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Introducción histórica a la Teoría Matemática del Control

# Historical Introduction to Mathematical Control Theory

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MATHEMATICAL CONTROL THEORY,  
or  
CONTROL ENGINEERING  
or simply  
CONTROL THEORY?

An interdisciplinary field of research in between Mathematics and Engineering with strong connections with Scientific Computing, Technology, Communications,...

## THE ORIGINS:

*“... if every instrument could accomplish its own work, obeying or anticipating the will of others ...*

*if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves.”*

Chapter 3, Book 1, of the monograph “Politics” by Aristotle (384-322 B. C.).

**Main motivation:** The need of automatizing processes to let the human being gain in liberty, freedom, and quality of life.

Nowadays:

The state equation

$$A(y) = f(v). \quad (1)$$

$y$  is the state to be controlled.

$v$  is the control. It belongs to the set of admissible controls  $\mathcal{U}_{ad}$ .

Roughly speaking the goal is to drive the state  $y$  close to a desired state  $y_d$ :

$$y \sim y_d.$$

In this general functional setting many different mathematical models feet:

- Linear versus nonlinear problems;
- Deterministic versus stochastic models;
- Finite dimensional versus infinite dimensional models;
- Ordinary Differential Equations (ODE) versus Partial Differential Equations (PDE).

Several kinds of different control problems may also fit in this frame depending on how the control objective is formulated:

- **Optimal control** (related with the **Calculus of Variations**)

$$\min_{v \in \mathcal{U}_{ad}} \|y - y_d\|^2.$$

- **Controllability**: Drive exactly the state  $y$  to the prescribed one  $y_d$ .

This is a more **dynamical notion**.

Several relaxed versions also arise: approximate controllability.

- **Stabilization or feedback control**. (real time control)

$$v = F(y); \quad A(y) = f(F(y)).$$

SOME OF THE KEY INGREDIENTS OF CONTROL THEORY

The concept of **feedback**. Inspired in the capacity of biological systems to self-regulate their activities.

Incorporated to Control Engineering in the twenties by the engineers of the “**Bell Telephone Laboratory**” but, at that time, it was already recognized and consolidated in other areas, such as Political Economics.

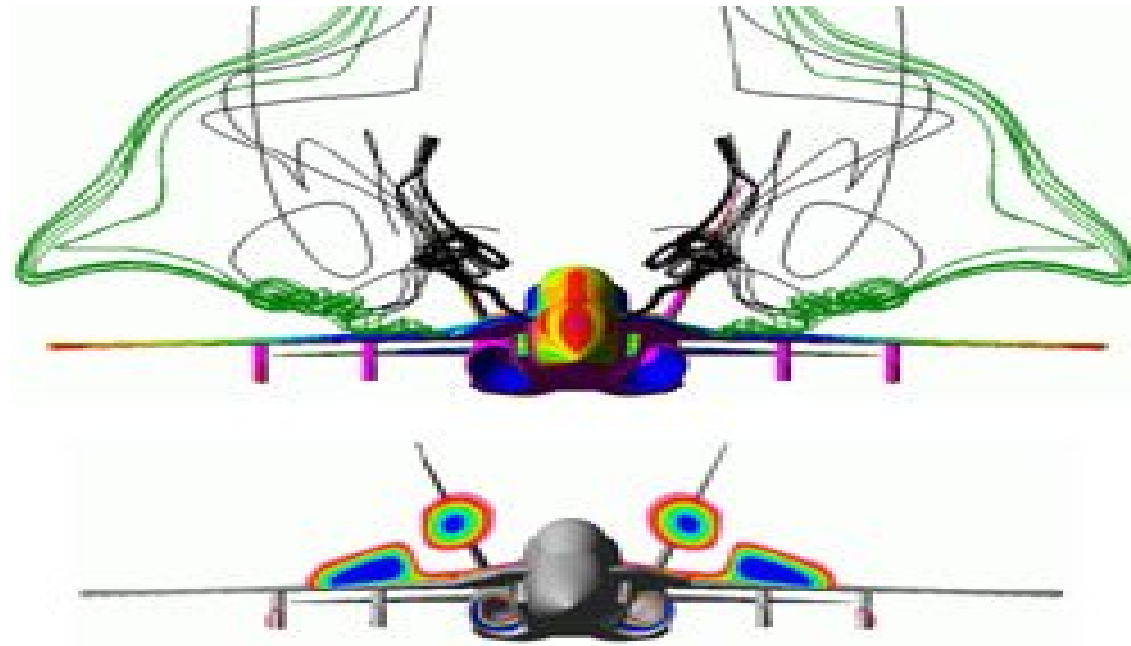
**Feedback process**: the one in which the state of the system determines the way the control has to be exerted in real time

Nowadays, feedback processes are ubiquitous in applications to Engineering, Economy also in Biology, Psychology, etc.

