The Location Problem for Compressor Stations in Pipeline Networks

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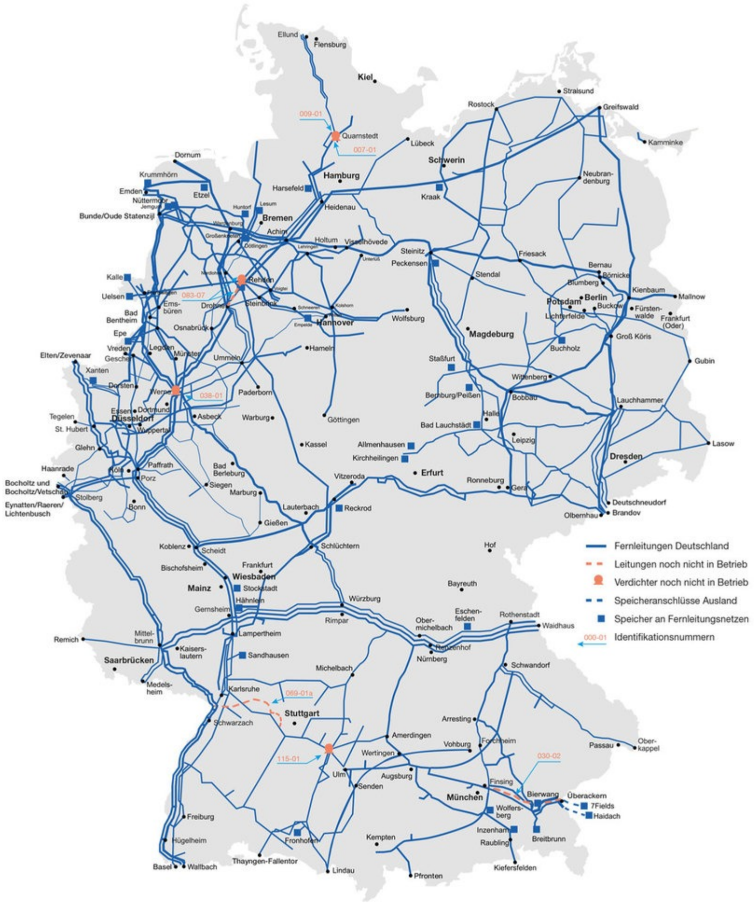
Motivation

Natural Gas Transport



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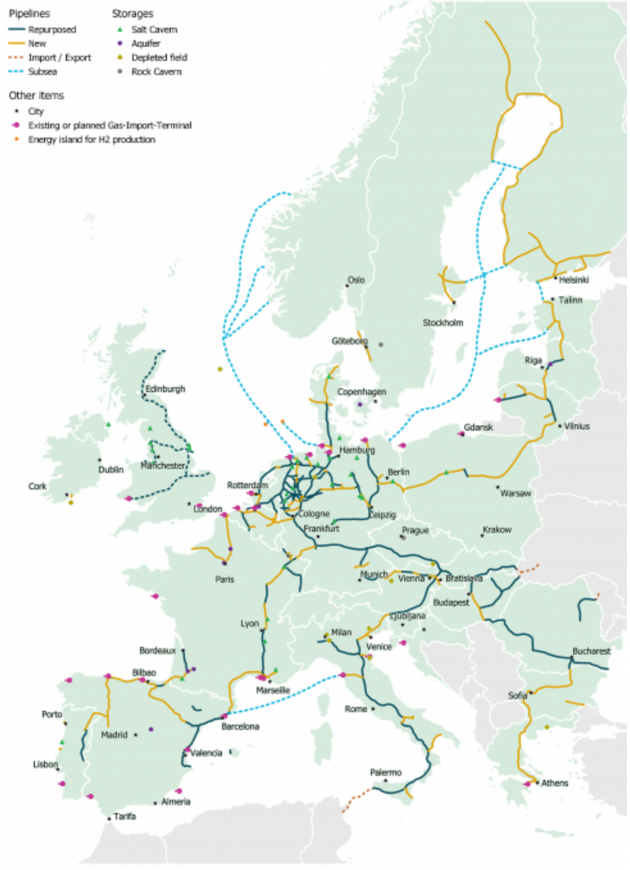

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Gas Network Modelling

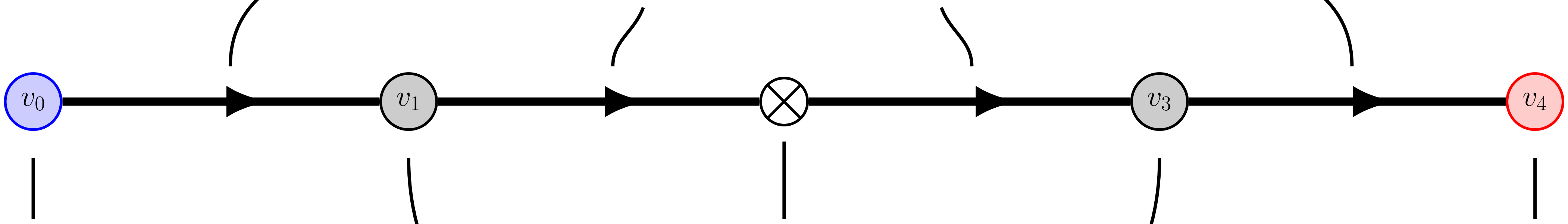
Gas Flow in Pipeline Networks

p	gas pressure	v	gas velocity	g	gravitational constant
ρ	gas density	λ/D	pipe friction	α	pipe slope

