Tracking controllability of heat and wave equations

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***Abstract*— We study the tracking controllability of the heat equation that, as we shall see, through transmutation, is intimately related to similar properties of the wave equation. More precisely, we seek for controls that, acting on a part of the boundary of the domain where the heat or wave process evolves, aims to assure that the trace on the complementary set tracks a given trajectory. We identify the dual observability problem, which consists on estimating the boundary sources, localized on a given subset of the boundary, out of boundary measurements on the complementary subset. Classical unique continuation and smoothing properties of the heat equation allow proving approximate tracking controllability properties and the smoothness of the class of trackable trajectories. We develop a new transmutation method which allows to transfer results on the sidewise controllability of the wave equation onto the tracking controllability of the heat one. This allows to achieve some estimates on the cost of control. The paper is complemented with the discussion of some possible variants of these results and a list of open problems.**

***Index Terms*— tracking controllability, sidewise controllability, heat equation, wave equation, transmutation**

I. Introduction

The initial motivation of this problem is to analyze the tracking or sidewise controllability problem for the heat equation:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T)\times\Omega, \\ y = v1_\gamma & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{on } \Omega, \end{cases} \tag{1}$$

when $\Omega \subset \mathbb{R}^d$ is a given open bounded domain, $T > 0$ a given time horizon, $\gamma \subset \partial\Omega$ a subset of the boundary, v the control and y_0 the initial value. Hereafter, we denote by 1_γ the characteristic function of the set γ of the boundary where the source term acts.

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The *(sidewise) tracking controllability problem* is formulated mathematically in the following way: given $\tilde\gamma \subset \partial\Omega$ (usually, but not necessarily, $\tilde\gamma \subset \partial\Omega\backslash\gamma$), and $w \in L^2((0,T)\times\tilde\gamma)$, to find a control $v \in L^2((0,T)\times\gamma)$ such that:

$$\partial_\nu y = w \qquad \text{on } (0,T)\times\tilde\gamma, \tag{2}$$

where ν denotes the normal vector to $\partial\Omega$ pointing outwards. In other words, we seek to control the flux on $(0,T)\times\tilde\gamma$ by acting on $(0,T)\times\gamma$. When such a control v exists, so that (2) is satisfied, the target w is said to be *reachable*.

Obviously, this kind of problems is not exclusive of the heat equation. One could formulate similar ones for other models like the wave equation. And, actually, one of the main results of this paper will show that the tracking controllability of the heat and wave equation can be related through a suitable subordination or transmutation principle.

In the particular one-dimensional case the reachable space has been analyzed in the context of motion planning in the pioneering work [12], by using power series representation methods. Other works on 1d parabolic equations in which boundary traces are controlled include [7], [18], and [26]. In the multi-dimensional setting the only known results are only valid for cylinders (see [19] and [20]), where separation of variables can be employed, reducing the problem to the $d = 1$ case.

In this paper, first, in Section II, by duality, we transform the tracking controllability problem on its dual observability one, which consists on identifying sources on a part of the boundary of the domain out of measurements on the control domain. This observability problem differs from classical ones on the fact that, normally, the initial data of the system are the objects to be identified. Duality, together with the Holmgren's Uniqueness Theorem, allows to prove easily the approximate tracking controllability property, i.e. the fact that (2) can be achieved for all target up to an arbitrarily small ε error.

Second, in Section III, using a new transmutation formula, inspired on the classical Kannai transform [10], we show that the tracking controllability of the heat equation is subordinated to the analog property of the wave equation. The tracking controllability of the wave equation has been mainly analyzed for $d = 1$, first in [15] and [16], with constructive methods; and then, in [25], with a duality approach that inspires this paper.

A third contribution of this paper, in Section IV, concerns the quantification of the cost of approximate controllability for the heat equation. This is done by carefully analyzing power series representations, a method that, as mentioned above, has already been used to tackle the tracking control of the $1d$ heat equation.

The results of this paper can be extended to many situations: the control may act on Neumann boundary conditions and aim at the Dirichlet trace, the heat equation may have space-dependent coefficients, etc. Section V is devoted to present some of these variants and some interesting and challenging open problems.

II. Framework for tracking controllability

In this section we study the framework for tracking controllability in an abstract setting and then apply it to study the controllability of the heat equation. We recall that the framework for the wave equation was already introduced in [25].