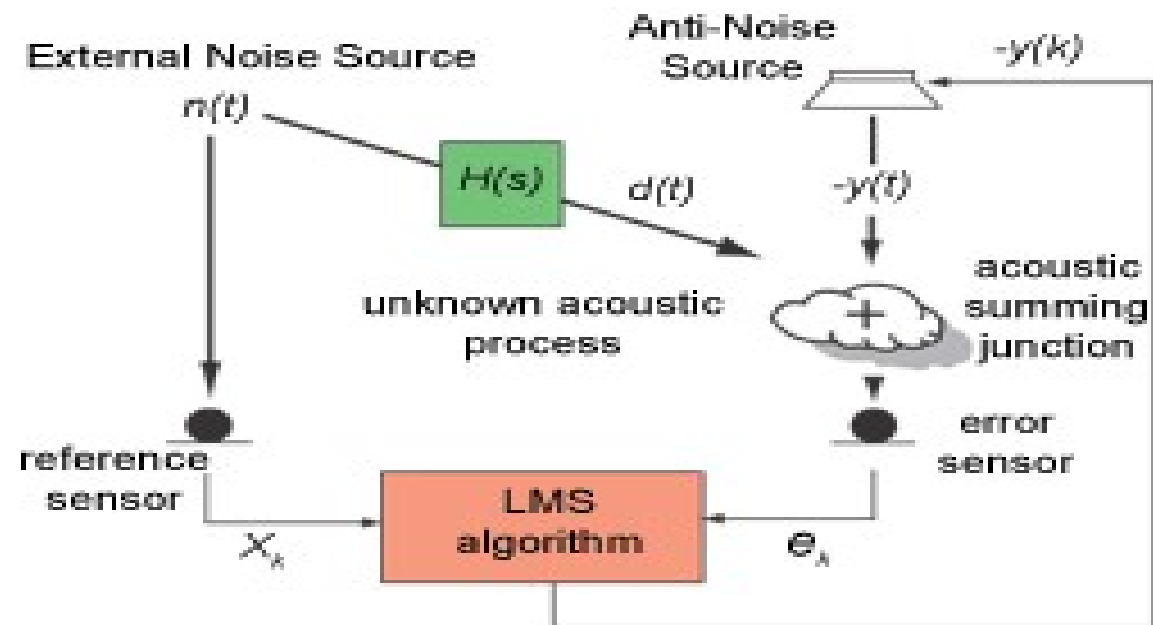
**Cause-effect principle** → **Cause-effect-cause principle**.



- The **thermostat**;
- The **control of aircrafts** in flight or vehicles in motion:



- Noise reduction:



[http://www.ind.rwth-aachen.de/research/noise\\_reduction.html](http://www.ind.rwth-aachen.de/research/noise_reduction.html)

Noise reduction is a subject to research in many different fields. Depending on the environment, the application, the source signals, the noise, and so on, the solutions look very different. Here we consider noise reduction for audio signals, especially speech signals, and concentrate on common acoustic environments such as an office room or inside a car. The goal of the noise reduction is to reduce the noise level without distorting the speech, thus reduce the stress on the listener and - ideally - increase intelligibility.

The need of **fluctuations**.

*“It is a curious fact that, while political economists recognize that for the proper action of the law of supply and demand there must be fluctuations, it has not generally been recognized by mechanics in this matter of the steam engine governor. The aim of the mechanical engineers, as is that of the political economist, should be not to do away with these fluctuations all together (for then he does away with the principles of self-regulation), but to diminish them as much as possible, still leaving them large enough to have sufficient regulating power.”*

H.R. Hall, *Governors and Governing Mechanisms*, The Technical Publishing Co., 2nd ed., Manchester 1907.

Accordingly, optimal controls and trajectory are often complex, non intuitive, difficult to compute or to guess.

Control Theory provides systematic mathematical tools to compute them.

An example: Lagrange multipliers.