Steady State Gas Transport

$$(\rho v)_x = 0$$

$$\left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha)$$



Source: Inlet Pressure

$$p = p_0$$

Compressor Station

$$\frac{p_{\text{out}}}{p_{\text{in}}} = u$$

Sink: Gas Outflow

$$(\rho v) = b$$

Coupling Conditions

Conservation of Mass: $\sum(\rho v)_{\text{in}} = \sum(\rho v)_{\text{out}}$, Continuity in Pressure: $p_{\text{in}} = p_{\text{out}}$

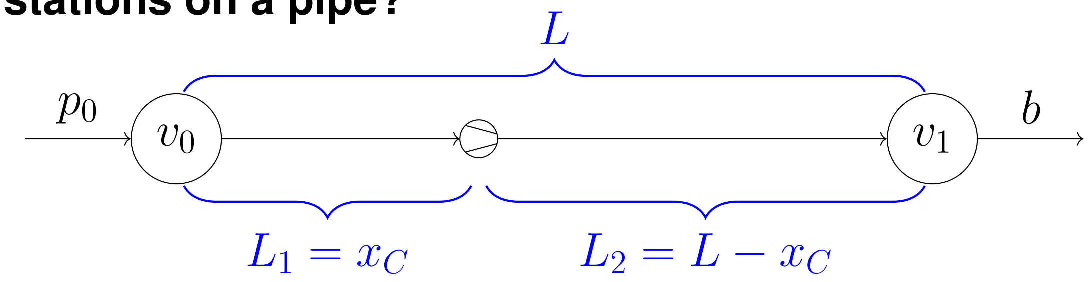
Optimal Compressor Location

Mathematical Modelling

- **The stationary gas flow for ideal gases on a single pipe is given by**

$$q(x) \equiv b \quad (const.), \qquad p(x) = p_0^2 - \phi\, b\, |b|\, x \quad \text{with} \quad \phi = \frac{\lambda}{D} R_S\, T, \quad x \in [0, L]$$

- **Where to place compressor stations on a pipe?**



- **The stationary gas flow with compressor station for ideal gas is given by**

$$p_1^2(x) = p_0^2 - \phi\, b\, |b|\, x \qquad x \in [0, L_1]$$

$$p_2^2(x) = u\, p_0^2 - \phi\, b\, |b|\, \big(u\, L_1 + x\big) \quad x \in [0, L_2]$$

- **Consider pressure bounds on the pipe**

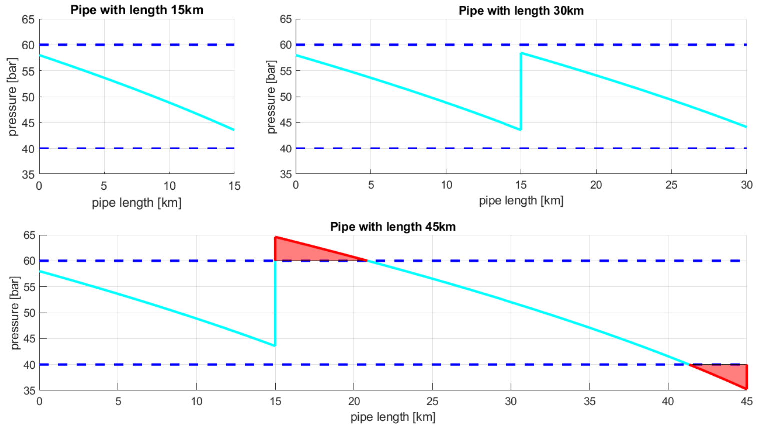
$$\begin{aligned} p_1(x) &\in [p_{\min}, p_{\max}] \\ p_2(x) &\in [p_{\min}, p_{\max}] \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} p_1(0) &\le p_{\max}, & p_1(L_1) &\ge p_{\min}, \\ p_2(0) &\le p_{\max}, & p_2(L_2) &\ge p_{\min} \end{aligned}$$

Optimal Compressor Location

Deterministic Optimization

For $f : \mathbb{R}^2 \to \mathbb{R}$ with f strictly monotonically increasing in the first argument, consider the optimization problem:

$$
\text{(OPT 1)} \quad \begin{cases} \min\limits_{u, x_C} & f(u, q), \\ \text{s.t.} & p_1(L_1) \geq p_{\text{min}}, \quad p_2(0) \leq p_{\text{max}}, \quad p_2(L_2) \geq p_{\text{min}}, \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}
$$

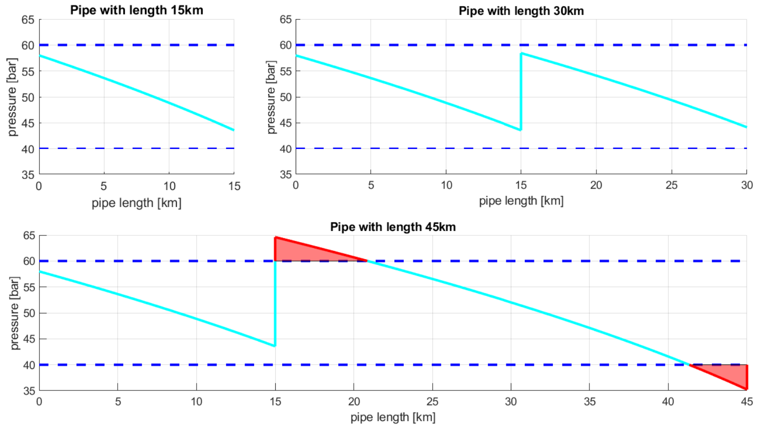


Optimal Compressor Location

Deterministic Optimization

For $f : \mathbb{R}^2 \to \mathbb{R}$ with f strictly monotonically increasing in the first argument, consider the optimization problem:

(OPT 1)
$$\begin{cases} \min\limits_{u, x_C} & f(u, q), \\ \text{s.t.} & p_1(L_1) \geq p_{\text{min}}, \quad p_2(0) \leq p_{\text{max}}, \quad p_2(L_2) \geq p_{\text{min}}, \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}$$



Lemma

Let $p_0 \in [p_{\text{min}}, p_{\text{max}}]$ and $b > 0$ be given.