A. An abstract setting

Let us consider the Hilbert spaces Y, U and W endowed with the scalar products $\langle\cdot,\cdot\rangle_Y$, $\langle\cdot,\cdot\rangle_U$ and $\langle\cdot,\cdot\rangle_W$ respectively. Let $A : D(A) \to Y$, $B : U \to Y$ and $E : D(A) \to W$ be linear operators. We can consider the tracking controllability problem in the abstract setting:

$$\begin{cases} y_t = Ay + Bu, \\ y(0) = y_0, \end{cases} \tag{3}$$

for the target:

$$Ey(t) = w(t) \text{ on } (\tau, T), \tag{4}$$

for some $\tau \geq 0$. As it is a classical in control problems, we consider the dual problem of (3)-(4), which reads as follows:

$$\begin{cases} -p_t = A^*p + E^*f, \\ p(T) = 0. \end{cases} \tag{5}$$

Proposition II.1 (Dual notion of approx. contr.): Let Y, U and W Hilbert spaces and $A : D(A) \to Y$, $B : U \to Y$ and $E : D(A) \to W$ be linear operators. Then, for all $w \in L^2(0,T;W)$ and $\varepsilon > 0$ there is a control $f \in L^2(0,T;U)$ such that the solution of (3) satisfies:

$$\|Ey - w\|_{L^2(0,T;W)} < \varepsilon, \tag{6}$$

if and only if for all $f \in L^2(0,T;U) \setminus \{0\}$ the solution p_f of (5) satisfy that:

$$B^*p_f \neq 0. \tag{7}$$

In fact, if (7) is satisfied for all $f \in L^2(0,T;U) \setminus \{0\}$ and if $y_0 = 0$, we obtain (6) applying the control $v = B^*p_f$, where f is the minimizer of:

$$J(f) = \frac{1}{2}\|B^*p_f\|^2_{L^2(0,T;U)} - \int_0^T \langle f, w\rangle_W dt + \varepsilon\|f\|_{L^2(0,T;W)}. \tag{8}$$

Proposition II.1 is based on the Hilbert Uniqueness Method, which is explained for instance in [17], and more recently, in [3, Section 2.3].

Proof: Let us first suppose that (7) is satisfied for all $f \in L^2(0,T;U) \setminus \{0\}$. By linearity, it suffices to prove the approximate controllability for $y_0 = 0$. From (7), J is strictly convex, continuous and coercive, so J has a unique minimizer $\tilde{f} \in L^2(0,T;U)$. Thus, we find for all $f \in L^2(0,T;U)$ and $\delta \neq 0$ that:

$$\begin{aligned} &\delta \int_0^T \langle B^*p_{\tilde{f}}, B^*p_f\rangle_U dt - \delta \int_0^T \langle f, w\rangle_W dt \\ &+ \varepsilon(\|\tilde{f} + \delta f\|_{L^2(0,T;W)} - \|\tilde{f}\|_{L^2(0,T;W)}) + O_{\delta\to 0}(\delta^2) \\ &= J(\tilde{f} + \delta f) - J(\tilde{f}) \geq 0. \end{aligned} \tag{9}$$

Moreover, if y is the solution of (3) with $y_0 = 0$ and $v = B^*p_{\tilde{f}}$, then:

$$\begin{aligned} 0 &= \int_0^T \langle y_t - Ay - BB^*p_{\tilde{f}}, p_f\rangle_Y dt \\ &= \int_0^T \langle y, E^*f\rangle_Y - \int_0^T \langle BB^*p_{\tilde{f}}, p_f\rangle_Y dt \end{aligned} \tag{10}$$

which implies that:

$$\int_0^T \langle B^*p_{\tilde{f}}, B^*p_f\rangle_U dt = \int_0^T \langle Ey, f\rangle_W dt. \tag{11}$$

Thus, considering (9)-(11), the solution of (3) with control $v = B^*p_{\tilde{f}}$ satisfies:

$$\begin{aligned} &\delta \int_0^T \langle Ey - w, f\rangle_W dt + O(\delta^2) \\ &\geq -\varepsilon(\|\tilde{f} + \delta f\|_{L^2(0,T;W)} - \|\tilde{f}\|_{L^2(0,T;W)}) \\ &\geq -\varepsilon|\delta|\|f\|_{L^2(0,T;W)}. \end{aligned} \tag{12}$$

Taking $\delta \to 0^+$ and $\delta \to 0^-$ we obtain from (12) that:

$$\left|\int_0^T \langle Ey - w, f\rangle_W dt\right| \leq \varepsilon\|f\|_{L^2(0,T;W)},$$

for all $f \in L^2(0,T;W)$, which implies (6).