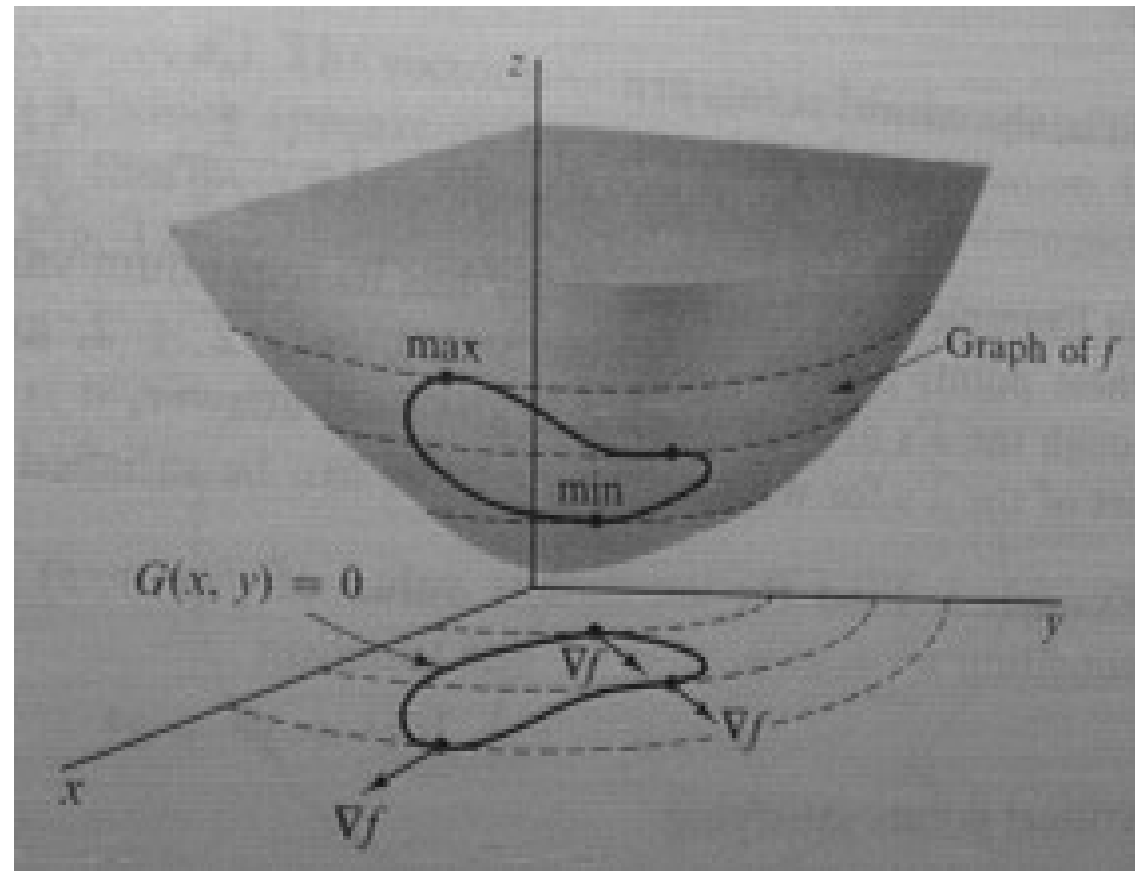
$$\min_{g(x)=c} f(x).$$

The answer: critical points  $x$  are those for which

$$\nabla f(x) = \lambda \nabla g(x)$$

for some real  $\lambda$ .

This is so because  $\nabla g(x)$  is the normal to the level set in which minimization occurs. A necessary condition for the point  $x$  to be critical is that  $\nabla f(x)$  points in this normal direction. Otherwise, if  $\nabla f(x)$  had a nontrivial projection over the level set  $g(x) = c$  there would necessarily exist a better choice of  $x$  for which  $f(x)$  would be even smaller.



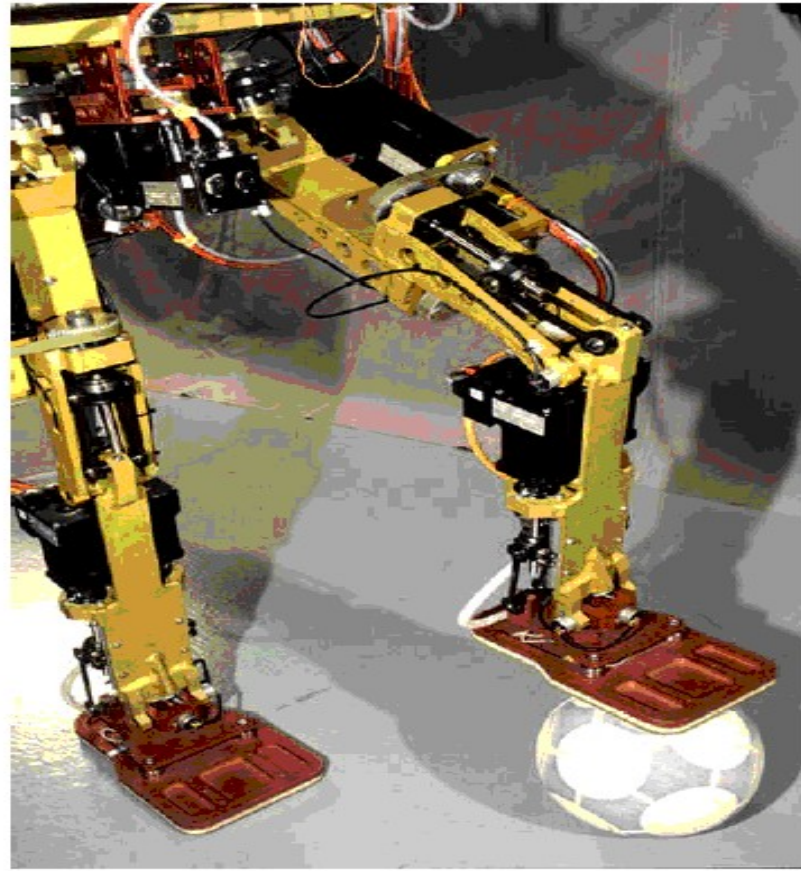
## Cybernetics.

“Cybernétique” was proposed by the French physicist [A.-M. Ampère](#) in the [XIX Century](#) to design the nonexistent science of process controlling. This was quickly forgotten until 1948, when [N. Wiener \(1894–1964\)](#) chose “Cybernetics” as the title of his book.