(i) For $L \leq \frac{p_0^2 - p_{\text{min}}^2}{\phi\ b\ |b|}$ every point (u, x_C) with $u = 1$ and $x_C \in [0, L]$ is a solution of the optimization problem (OPT 1).

(ii) For $\frac{p_0^2 - p_{\text{min}}^2}{\phi\ b\ |b|} < L \leq \frac{p_0^2 + p_{\text{max}}^2 - 2\ p_{\text{min}}^2}{\phi\ b\ |b|}$ the optimization problem (OPT 1) has a unique solution (u^*, x_C^*) with $u^* > 1$ and $x_C \in [0, L]$.

(iii) For $L > \frac{p_0^2 + p_{\text{max}}^2 - 2\ p_{\text{min}}^2}{\phi\ b\ |b|}$ the optimization problem (OPT 1) does not have a solution.

Optimal Compressor Location

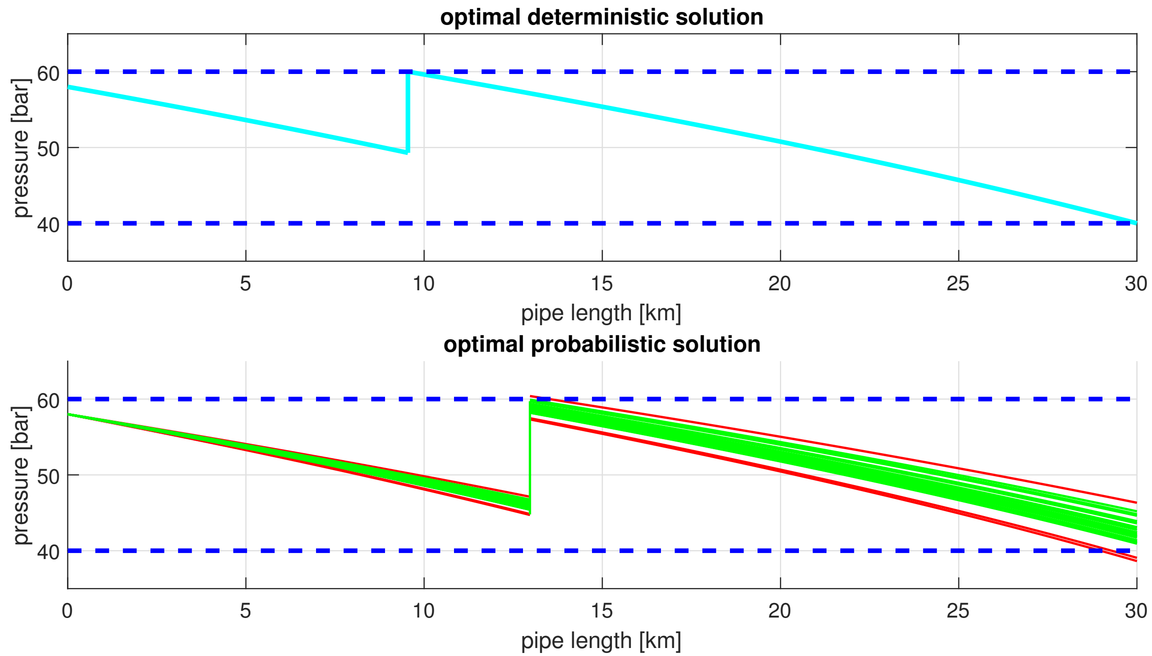
Probabilistic Optimization

Gas outflow b is random in the sense that

$$b = \xi(\omega), \quad \xi \sim \mathcal{N}(\mu, \sigma)$$

Consider the following optimization problem:

$$\text{(OPT 2)} \quad \begin{cases} \min\limits_{u, x_C} & f(u, q), \\ \text{s.t.} & \mathbb{P}\begin{pmatrix} p_1(L_1) & \geq & p_{\text{min}} \\ p_2(0) & \leq & p_{\text{max}} \\ p_2(L_2) & \geq & p_{\text{min}} \\ p_2(L_2) & \leq & p_{\text{max}} \end{pmatrix} \geq \alpha, \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}$$



Lemma

If (u^*, x_C^*) with $u^* > 1$ is a solution of (OPT 2), then the probabilistic constraint is active.

Optimal Compressor Location

Probabilistic Optimization

Theorem

Let $p_0 \in \left[\, p_{\text{min}},\ p_{\text{max}} \,\right]$ be given.

(i) If there exists a pair (u, x_C) with $u = 1$ and $x_C \in [0, L]$, that satisfies the constraints of (OPT 2), then every pair (u, x_C) with $u = 1$ and $x_C \in [0, L]$ is a solution of (OPT 2).

(ii) If there exist a pair (u, x_C), that satisfies the constraints of (OPT 2) and if (u, x_C) with $u = 1$ is infeasible for at least one $x_C \in [0, L]$, then there exists at least one solution (u^*, x_C^*) of (OPT 2) with $u^* > 1$ and $x_C^* \in [0, L]$.

Optimal Compressor Location

Probabilistic Optimization

Theorem

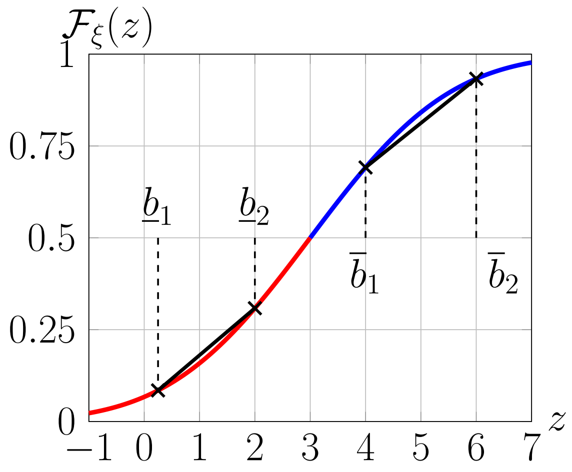
Let $p_0 \in \left[p_{\text{min}},\ p_{\text{max}} \right]$ be given.