Reciprocally, if $B^*p_f = 0$ for some $f \neq 0$, considering (10), Ey is orthogonal to f for all $v \in L^2(0,T;U)$, so under that hypothesis the system (3) is not approximately controllable. ∎

In a similar way, based on the Hilbert Uniqueness Method, we can obtain the duality result for exact controllability:

Proposition II.2 (Dual notion of exact contr.): Let Y, U and W Hilbert spaces and $A : D(A) \to Y$, $B : U \to Y$ and $E : D(A) \to W$ be linear operators. Then, for all $w \in L^2(0,T;W)$ and $\varepsilon > 0$ there is a control $f \in L^2(0,T;U)\setminus\{0\}$ such that the solution of (3) satisfies:

$$Ey = w, \tag{13}$$

if and only if

$$\sup_{f \in L^2(0,T;U)\setminus\{0\}} \frac{\|f\|_{L^2(0,T;W)}}{\|B^*p_f\|_{L^2(0,T;U)}} < +\infty, \tag{14}$$

for p the solution of (5). In fact, if (14) is satisfied, we obtain (13) applying the control $v = \partial_\nu p_f$, where f is the minimizer of:

$$J(f) = \frac{1}{2}\|B^* p_f\|^2_{L^2(0,T;U)} - \int_0^T \langle f, w\rangle_W dt. \quad (15)$$

Remark II.3: The existence of Carleman inequalities could allows us to prove some kind of exact controllability for the wave equation, an existence which remains as a challenging open problem.

B. Tracking of the heat equation

A consequence of Proposition II.1 is the approximate controllability of the heat equation:

Proposition II.4 (Approx. contr.): Let Ω be an analytic domain, $\gamma \subset \partial\Omega$ be relatively open and non-empty, and $\tilde{\gamma} \subset\subset \partial\Omega \setminus \gamma$. Then, for all $w \in L^2((0,T) \times \tilde{\gamma})$ and $\varepsilon > 0$ there is a control $v \in L^2((0,T) \times \gamma)$ such that the solution of (1) satisfies (6).

Proof: The dual problem of (1)-(2) reads as follows:

$$\begin{cases} -p_t - \Delta p = 0 & \text{in } (0,T) \times \Omega, \\ p = f 1_{\tilde{\gamma}} & \text{on } (0,T) \times \partial\Omega, \\ p(T) = 0 & \text{on } \Omega. \end{cases} \quad (16)$$

By Proposition II.2, it suffices to prove that $\partial_\nu p_f = 0$ on γ implies that $f = 0$. This result follows from Holmgren's Uniqueness Theorem (see, for instance, [11]), which implies that the solution of the heat equations are analytic, and by Hopf's Lemma (which states that if $p = 0$ on a part of the boundary and $p > 0$ on a neighbourhood of the boundary, then $\partial_\nu p$ on that part of the boundary). In fact, if p and $\partial_\nu p$ are both null on γ, all the derivatives of p derivatives are null on γ. Otherwise, their is $\bar{x} \in \gamma$ such that p is strictly positive or negative in an open $\tilde{\Omega} \subset \Omega$ such that $\bar{x} \in \partial\tilde{\Omega}$, which by Hopf's lemma implies that $\partial_\nu p(\bar{x}) \neq 0$. Finally, the result follows from analyticity. ∎

Remark II.5 (Regularity of the trackable space): Due to the regularizing effect of the heat equation, we cannot expect that the trackable space contains irregular traces if $\tilde{\gamma} \subset\subset \partial\Omega \setminus \gamma$. Indeed, with a classic bootstrapping argument as in [6, Lemma 2.5], we can show that the reachable space must be regular (notably, if Ω is a C^∞ domain, the trace must be C^∞).

III. Transmutation for tracking controllability

In this section we relate the tracking controllability of the heat equation and of the wave equation by using a variant of the Kannai transform (see [10], [21] and [22]). We recall that the Kannai consists on averaging the solutions of the wave equation with the kernel:

$$k(t,s) := \frac{e^{-s^2/(4t)}}{\sqrt{4\pi t}}. \quad (17)$$

We recall that k is the fundamental solution of the heat equation; i.e. it satisfies:

$$\partial_t k = \partial_{ss} k; \quad k(0,s) = \delta_0(s). \quad (18)$$

As we show in Section III-A, the Kannai transform links the heat and wave equations.