Wiener defined Cybernetics as “ [the science of control and communication in animals and machines](#)”.

In this way, he established the connection between Control Theory and Physiology and anticipated that, in a desirable future, engines would obey and imitate human beings.





In mathematical terms this corresponds to **duality** in **convex analysis**.

To each optimization problem it corresponds a dual one. Solving the primal one is equivalent to solving the dual one, and viceversa. But often in practice one is much easier to solve than the other one.

This duality principle is to be used to always solve the easy one.

**PRIMAL = DUAL**

**CONTROL = COMMUNICATION**

# ORIGINS

- Irrigation systems, ancient Mesopotamia, 2000 BC.
- Harpenodaptai, ancient Egypt, the string stretchers.
  - \* Primal: The minimal distance between two points is given by the straight line.
  - \* Dual: The maximal distance between the extremes of a cord is obtained when the cord is along a straight line.

In mathematical terms, things are not easy:

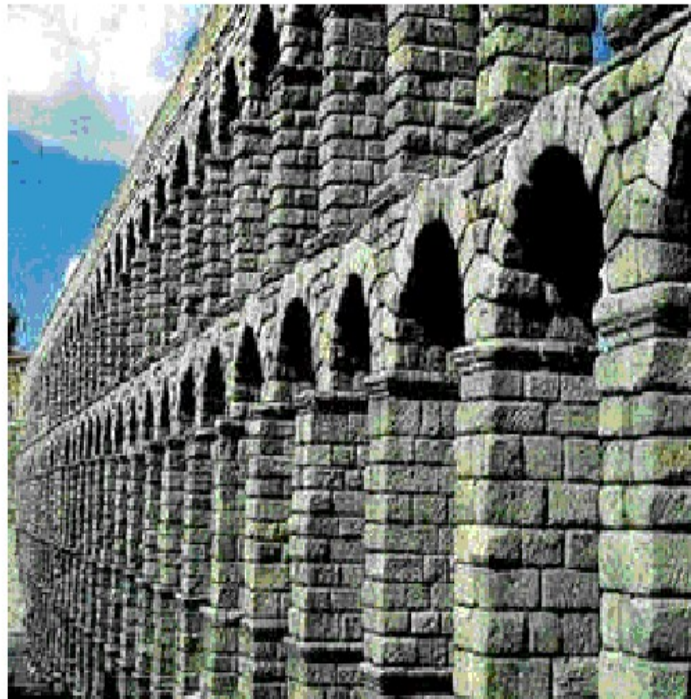
To minimize the functional

$$\int_0^1 ||x'(t)|| dt$$

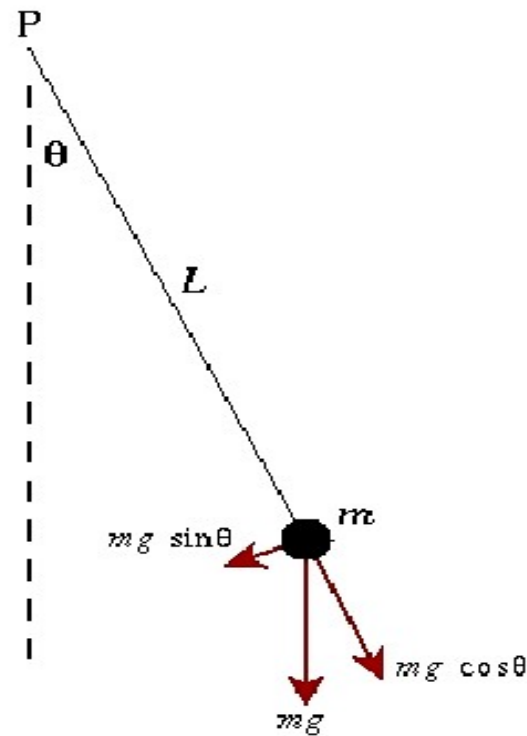
among the set of parametrized curves  $x : [0, 1] \rightarrow \mathbf{R}$ , such that  $x(0) = A$  and  $x(1) = B$ .

We easily end up working in the BV class of functions of bounded variation, out of the most natural and simple context of Hilbert spaces.