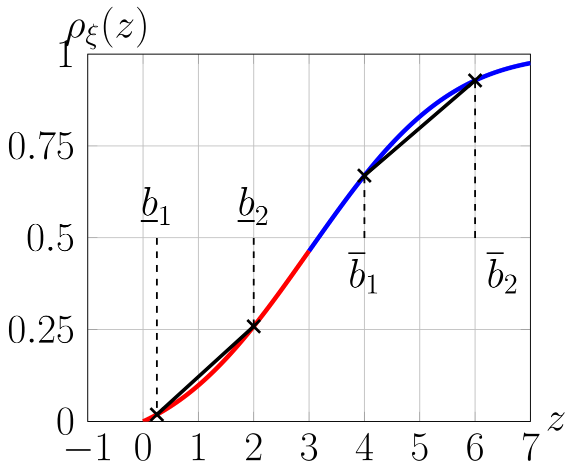
(i) If there exists a pair (u, x_C) with $u = 1$ and $x_C \in [0, L]$, that satisfies the constraints of (OPT 2), then every pair (u, x_C) with $u = 1$ and $x_C \in [0, L]$ is a solution of (OPT 2).

(ii) If there exist a pair (u, x_C), that satisfies the constraints of (OPT 2) and if (u, x_C) with $u = 1$ is infeasible for at least one $x_C \in [0, L]$, then there exists at least one solution (u^*, x_C^*) of (OPT 2) with $u^* > 1$ and $x_C^* \in [0, L]$.

Theorem

Let $\alpha > \frac{1}{2}$ be given. For a Gaussian distribution, *Statement (ii) in the last Theorem* guarantees the existence of a unique solution (u^*, x_C^*) of (OPT 2) with $u^* > 1$ and $x_C^* \in [0, L]$.





Optimal Compressor Location

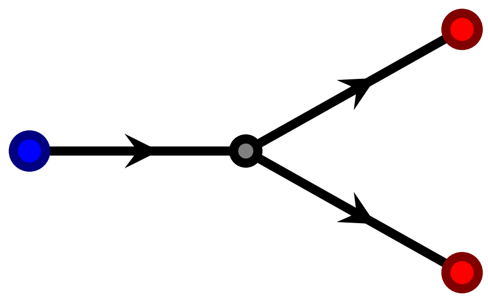
Mathematical Modelling on Networks

- **Consider a connected, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$ and set of edges $\mathcal{E}$**
- **Binary variables δ_i states if a compressor location is located on edge e_i**
- **The stationary gas flow for ideal gas on pipe e_i is given by**

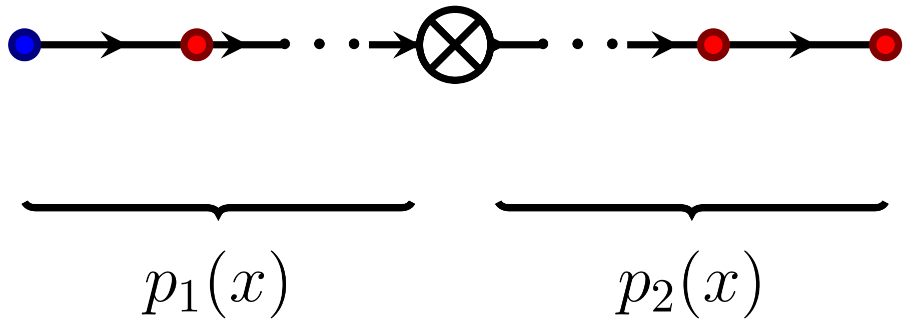
$$p_{i,1}^2(x) = p_{i,1}^2(0) - \phi\, q_i\, |q_i|\, x \qquad x \in [0, \delta_i\, x_{C,i}]$$

$$p_{i,2}^2(x) = \big(1 - \delta_i + \delta_i\, u_i\big) p_{i,1}^2(L_{i,1}) - \phi\, q_i\, |q_i|\, x \qquad x \in [0, L - \delta_i\, x_{C,i}]$$

- **Uniqueness in general strongly depends on the graph topology**



(a) Scheme of a symmetric graph with one source and two sinks



(b) Scheme of a linear graph with one source and n sinks

Optimal Compressor Location

Deterministic Optimization on Networks

For $f \in \mathcal{C}(\mathbb{R}^{2m})$, $g \in \mathcal{C}(\mathbb{R})$ and $\gamma_1, \gamma_2 > 0$, consider the deterministic optimization problems

$$\text{(OPT 3)} \quad \begin{cases} \min\limits_{u, x_C, \delta} & f(u,q), \\ \text{s.t.} & \text{for all } i = 1, \cdots, m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C, \end{cases}$$

$$\text{(OPT 4)} \quad \begin{cases} \min\limits_{u, x_C, \delta, n_C} & \gamma_1 \, f(u,q) + \gamma_2 \, g(n_C), \\ \text{s.t.} & \text{for all } i = 1, \cdots, m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C. \end{cases}$$

Optimal Compressor Location

Deterministic Optimization on Networks

For $f \in \mathcal{C}(\mathbb{R}^{2m})$, $g \in \mathcal{C}(\mathbb{R})$ and $\gamma_1, \gamma_2 > 0$, consider the deterministic optimization problems

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$$

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$$

Theorem

For all $v_i \in \mathcal{V}_{\text{in}}$ let $p_{i,0} \in [p_{\min}, p_{\max}]$ be given and for all $v_i \in \mathcal{V}_{\text{out}}$ let $b_i \geq 0$ be given. Further let a number $n_C \in \{0, \cdots, m\}$ be given.