We prove in this section that irregular traces independent of time cannot be reached. Notably, let us consider the following control problem for the wave equation:

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } \mathbb{R} \times \Omega, \\ z(t,\cdot) = g 1_\gamma & \text{on } \mathbb{R} \times \partial\Omega, \\ z(0,\cdot) = z_0 & \text{on } \Omega, \\ z_t(0,\cdot) = z_1 & \text{on } \Omega, \end{cases} \quad (19)$$

Here, $\Omega \subset \mathbb{R}^d$ is a C^2 domain, g is the control and $(z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ the initial states. For this section we define the functional space:

$$\mathcal{E}(\mathbb{R}; H) := \Big\{ g \in L^\infty_{\text{loc}}(\mathbb{R}; H) : \forall \delta > 0\ \exists C_\delta > 0 : \|g(t)\|_H \leq C_\delta e^{\delta t^2}\ \forall t \in \mathbb{R} \Big\},$$

for a given Hilbert space H.

The main result that we prove is the following:

Theorem III.1 (Untrackable traces for the wave eq.): Let Ω be a C^2 domain, $\gamma \subset \partial\Omega$, and $\tilde{\gamma} \subset\subset \partial\Omega \setminus \gamma$. Let $w \in L^\infty(\tilde{\gamma}) \setminus C^0(\tilde{\gamma})$. Then, for any $\tau > 0$ there is no $g \in \mathcal{E}(\mathbb{R}; L^2(\gamma))$, $z_0 \in L^2(\Omega)$ and $z_1 \in H^{-1}(\Omega)$ such that the solution z of (19) satisfies:

$$\partial_\nu z(t,x) = w(x) \text{ on } (\tau, \infty) \times \tilde{\gamma}$$

This is done with the Kannai transform. Indeed, this is the first time that the Kannai transform is used to obtain negative results about tracking controllability. For that, we consider targets that present discontinuities on the space variable. This means that our result apply to all the dimensions except dimension 1.

A. The Kannai transform

The importance of the Kannai transform is reflected by following link between the control problems (1) and (19):

Lemma III.2 (Kannai transform): Let Ω be a C^2 domain, $\gamma \subset \partial\Omega$, $g \in \mathcal{E}(\mathbb{R}, L^2(\gamma))$, $z_0 \in L^2(\Omega)$, $z_1 \in H^{-1}(\Omega)$ and z the solution of (19). Then,

$$y(t,x) = \int_{-\infty}^{\infty} k(t,s) z(s,x) ds, \quad (20)$$

is a solution of (1) for $T = \infty$, $y_0 = z_0$,

$$v(t,x) := \int_{-\infty}^{\infty} k(t,s) g(s,x) ds,$$

and it satisfies:

$$\partial_\nu y(t,x) = \int_{-\infty}^{\infty} k(t,s) \partial_\nu z(s,x) ds.$$

Proof: It is trivial that the function y given by (20) satisfy the boundary conditions of (1). Moreover, it satisfies the initial condition because of (18). Finally, it is a solution of the heat equation. Indeed, if $g \in \mathcal{D}(\mathbb{R} \setminus \{0\} \times \gamma)$, then for all $t \in (0,T)$

and $x \in (0, L)$ the following equality holds:

$$\begin{aligned} y_t &= \int_{-\infty}^{\infty} k_t(t,s) z(s,x) ds = \int_{-\infty}^{\infty} k_{ss}(t,s) z(s,x) ds \\ &= \int_{-\infty}^{\infty} k(t,s) z_{ss}(s,x) ds = \int_{-\infty}^{\infty} k(t,s) \Delta z(s,x) ds \\ &= \Delta \left(\int_{-\infty}^{\infty} k(t,s) z(s,x) ds \right) = \Delta y. \end{aligned}$$

We have used (18) in the second equality. Moreover, the integration by parts on the third equality is rigorous because k decays when $s \to \infty$. Finally, for any $g \in \mathcal{E}(\mathbb{R}; L^2(\gamma))$ we can prove that $\int_{-\infty}^{\infty} k_{ss}(t,s) z(s,x) ds = \Delta \int_{-\infty}^{\infty} k(t,s) z(s,x) ds$ with a density argument, since for all $t > 0$ the function $e^{-s^2/(4t)} z$ decays quadratic exponentially when $s \to \infty$, so $y_t = \Delta y$. ∎

B. Proof of Theorem III.1

Let us now prove Theorem III.1. For that, we first prove the following result:

Lemma III.3: Let Ω be a C^2 domain, $\gamma \subset \partial\Omega$, and $\tilde{\gamma} \subset\subset \partial\Omega \setminus \gamma$ be relatively open. Let $w \in L^\infty(\tilde{\gamma})$ such that $w \notin C^0(\tilde{\gamma})$. Then, for any $\tau > 0$ there is no $g \in \mathcal{E}(\mathbb{R}; L^2(\gamma))$, $z_0 \in L^2(\Omega)$ and $z_1 \in H^{-1}(\Omega)$ such that the solution z of (19) satisfies:

$$\begin{cases} \partial_\nu z(t,x) = w_1 & \text{on } ((-\infty,-\tau) \cup (\tau,\infty)) \times \tilde{\gamma}_1, \\ \partial_\nu z(t,x) = -w_2 & \text{on } ((-\infty,-\tau) \cup (\tau,\infty)) \times \tilde{\gamma}_2. \end{cases} \tag{21}$$

Proof: We are going to prove Lemma III.3 by reductio ad absurdum. Let us consider the function:

$$y(t,x) = \int_{-\infty}^{\infty} k(t,s) z(s,x) ds,$$

for k given by (17). By Lemma III.2, y is a solution of (1) for $T = \infty$,

$$v(t,x) := \int_{-\infty}^{\infty} k(t,s) g(s,x) ds$$

and it satisfies:

$$\partial_\nu y(t,x) = \int_{-\infty}^{\infty} k(t,s) \partial_\nu z(s,x) ds \text{ on } \mathbb{R} \times \tilde{\gamma}. \tag{22}$$

Because of the regularizing effect of the heat equation, $\partial_\nu u \in C^0([0,T] \times \tilde{\gamma})$. However, the following limit holds on $L^\infty(\tilde{\gamma})$:

$$\begin{aligned} &\lim_{t\to\infty} \int_{-\infty}^{\infty} k(t,s) \partial_\nu z(s,x) ds \\ &\quad = \lim_{t\to\infty} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{\sqrt{\pi}} \partial_\nu z\left(s\sqrt{4t}, x\right) ds = w_1. \end{aligned} \tag{23}$$

Since $C^0(\tilde{\gamma})$ is close in $L^\infty(\tilde{\gamma})$, we arrive at a contradiction. ∎

From Lemma III.3 we deduce the proof of Theorem III.1:

Proof: The proof of Theorem III.1 is based on a symmetry reasoning. In fact, if such a solution z exists we define:

$$\tilde{z} := \begin{cases} z(t,x) & t \geq 0, \\ z(-t,x) & t < 0. \end{cases}$$

Clearly, $\tilde{z}$ satisfies (19) for:

$$\tilde{g}(t) = g(t) 1_{t\geq 0} + g(-t) 1_{t<0}$$

and (21), which contradicts the results of Theorem III.1. ∎

Remark III.4 (Implications of Theorem III.1): Theorem III.1 has one of the following implications: either the trackable space on the interval (τ, ∞) does not contain some discontinuous functions or the cost for getting those traces is exponentially large in the sense that there is $\delta > 0$ such that for all $C > 0$ there is a sequence t_n such that $\|g\|_{L^2(\gamma)} \geq Ce^{\delta t_n^2}$. Determining which assertion holds, though, is an open problem.

IV. Controls for the 1d heat equation

In this section we study the tracking controllability of the 1d heat equation by using the flatness approach and duality. Notably, we study the solutions of:

$$\begin{cases} y_t - \partial_{xx} y = 0 & \text{in } (0,T) \times (0,L), \\ y(\cdot, 0) = 0 & \text{on } (0,T), \\ y(\cdot, L) = v & \text{on } (0,T), \\ y(0) = 0 & \text{on } (0,L). \end{cases} \tag{24}$$

First, we recall how to compute explicitly the controls so that the solution of (24) satisfies:

$$\partial_x y(\cdot, 0) = w \qquad \text{on } (0,T), \tag{25}$$

where w is a flat function. Then, we use those controls to get an upper bound on the cost for approximate tracking controllability. Notably, we derive an upper bound for the minimum cost of the control which acts on $(0,T) \times \{L\}$ so that:

$$\|\partial_x y(\cdot, 0) - w\|_{C^0([0,T])} \leq \varepsilon, \tag{26}$$

with the controls obtained explicitly with the flatness method.