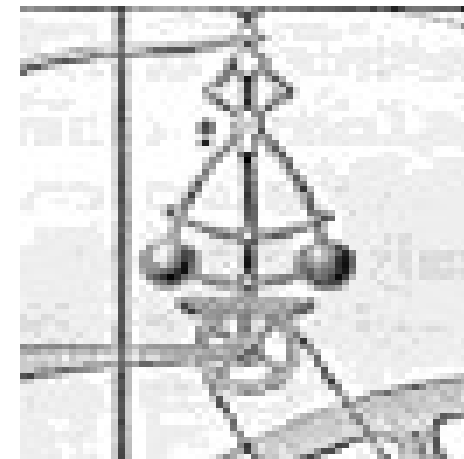
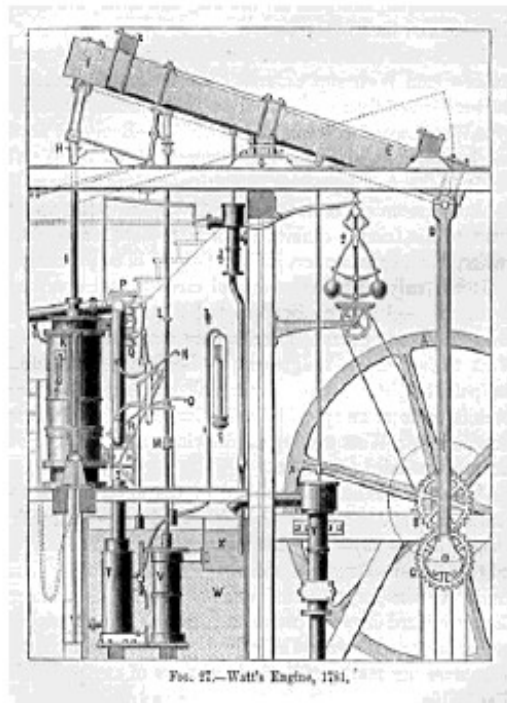
- **Roman aqueducts.** Systems of water transportation endowed with valves and regulators.



- **The pendulum.** The works of Ch. Huygens and R. Hooke, in the end of the XVII century, the goal being measuring in a precise way location and time, so precious in navigation.



- **Regulator of windmills.** Applied later by **J. Watt (1736-1819)**, to the steam engine, the motor of industrial revolution.



The first mathematical rigorous analysis of the stability properties of the steam engine was done by Lord J. C. Maxwell, in 1868.

The explanation of some erratic behaviors was explained. Until then it was not well understood why apparently more elaborated and perfect regulators could have a bad behavior.

The reason is now referred to as the overdamping phenomenon. Consider the equation of the pendulum:

$$x'' + x = 0.$$

This describes a pure conservative dynamics: the energy

$$e(t) = \frac{1}{2}[x^2(t) + |x'(t)|^2]$$



is constant in time.

Let us now consider the dynamics of the pendulum in presence of a **friction** term:

$$x'' + x = -kx',$$

$k$  being a positive constant  $k > 0$ .

The energy decays exponentially. **But the decay rate does not necessarily increase with the damping parameter  $k$  despite of the following energy dissipation law:**

$$\frac{de(t)}{dt} = -k|x'(t)|^2.$$

Indeed, computed the eigenvalues of the characteristic equation one finds:

$$\lambda_{\pm} = [-k \pm \sqrt{k^2 - 4}]/2.$$

It is easy to see that  $\lambda_+$  increases as  $k > 2$  increases.

This confirms the prediction that optimal controls and strategies are often complex and that they do not necessarily obey to the very first intuition.

- **Automatic control.** The number of applications rapidly increased in the thirties covering different areas like amplifiers in telecommunications, distribution systems in plants, stabilization of aeroplanes, electrical mechanisms in paper production, petroleum and steel industry,...

By that time there were two clear and distinct approaches:

- **State space approach**, based on modelling by means of Ordinary Differential equations (ODE);
- **The frequency domain** approach, based in the Fourier representation of signals.

PHYSICAL SPACE  $\equiv$  FREQUENCY SPACE

But after the second world war it was discovered that most physical systems were **nonlinear** and **nondeterministic**.

IMPORTANT CONTRIBUTIONS WERE MADE IN THE 60's:

- **Kalman** and his theory of **filtering and algebraic approach** to the control of systems;
- **Pontryagin** and his **maximum principle**: A generalization of Lagrange multipliers.
- **Bellman** and his principle of **dynamic programming**:  
A trajectory is optimal if it is optimal at every time.

## Kalman and the controllability of finite dimensional linear systems

Let  $n, m \in \mathbb{N}^*$  and  $T > 0$ . Consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0, T), \\ x(0) = x^0. \end{cases} \quad (2)$$

$A$  is a real  $n \times n$  matrix,  $B$  is a real  $n \times m$  matrix and  $x^0$  a vector in  $\mathbb{R}^n$ . The function  $x : [0, T] \longrightarrow \mathbb{R}^n$  represents the *state* and  $u : [0, T] \longrightarrow \mathbb{R}^m$  the *control*. Both are vector functions of  $n$  and  $m$  components respectively depending exclusively on time  $t$ . Obviously, in practice  $m \leq n$ . The most desirable goal is, of course, controlling the system by means of a minimum number  $m$  of controls.

Given an initial datum  $x^0 \in \mathbb{R}^n$  and a vector function  $u \in L^2(0, T; \mathbb{R}^m)$ , system (2) has a unique solution  $x \in H^1(0, T; \mathbb{R}^n)$  characterized by the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0, T]. \quad (3)$$

System (2) is **exactly controllable** in time  $T > 0$  if given any initial and final one  $x^0, x^1 \in \mathbb{R}^n$  there exists  $u \in L^2(0, T, \mathbb{R}^m)$  such that the solution satisfies  $x(T) = x^1$ .