(i) If a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ with $u_j = 1$ for all $j = 1, \cdots, m$ satisfies the constraints of (OPT 3), every triple $(\mathbf{1}_m, x_C, \delta)$ with $x_{C,j} \in [0, L_j]$, $\delta_j \in \{0,1\}$ and $\sum_{j=1}^{m} \delta_j = n_C$ is a solution of (OPT 3).

(ii) If there exists a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ that satisfies the constraints in (OPT 3), and if $(\mathbf{1}_m, x_C, \delta)$ is infeasible for at least one pair (x_C, δ) with $x_{C,j} \in [0, L_j]$ and $\delta_j \in \{0,1\}$, the optimization problem (OPT 3) has at least one solution.

Optimal Compressor Location

Deterministic Optimization on Networks

For $f \in \mathcal{C}(\mathbb{R}^{2m})$, $g \in \mathcal{C}(\mathbb{R})$ and $\gamma_1, \gamma_2 > 0$, consider the deterministic optimization problems

$$
\text{(OPT 3)} \quad \begin{cases} \min\limits_{u, x_C, \delta} & f(u,q), \\ \text{s.t.} & \text{for all } i = 1, \cdots, m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C, \end{cases}
$$

$$
\text{(OPT 4)} \quad \begin{cases} \min\limits_{u, x_C, \delta, n_C} & \gamma_1 \, f(u,q) + \gamma_2 \, g(n_C), \\ \text{s.t.} & \text{for all } i = 1, \cdots, m, \text{ we have} \\ & p_{i,1}(L_{i,1}) \geq p_{\min}, \\ & p_{i,2}(0) \leq p_{\max}, \\ & p_{i,2}(L_{i,2}) \geq p_{\min}, \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C. \end{cases}
$$

Theorem

For all $v_i \in \mathcal{V}_{\text{in}}$ let $p_{i,0} \in [p_{\min}, p_{\max}]$ be given and for all $v_i \in \mathcal{V}_{\text{out}}$ let $b_i \geq 0$ be given.

(i) If a quadruple $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \cdots, m\}$ with $u_j = 1$ for all $j = 1, \cdots, m$ satisfies the constraints of (OPT 4), every quadruple $(\mathbf{1}_m, x_c, \delta, n_C)$ with $x_C \in [0, L_j]$, $\delta_j \in \{0,1\}$, $n_C \in \{0, \cdots, m\}$ and $\sum_{j=1}^m \delta_j = n_C$ is a solution of (OPT 4).

(ii) If there exists a quadruple $(u, x_C, \delta, n_C) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \{0, \cdots, m\}$, that satisfies the constraints in (OPT 4), and if $(\mathbf{1}_m, x_C, \delta, n_C)$ is infeasible for at least one triple $(x_C, \delta, n_C$ with $x_C \in [0, L_j]$, $\delta_j \in \{0,1\}$ and $n_C \in \{0, \cdots, m\}$, the optimization problem (OPT 4) has at least one solution.

Optimal Compressor Location

Probabilistic Optimization on Networks

Consider the probabilistic constrained optimization problems

$$
\text{(OPT 5)}\quad \begin{cases}
\min\limits_{u,x_C,\delta} \; f(u,q), \\
\text{s.t. } \mathbb{P}\begin{pmatrix} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{pmatrix} \forall k = 1,\cdots,m \geq \alpha, \\
\text{and for all } i = 1,\cdots,m, \text{ we have} \\
u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\
\sum_{j=1}^{m} \delta_j = n_C,
\end{cases}
$$

$$
\text{(OPT 6)}\quad \begin{cases}
\min\limits_{u,x_C,\delta,n_C} \; \gamma_1 \, f(u,q) + \gamma_2 \, g(n_C), \\
\text{s.t. } \mathbb{P}\begin{pmatrix} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{pmatrix} \forall k = 1,\cdots,m \geq \alpha, \\
\text{and for all } i = 1,\cdots,m, \text{ we have} \\
u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\
\sum_{j=1}^{m} \delta_j = n_C.
\end{cases}
$$

Optimal Compressor Location

Probabilistic Optimization on Networks

Consider the probabilistic constrained optimization problems

(OPT 5)
$$\left\{\begin{array}{ll} \min\limits_{u,x_C,\delta} & f(u,q), \\ \text{s.t.} & \mathbb{P}\left(\begin{array}{lcl} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{array} \quad \forall k=1,\cdots,m \right) \geq \alpha, \\ & \text{and for all } i=1,\cdots,m, \text{ we have} \\ & u_i \geq 1, \quad x_{C,i} \in [0,L_i], \quad \delta_i \in \{0,1\}, \\ & \sum\limits_{j=1}^{m} \delta_j = n_C, \end{array}\right.$$

(OPT 6)
$$\left\{\begin{array}{ll} \min\limits_{u,x_C,\delta,n_C} & \gamma_1\, f(u,q) + \gamma_2\, g(n_C), \\ \text{s.t.} & \mathbb{P}\left(\begin{array}{lcl} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{array} \quad \forall k=1,\cdots,m \right) \geq \alpha, \\ & \text{and for all } i=1,\cdots,m, \text{ we have} \\ & u_i \geq 1, \quad x_{C,i} \in [0,L_i], \quad \delta_i \in \{0,1\}, \\ & \sum\limits_{j=1}^{m} \delta_j = n_C. \end{array}\right.$$

Theorem

Let $p_{i,0} \in \left[\, p_{\text{min}},\ p_{\text{max}} \,\right]$ be given for every node $v_i \in \mathcal{V}_{\text{in}}$. Further let a number $n_C \in \{0,\cdots,m\}$ be given.