In particular, we prove the following result:

Theorem IV.1 (Cost for approx. contr.): Let $L > 0$, $T > 0$ and $w \in W^{1,\infty}(0,T)$ a function satisfying $w(0) = 0$. Then, for all $s \in (0,1)$ there is a constant $C > 0$ such that for all $\varepsilon > 0$ we can obtain a control v_ε satisfying:

$$\begin{aligned} &\|v_\varepsilon\|_{C^0([0,T])} \\ &\leq C \exp\left[C \left(\frac{\|w\|_{W^{1,\infty}(0,T)}}{\varepsilon} \right)^{1/s} \right] \|w\|_{W^{1,\infty}(0,T)} \end{aligned} \tag{27}$$

and such that the solution of (24) satisfies (26).

The proof of Theorem IV.1 consists on approximating the targets with Gevrey function by convolution with cut-off functions. The requirement of being w more regular than the space in which the norm is taken is common on parabolic control theory (see, for instance, [8]). In this paper we use this to approximate the target with an element of the trackable space in a uniform way with the C^0 norm. Before proving Theorem IV.1 we first recall some results of Gevrey functions, and then perform some technical estimates on an auxiliary function.

A. Previous results

In this section we recall the known controllability results to flat outputs in one-dimensional domains and the existence of Gevrey cut-off functions. By definition w is a Gevrey function of order s if and only if $w \in C^\infty([0,T])$ and satisfies for some $C, R > 0$:

$$|w^{(i)}(t)| \leq C\frac{(i!)^s}{R^i}, \quad \forall t \in [0,T], \forall i \geq 0.$$

Here we consider $w \in [1,2)$. If $s = 1$, then the Gevrey function is analytic.

Lemma IV.2 (Controls for flat targets): Let $s \in [0,2)$, $L > 0$, $T > 0$ and $w \in C^\infty([0,T])$ be a Gevrey function of order s satisfying $w^{(i)}(0) = 0$ for all $i \in \mathbb{N}$. Then, there is v a Gevrey function of order s such that the solution of (24) satisfies (25). The proof of Lemma IV.2 is mainly contained in [12]. We recall that they consider the controls:

$$v(t) = \sum_{i\geq 0} \frac{L^{2i+1}}{(2i+1)!} w^{(i)}(t).$$

and that the solution of (24) are given by:

$$y(t,x) = \sum_{i\geq 0} \frac{x^{2i+1}}{(2i+1)!} w^{(i)}(t). \tag{28}$$

Next, we recall the existence of cut-off functions that belong to a Gevrey space:

Lemma IV.3 (Gevrey cut-off functions): Let $s > 1$. There is a cut-off function ξ supported in $[0,1]$ of Gevrey order s and satisfying $\int_0^1 \xi(t)dt = 1$.

We may construct the function in Lemma IV.3 by considering that for all $s > 0$ the function $\exp\left(\frac{-1}{((1-t)t)^s}\right) 1_{(0,1)}$ is of order $1 + 1/s$. This was proved first in [24], and an English version of the proof can be consulted in [26, Lemma 4]. Their proofs are based on Cauchy integral formula and on Stirling formula.

B. Upper bounds for auxiliary functions

In order to quantify the cost we introduce the auxiliary functions:

$$\mathcal{G}_s(x) := \sum_{i\geq 0} \frac{x^i}{(i!)^s}. \tag{29}$$

We remark that these functions have an exponential bound:

Lemma IV.4 (Upper bounds for $\mathcal{G}_s$): Let $s > 0$. Then, there is $C > 0$ depending on s such that:

$$\mathcal{G}_s(x) \leq C \exp\left(Cx^{\frac{1}{s}}\right), \quad \forall x \geq 0. \tag{30}$$

Lemma IV.4 is proved by estimating $\partial_x[\ln(\mathcal{G}_s(x))]$ with Stirling's formula and a splitting of the lower and the higher order terms of the sum:

Proof: In order to prove (30) it suffices to prove that there is $C > 0$ such that:

$$\mathcal{G}_s'(x) \leq Cx^{\frac{1-s}{s}} \mathcal{G}_s(x), \quad \forall x \geq 1. \tag{31}$$

For that purpose, we remark that:

$$\mathcal{G}_s'(x) = \sum_{i\geq 1} i^{1-s} \frac{x^{i-1}}{((i-1)!)^s} = \sum_{i\geq 0} (i+1)^{1-s} \frac{x^i}{(i!)^s}.$$

In order to prove (31) we split the terms into $i < 2x^{1/s}e$ and $i \geq 2x^{1/s}e$. On the one hand, if $i \geq 2x^{1/s}e$, from Stirling's formula we get that:

$$\frac{x^i}{(i!)^s} \leq C\frac{x^i e^{is}}{i^{is}} = C\left(\frac{x^{1/s}e}{i}\right)^{is} \leq C2^{-is}. \tag{32}$$