According to this definition the aim of the control process consists in driving the solution  $x$  from the initial state  $x^0$  to the final one  $x^1$  in time  $T$  by acting on the system through the control  $u$ .

**Example 1.** Consider the case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (4)$$

Then the system

$$x' = Ax + Bu$$

can be written as

$$\begin{cases} x'_1 = x_1 + u \\ x'_2 = x_2, \end{cases}$$

or equivalently,

$$\begin{cases} x'_1 = x_1 + u \\ x_2 = x_2^0 e^t, \end{cases}$$

where  $x^0 = (x_1^0, x_2^0)$  are the initial data.

This system is **not controllable** since the control  $u$  does not act on the second component  $x_2$  of the state which is completely determined by the initial data  $x_2^0$ .



**Example 2.** By the contrary, the equation of the harmonic oscillator is controllable

$$x'' + x = u. \quad (5)$$

The matrices  $A$  and  $B$  are now respectively

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Once again, we have at our disposal only one control  $u$  for both components  $x$  and  $y$  of the system. But, unlike in Example 1, now the control acts in the second equation where both components are present.

The answer to the problem does not depend simply on the number of controls  $M$  and the number of states  $N$  to be controlled, but rather on how the controller  $B$  is chosen in terms of the dynamics  $A$  of the matrix.

Más vale maña que fuerza.

## Observability property

Let  $A^*$  be the adjoint matrix of  $A$ , i.e. the matrix with the property that  $\langle Ax, y \rangle = \langle x, A^*y \rangle$  for all  $x, y \in \mathbb{R}^n$ . Consider the following homogeneous **adjoint system** of (2):

$$\begin{cases} -\varphi' = A^*\varphi, & t \in (0, T) \\ \varphi(T) = \varphi_T. \end{cases} \quad (6)$$

Multiplying the state equation by the adjoint state  $\varphi$ , if  $x(T) \equiv 0$  we get

$$\int_0^T \langle u, B^*\varphi \rangle dt + \langle x^0, \varphi(0) \rangle = 0, \quad \forall \varphi. \quad (7)$$

*This is a linear system of  $n$  equations. The unknown  $u$  belongs to  $(L^2(0, T))^m$ , which is an infinite-dimensional space.*

Identity (7) is in fact an optimality condition for the **critical points of the quadratic functional**  $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,

$$J(\varphi_T) = \frac{1}{2} \int_0^T |B^* \varphi|^2 dt + \langle x^0, \varphi(0) \rangle$$

where  $\varphi$  is the solution of the adjoint system (6) with initial data  $\varphi_T$  at time  $t = T$ .

**Lemma 1** *Suppose that  $J$  has a minimizer  $\hat{\varphi}_T \in \mathbb{R}^n$  and let  $\hat{\varphi}$  be the solution of the adjoint system (6) with initial data  $\hat{\varphi}_T$ . Then*

$$u = B^* \hat{\varphi} \tag{8}$$

*is a control of system (2) with initial data  $x^0$ .*

A necessary and sufficient condition for that is  $J$  to be **coercive**.

$$\int_0^T |B^* \varphi|^2 dt \geq c |\varphi(0)|^2, \quad (9)$$

for all  $\varphi_T \in \mathbb{R}^n$ ,  $\varphi$  being the corresponding solution of (6).

In the sequel (9) will be called the **observation** or **observability inequality**. When it holds the adjoint system is **observable**.

It guarantees that the solution of the adjoint problem at  $t = 0$  is uniquely determined by the observed quantity  $B^* \varphi(t)$  for  $0 < t < T$ . In other words, the information contained in this term completely characterizes the solution of (6).

CONTROLLABILITY OF THE STATE EQUATION

$\equiv$

OBSERVABILITY OF THE ADJOINT SYSTEM.

This is a rigorous mathematical expression of Wiener's principle of Cybernetics: **communication=control**.

## Kalman's controllability condition

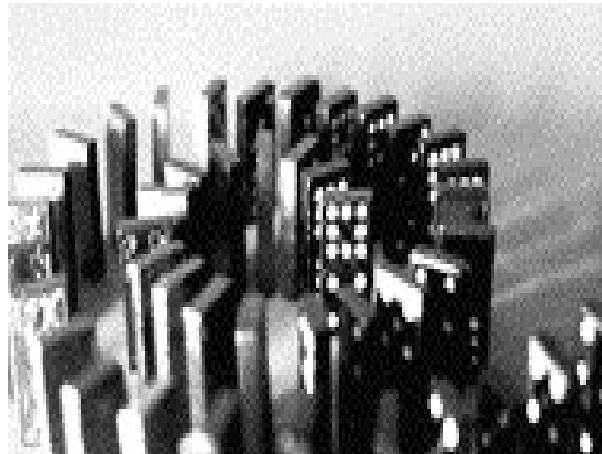
**Theorem 1** *System (2) is exactly controllable in some time  $T$  if and only if*

$$\text{rank}[B, AB, \dots, A^{n-1}B] = n. \quad (10)$$

*Consequently, if system (2) is controllable in some time  $T > 0$  it is controllable in any time.*

The “miracle” is that an strategic choice of  $B$ , even if it represents a single control ( $M = 1$ ) may allow controlling an arbitrarily large number of components ( $N$  as large as we wish).