(i) If there exists a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ with $u = \mathbf{1}_m$, that satisfies the constraints of (OPT 5), then every triple $(\mathbf{1}_m, x_C, \delta)$ with $x_{C,j} \in [0, L_j]$, $\delta_j \in \{0,1\}$ and $\sum_{j=1}^m \delta_j = n_C$ is a solution of (OPT 5).

(ii) If there exists a triple $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$, that satisfies the constraints of (OPT 6), and if $(\mathbf{1}_m, x_C, \delta)$ is infeasible for at least one pair (x_C, δ) with $x_{C,j} \in [0, L_j]$ and $\delta \in \{0,1\}$, then (OPT 5) has at least one solution.

Optimal Compressor Location

Probabilistic Optimization on Networks

Consider the probabilistic constrained optimization problems

(OPT 5)
$$\begin{cases} \min\limits_{u,x_C,\delta} & f(u,q), \\ \text{s.t.} & \mathbb{P}\left(\begin{array}{lll} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{array} \quad \forall k = 1,\cdots,m \right) \geq \alpha, \\ & \text{and for all } i = 1,\cdots,m, \text{ we have} \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C, \end{cases}$$

(OPT 6)
$$\begin{cases} \min\limits_{u,x_C,\delta,n_C} & \gamma_1\, f(u,q) + \gamma_2\, g(n_C), \\ \text{s.t.} & \mathbb{P}\left(\begin{array}{lll} p_{k,1}(L_{k,1}) & \geq & p_{\text{min}} \\ p_{k,2}(0) & \leq & p_{\text{max}} \\ p_{k,2}(L_{k,2}) & \geq & p_{\text{min}} \\ p_{k,2}(L_{k,2}) & \leq & p_{\text{max}} \end{array} \quad \forall k = 1,\cdots,m \right) \geq \alpha, \\ & \text{and for all } i = 1,\cdots,m, \text{ we have} \\ & u_i \geq 1, \quad x_{C,i} \in [0, L_i], \quad \delta_i \in \{0,1\}, \\ & \sum_{j=1}^{m} \delta_j = n_C. \end{cases}$$

Theorem

Let $p_{i,0} \in \left[\, p_{\text{min}},\ p_{\text{max}} \,\right]$ be given for every node $v_i \in \mathcal{V}_{\text{in}}$.

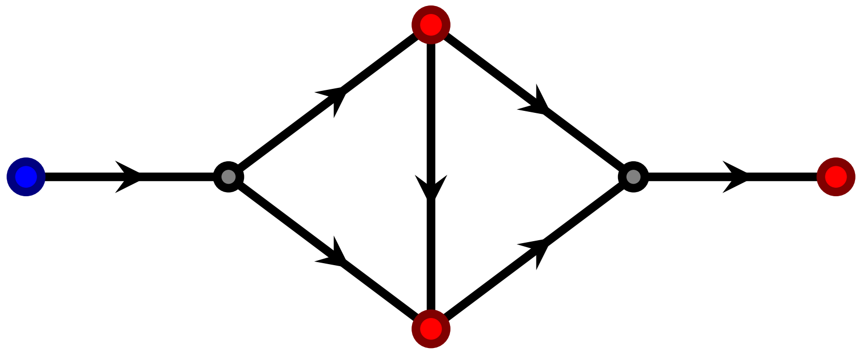
(i) If a quadruple $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0,\cdots,m\}$ with $u = \mathbf{1}_m$, that satisfies the constraints of (OPT 6), then every quadruple $(\mathbf{1}_m, x_C, \delta, n_C)$ with $x_{C,j} \in [0, L_j]$, $\delta_j \in \{0,1\}$, $n_C \in \{0,\cdots m\}$ and $\sum_{j=1}^m \delta_j = n_C$ is a solution of (OPT 6).

(ii) If there exists a quadruple $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0,\cdots,m\}$, that satisfies the constraints of (OPT 6), and if $(\mathbf{1}_m, x_C, \delta, n_C)$ is infeasible for at least one triple (x_C, δ, n_C) with $x_{C,j} \in [0, L_j]$, $\delta \in \{0,1\}$ and $n_C \in \{1,\cdots,m\}$, then (OPT 6) has at least one solution.

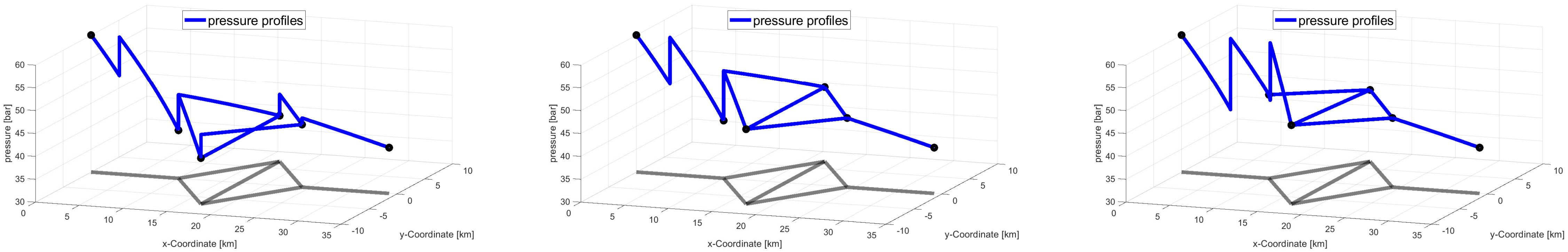
Optimal Compressor Location

A Numerical Example on a Diamond Graph

A scheme of a diamond graph with 1 source (blue) and 3 sinks (red):