Thus, from (32) we obtain for all $x \geq 1$ that:

$$\sum_{i\geq 2x^{1/s}e} (i+1)^{1-s} \frac{x^i}{(i!)^s} \leq C \leq Cx^{\frac{1-s}{s}} \mathcal{G}_s(x). \tag{33}$$

On the other hand, we find for all $x \geq 1$ that:

$$\begin{aligned}\sum_{i<2x^{1/s}e} (i+1)^{1-s} \frac{x^i}{(i!)^s} &\leq \sum_{i<2x^{1/s}e} (4e)^{1-s} x^{\frac{1-s}{s}} \frac{x^i}{(i!)^s} \\ &\leq Cx^{\frac{1-s}{s}} \mathcal{G}_s(x).\end{aligned} \tag{34}$$

Therefore, (31) follows from (33) and (34). ■

C. Conclusion of the proof of Theorem IV.1

We now have all the ingredients to compute the upper bound of the cost of approximate controllability. The proof of Theorem IV.1 is divided in 2 steps: first we approximate the target w by convolution with the cut-off function given in Lemma IV.3, and secondly we apply the control given in Lemma IV.2 and compute estimates on it with Lemma IV.4.

Proof: Step 1: approximating the normal trace. Let $s \in (0,1)$ and $w \in W^{1,\infty}(0,T)$ a function satisfying $w(0) = 0$. By linearity, we can suppose that:

$$\|w\|_{W^{1,\infty}(0,T)} = 1. \tag{35}$$

We define $\xi_\varepsilon := \varepsilon^{-1}\xi(x\varepsilon^{-1})$, for ξ the Gevrey function of order $2 - s$ given in Lemma IV.3 and:

$$w_\varepsilon := \tilde{w} * \xi_\varepsilon = \int_{t-\varepsilon}^{t} \tilde{w}(t')\xi_\varepsilon(t-t')dt', \tag{36}$$

for $\tilde{w}$ the prolongation of w by 0 to $\mathbb{R}^-$. Since ξ is supported in $[0,1]$, $w_\varepsilon = 0$ in $(-\infty, 0]$, so w_ε annihilates at $t = 0$. Moreover, from $w(0) = 0$ we get that:

$$|w_\varepsilon(t) - w(t)| \leq \sup_{t'\in(0,\varepsilon)} |\tilde{w}(t-t') - w(t)| \leq \varepsilon\|w\|_{W^{1,\infty}(0,T)}.$$

Thus, from (35) we obtain that:

$$\|w_\varepsilon - w\|_{C^0([0,T])} \leq \varepsilon. \tag{37}$$

Since ξ is a Gevrey function of order $2-s$ we can show easily that w_ε is a Gevrey function of order $2-s$. In fact, considering (35) and (36) we get that:

$$\|w_\varepsilon\|_{C^i([0,T])} \leq \varepsilon^{-i}\|\xi\|_{C^i([0,1])}, \quad \forall i \in \mathbb{N}. \tag{38}$$

Thus, from the assumption that ξ is a Gevrey function of order $2 - s$ and (38) we deduce that:

$$\|w_\varepsilon\|_{C^i([0,T])} \leq \left(\frac{C}{\varepsilon}\right)^i (i!)^{2-s}, \quad \forall i \in \mathbb{N}. \tag{39}$$

Step 2: estimation of the control. From Lemma IV.2 we obtain that $\partial_x y(\cdot, 0) = w_\varepsilon$ if we apply the control:

$$v_\varepsilon(t) = \sum_{i\geq 0} \frac{L^{2i+1}}{(2i+1)!} w_\varepsilon^{(i)}(t).$$

In particular, from (39) we find that:

$$\|v_\varepsilon\|_{C^0([0,T])} \leq \sum_{i\geq 0} \left(\frac{C}{\sqrt{\varepsilon}}\right)^{2i} \frac{(i!)^{2-s}}{(2i+1)!},$$

for C a constant independent of i. Next, we consider that:

$$\frac{(i!)^{2-s}}{(2i+1)!} \leq \frac{1}{(i!)^s}, \tag{40}$$

since:

$$\frac{(2i)!}{(i!)^2} = \binom{2i}{i} > 1.$$

Thus,

$$\begin{aligned}\|v_\varepsilon\|_{C^0([0,T])} &\leq \sum_{i\geq 0} \left(\frac{C}{\sqrt{\varepsilon}}\right)^{2i} \frac{(i!)^{2-s}}{(2i+1)!}\\ &\leq \sum_{i\geq 0} \left(\frac{C}{\sqrt{\varepsilon}}\right)^{2i} \frac{1}{(i!)^s}\\ &\leq \sum_{i\geq 0} \left(\frac{C}{\varepsilon}\right)^{i} \frac{1}{(i!)^s} = \mathcal{G}_s(C\varepsilon^{-1}).\end{aligned}$$

Hence, we obtain (27) from (30). ■

V. Open problems

First of all, we would like to remark that our technique can clearly work when we have other boundary conditions, such as Neumann or Robin.

