} + B g^{i+1}, \end{cases}$

Ajoint Equations $\begin{cases} \rho^i = \rho^{i+1} + \delta v^{i+1} \cdot \nabla \rho^{i+1}, \quad \rho^{i+1} = \Lambda^{-2}\theta^{i+1}, \\ w^i = w^{i+1} + \delta A w^i + \delta \mathbb{P}(\theta^i \nabla \rho^i), \end{cases}$

Optimality Condition: $\boxed{g^{i+1} = \frac{1}{\gamma} w^i|_\Gamma.}$

Well-posedness and stability of the closed-loop system

- The closed-loop system reads

$$\begin{cases} \partial_t \theta = -v \cdot \nabla \theta, \quad \theta(0) = \theta_0, \\ \partial_t v = Av + Bg, \quad v(0) = v_0, \\ g = -\gamma^{-1} \delta B^* (I - \delta A)^{-1} \mathbb{P}(\theta \nabla \eta) \quad \text{(sub-optimal)}, \end{cases}$$

  where $\eta = (I - \Delta)^{-1} \theta$, γ and h are the fixed parameters.

- Let $B = \mathbb{P} I$ (internal control). Then

$$\partial_t v = Av - \gamma^{-1} \delta (I - \delta A)^{-1} \mathbb{P}(\theta \nabla \eta).$$

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- Let $B = \mathbb{P}I$ (internal control). Then

$$\partial_t v = Av - \gamma^{-1} \delta (I - \delta A)^{-1} \mathbb{P}(\theta \nabla \eta).$$

- **Well-posedness:** For $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times \mathscr{D}(A)$, there exists a unique solution (θ, v) satisfying

$$(\theta, v) \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)) \times L^\infty(0, T; V_n^2(\Omega)) \cap L^2(0, T; V_n^3(\Omega))$$

for any $T > 0$.

Well-posedness and stability (cont'd)

- Applying energy estimates yields

$$\frac{d}{dt}\text{ Total Energy } = \frac{d}{dt}\|\theta\|^2_{(H^1(\Omega))'} + \frac{\gamma}{h}\frac{d}{dt}\|v\|^2_{H^1(\Omega)} \leq -C\|v\|^2_{H^1(\Omega)} < 0.$$

- **Long-time behavior:**
  1. $\|v\|_{L^2}, \|\nabla v\|_{L^2}, \|Av(t)\|_{L^2}, \|\partial_t v(t)\|_{L^2} \to 0$ as $t \to +\infty$;
  2. $\|\theta\|_{(H^1(\Omega))'} \to c_0$ as $t \to \infty$, and
     $c_0 < \sqrt{\|\Lambda\eta_0\|_{L^2} + \frac{\gamma}{h}\|v_0\|^2_{L^2} + \gamma(2\|\mathbb{D}v_0\|^2_{L^2} + k\|v_0 \cdot \tau\|^2_{L^2})}$;
  3. $\|g\|_{L^2} \to 0$ as $t \to +\infty$;
  4. $\|\theta\nabla\eta - \nabla q\|_{(H^2(\Omega))'} \to 0$ as $t \to +\infty$.

Numerical simulation of the closed-loop system

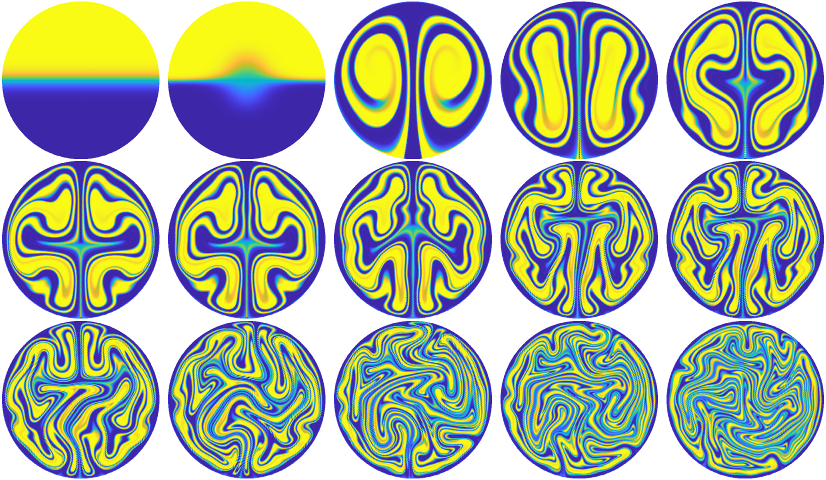
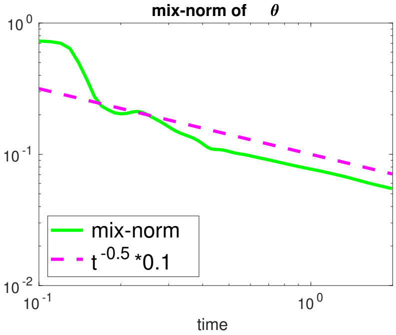


Fig. 5. $\theta_0 = \tanh(y/0.1)$. Density evolution for $t \in [0, 2]$, $h = 0.0125, \delta = 0.1, \gamma =$1e-6

Mixing decay rate in time



Ongoing work

- Investigate the optimality of the feedback laws
- Justify the polynomial decay rate of the mix-norm $\|\theta\|_{(H^1(\Omega))'}$ in time and its relation to the control actuation
- Analyze the asymptotic behavior of the closed-loop system by localized internal control and Navior slip boundary control

Thank you for your attention!

Questions?