Variable	Letter	Value	Unit
inlet pressure	p_0	60	bar
lower pressure bound	p_{min}	40	bar
upper pressure bound	p_{max}	60	bar
gas outflow (=mean value)	b (=μ)	$[90,\ 60,\ 120]$	kg/m^{2}s
covariance matrix	Σ	diag$(2.25,\ 2.25,\ 2.25)$	
speed of sound in the gas	c	343	m/s
pipe friction coefficient	λ^F	0.1	
pipe diameter	D	0.5	m
specific gas constant	R_S	515	J/kg K
gas temperature	T	293	K
probability level	α	0.8	



Optimal Compressor Location

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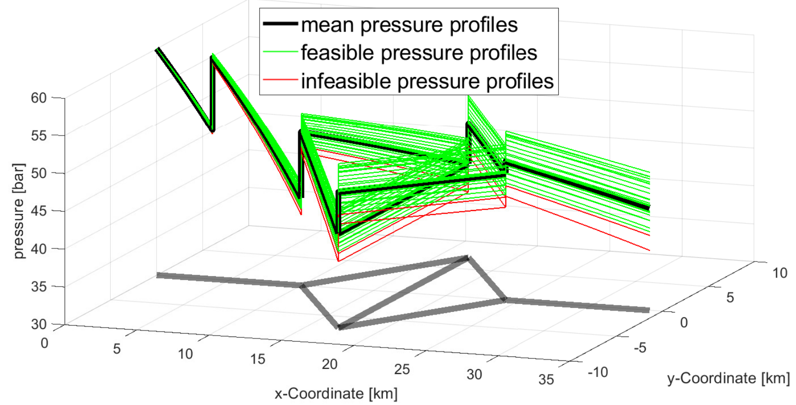
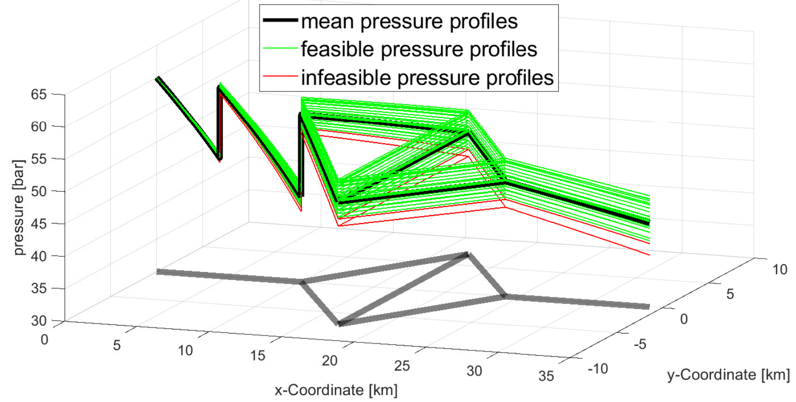
Deterministic Scenarios		
$n_C = 7$, objective value $4.5724 \cdot 10^5$	$n_C = 3$, objective value $6.9940 \cdot 10^5$	$n_C = 2$, objective value $1.5803 \cdot 10^6$
$u^* = \begin{bmatrix} 1.3531 \\ 1.4224 \\ 1.4161 \\ 1 \\ 1.2479 \\ 1.2725 \\ 1.0661 \end{bmatrix}$ $x_C^* = \begin{bmatrix} 3239.48 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$u^* = \begin{bmatrix} 1.4490 \\ 1.5791 \\ 1.5811 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_C^* = \begin{bmatrix} 3846.21 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$u^* = \begin{bmatrix} 1.8263 \\ 1 \\ 1.6044 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_C^* = \begin{bmatrix} 5616.11 \\ 0 \\ 582.84 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
Probabilistic Scenarios		
$n_C = 7$, objective value $5.7952 \cdot 10^5$	$n_C = 3$, objective value $8.9719 \cdot 10^5$	$n_C = 2$, no solution
$u^* = \begin{bmatrix} 1.4202 \\ 1.4665 \\ 1.4655 \\ 1 \\ 1.2674 \\ 1.2739 \\ 1.0710 \end{bmatrix}$ $x_C^* = \begin{bmatrix} 3864.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$u^* = \begin{bmatrix} 1.5028 \\ 1.6791 \\ 1.6812 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_C^* = \begin{bmatrix} 4365.68 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	No solution exists due to an empty admissible set



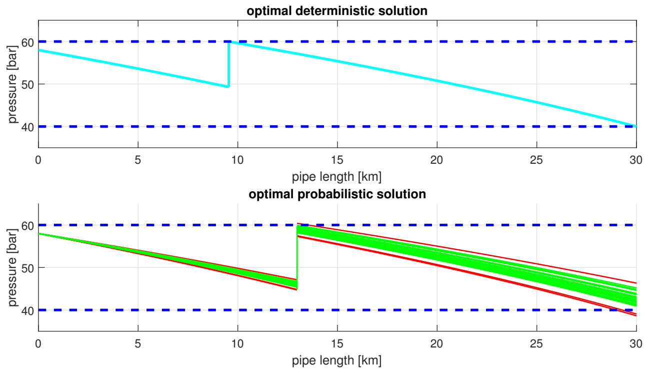


Optimal Compressor Location

A Numerical Example for Transient Gas Flow

- **Consider the probabilistic example for the stationary flow on a single pipe**
- **Solve the probabilistic constraint optimization problem for $\alpha = 90\%$**

$$
\text{(OPT 2)} \quad \begin{cases} \min\limits_{u, x_C} & f(u,q), \\ \text{s.t.} & \mathbb{P}\begin{pmatrix} p_1(L_1) & \geq & p_{\text{min}} \\ p_2(0) & \leq & p_{\text{max}} \\ p_2(L_2) & \geq & p_{\text{min}} \\ p_2(L_2) & \leq & p_{\text{max}} \end{pmatrix} \geq \alpha \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}
$$