This is the so-called **trigger effect** or **efecto dominó**.





**Remark 1** *The set of controllable pairs  $(A, B)$  is open and dense.*  
*This means that*

- *Most systems are controllable;*
- *The controllability property is robust, i. e. it is invariant under small perturbations of  $A$  and/or  $B$ .*

## RECENT IMPORTANT FURTHER DEVELOPMENTS:

- Nonlinear problems;

Lie brackets: Think on how park or unpark your car...

- Stochastic models;

Human beings introduce more uncertainty in already uncertain systems...

- Infinite dimensional systems = Partial Differential Equations (PDE), also referred to as Distributed Parameter Systems. The models in Continuum Mechanics....

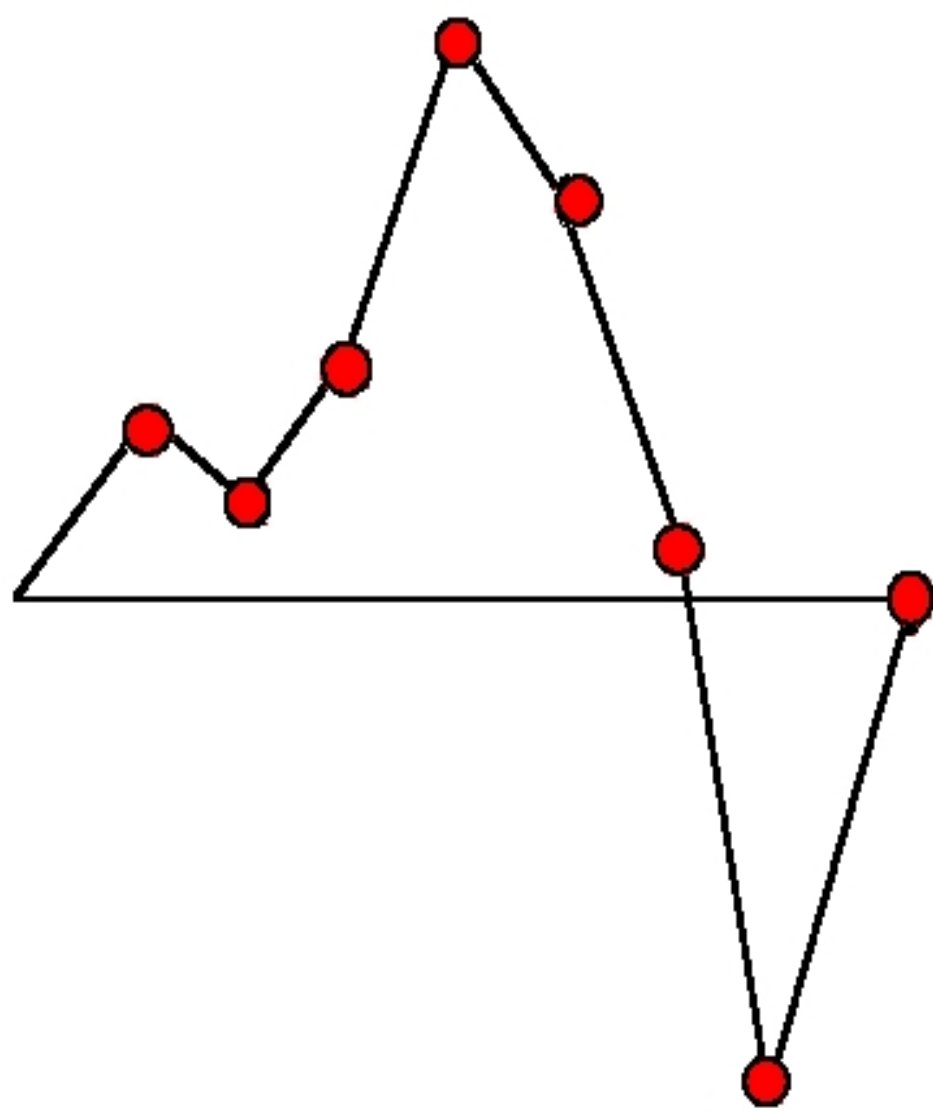
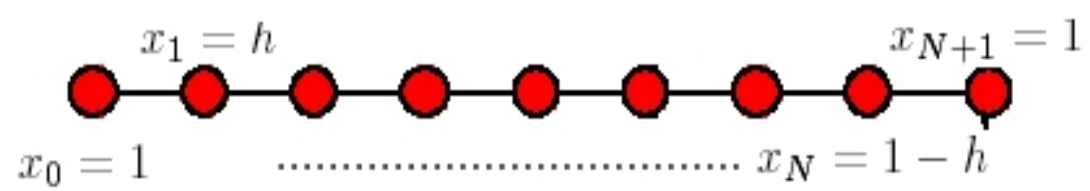
In the PDE context itself a lot of work has been done distinguishing elliptic, parabolic and hyperbolic equations, and also for more sophisticated systems arising in fluid-structure interaction, thermoelasticity, multi-link structures,...