The method and results in this paper lead to some interesting open problems and could be extended in various directions (in addition to the ones proposed in Remark III.4) that we briefly describe:

- **Multi-dimensional domains.** It is an open problem to get more precise information about controls for multi-dimensional domains, both for the heat and wave equations. The ideal scenario is to have explicit formulas; though it is more realistic to search for quantitative estimates for the cost of approximate controllability. In fact, generalizing the flatness method for the heat equation and characteristic or energy methods for the wave equation in non-cylindrical domains seem challenging tasks.
- **Optimality on the cost of approximate controllability.** One relevant open problem is whether we can sharpen the bounds given in Theorem IV.1 to $\exp(C\varepsilon^{-1})$ or $\exp(C\varepsilon^{-1/2})$, which is more in line with the known bounds for classical approximate controllability of parabolic equations. Indeed, the cost of getting at ε distance in the L^2 norm to a function $y^T \in H^2(0,L)\cap H_0^1(0,L)$ in the heat equation with constant coefficients is bounded by $\exp(C\varepsilon^{-1/2})$ (see [8]). In addition, a cost of $\exp(C\varepsilon^{-1})$ has been obtained for more general heat equations (see [8], [23] and [2]), for the semi-linear heat equation (see [27]), for the Ginzburg-Landau equation (see [1]) and for the hypoelliptic heat equation (see [13]). Their proofs are based on an observability inequality with appropriate weights for the null controllability, so we cannot replicate them for the control problem (1).
- **Observability estimates for the heat equation.** The obtention of observability inequalities for system (16) remains open. In fact, it is interesting to see if we can obtain an observability inequality from Lemma IV.2. The main difficulty is that Gevrey functions do not form a Banach space and its dual is the space of ultra-distributions (see, for example, [4]).
- **Tracking plus null control.** It is an interesting problem to determine if we may take the heat equation to equilibrium in addition to controlling the trace. It is clear that for all $\delta > 0$ we can control the trace on $(0, T-\delta)$ and then obtain $y(T,\cdot) = 0$, as the heat equation is controllable at arbitrary small times (see, for instance, [14] and [9]). However, because of continuity, this is not true when aiming both at the trace on $(0,T)$ and at the final state. However, it is a relevant problem to determine if this is true with proper compatibility conditions.
- **Other parabolic equations.** A possible extension is to determine both theoretically and numerically if analog properties are satisfied by more complex parabolic equations like the heat equation with time and space dependent parameters, Stokes equation and Stefan equation, or if there is any remarkable difference.
- **Sidewise controllability of the wave equation.** Sidewise observability estimates for the wave equation are also an open problem. This includes establishing observability inequalities under suitable concavity and microlocal geometric conditions on the support of the source and the measurement set, for sources fulfilling suitable pseudo-differential conditions. Some breakthroughs are done in [5].

VI. Conclusions

In this paper we have studied the tracking controllability for the heat equation and its relation with the sidewise controllability of the wave equation. First, duality allows to prove approximate controllability results in multi-dimensional domains. Second, the transmutation method provides a way for obtaining controls for the heat equation, which are useful for numerical purposes, and a way for proving that irregular traces on the wave equation require controls with a norm increasing exponentially. Third, revisiting the flatness approach we obtain the cost of approximate controllability for the 1d heat equation and, combining this with duality methods, we control the trajectories on two interior points. In the future, efforts should be devoted to the understanding of controllability in multi-dimensional domains and of more complex parabolic and hyperbolic equations: finding in that complex setting a method that can provide explicit formulas or quantitative estimates for the controls remains an interesting while challenging issue. Also, efforts should be devoted to obtaining Carleman inequalities for proving the tracking null controllability of the heat equation.

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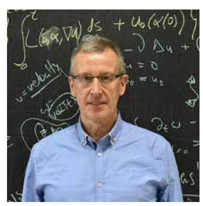

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