Optimal Compressor Location

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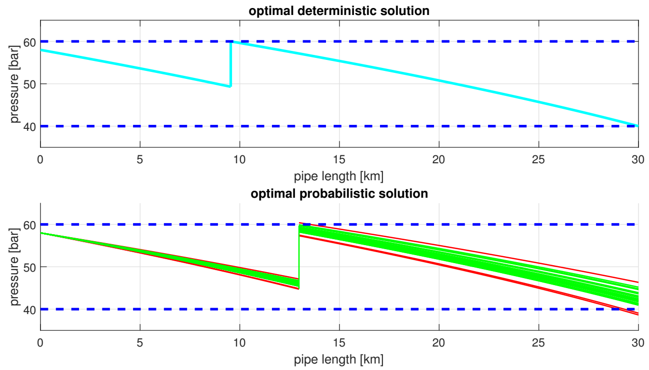
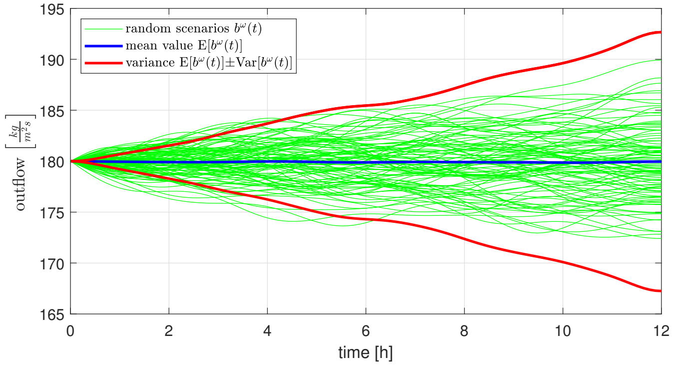


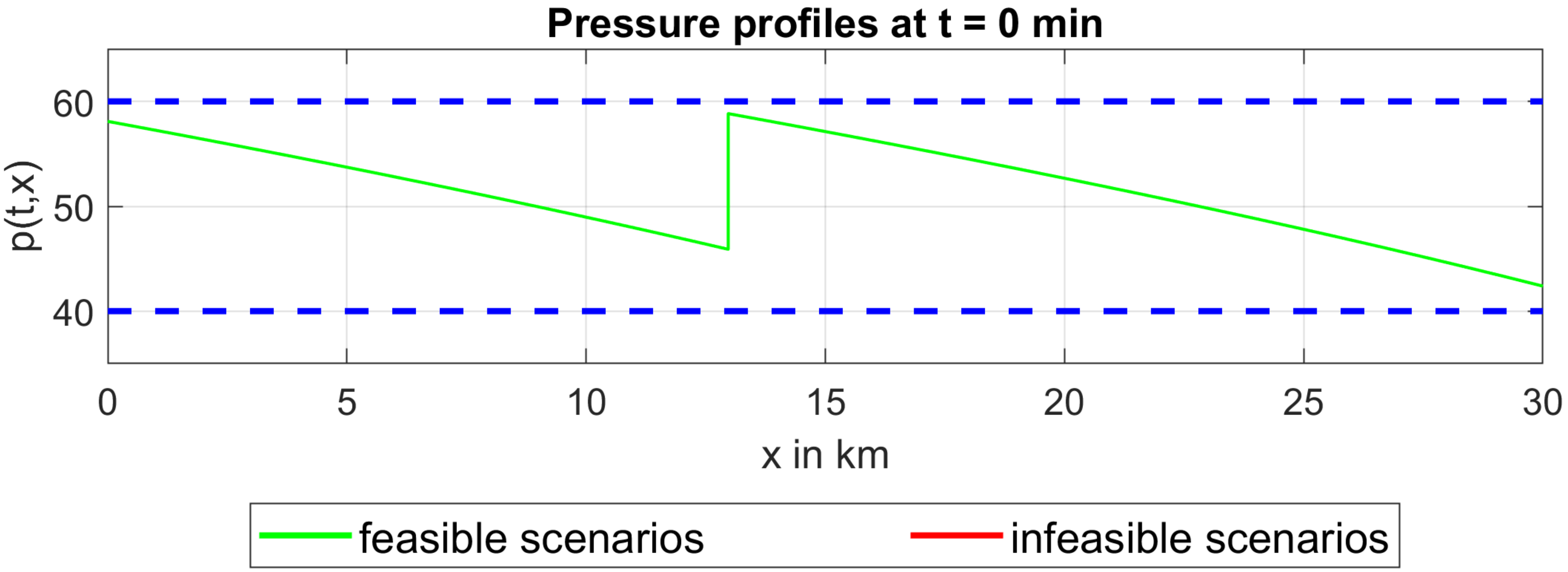
- **Randomize the boundary data in time by a *Wiener process***
- **The probabilistic robustness of the steady state control is**

$$\mathbb{P}\begin{pmatrix} p_1(t, L_1) & \geq & p_{\text{min}} & \\ p_2(t, 0) & \leq & p_{\text{max}} & \forall t \in [0,T] \\ p_2(t, L_2) & \geq & p_{\text{min}} & \\ p_2(t, L_2) & \leq & p_{\text{max}} & \end{pmatrix} = 85.74\%$$



Optimal Compressor Location

A Numerical Example for Transient Gas Flow



Optimal Compressor Location

References

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- M. Schuster: *Nodal Control and Probabilistic Constrained Optimization.* PhD thesis, FAU Erlangen-Nürnberg, Germany, 2021, `https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/searchtype/latest/docId/410/start/2/rows/10`