But, there is an important jump between the finite-dimensional theory of Kalman and the PDE theory.

## FROM FINITE TO INFINITE DIMENSIONS

This problem is conceptually and technically difficult. This is particularly the case for hyperbolic problems in which the dynamics is purely conservative.

The situation is better for parabolic equations in which the intrinsic dissipative nature of the system introduces some simplification of the dynamics.



## IS PDE CONTROL RELEVANT?

The answer is, definitely, **YES**.

Let us mention some examples in which the wave equation is involved in a way or another.

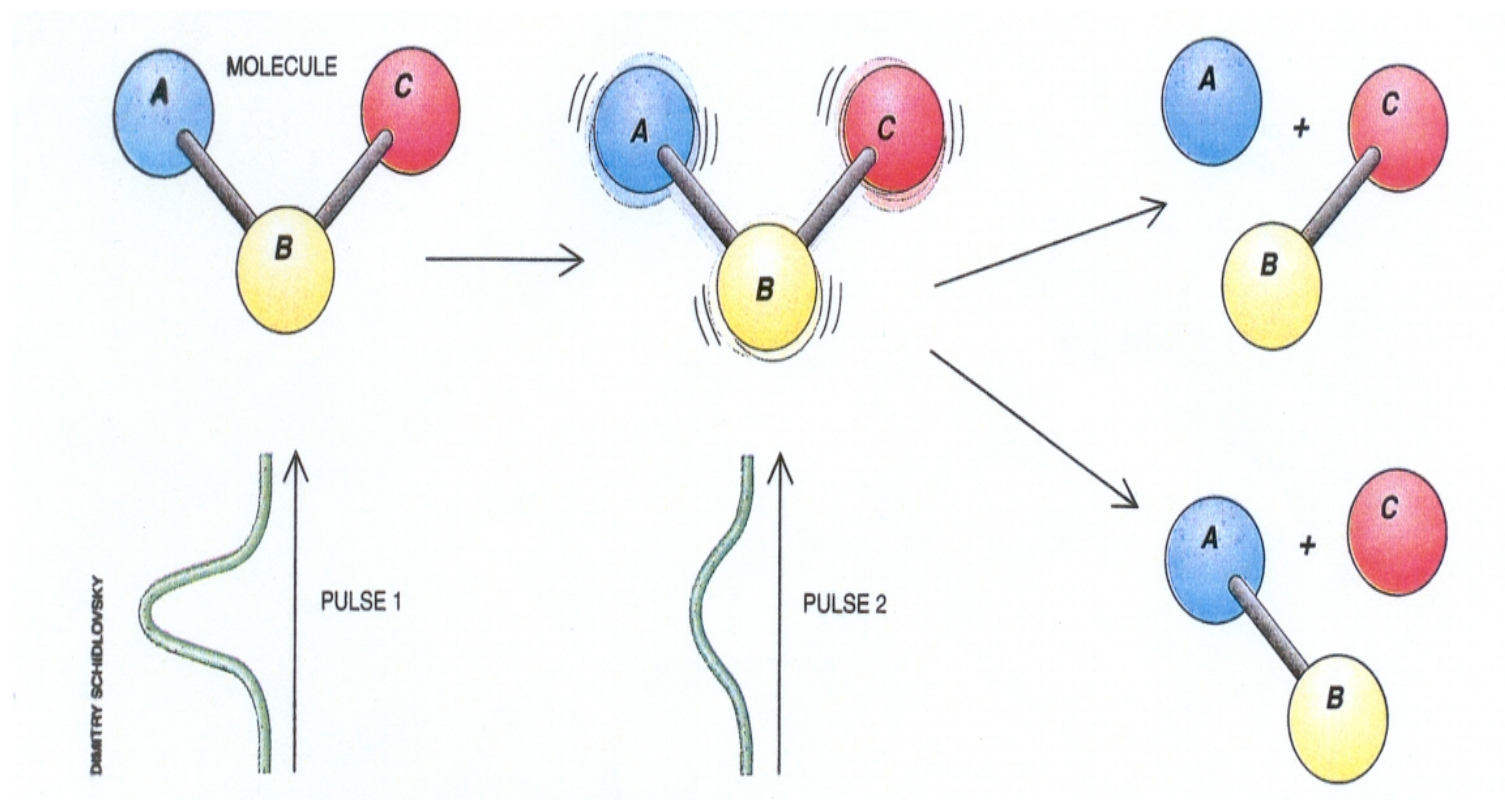
- **Noise reduction in cavities and vehicles.**

Typically, the models involve the wave equation for the **acoustic waves** coupled with some other equations modelling the **dynamics of the boundary structure**, the action of **actuators**, possibly through **smart mechanisms** and materials.

- Quantum control and Computing.

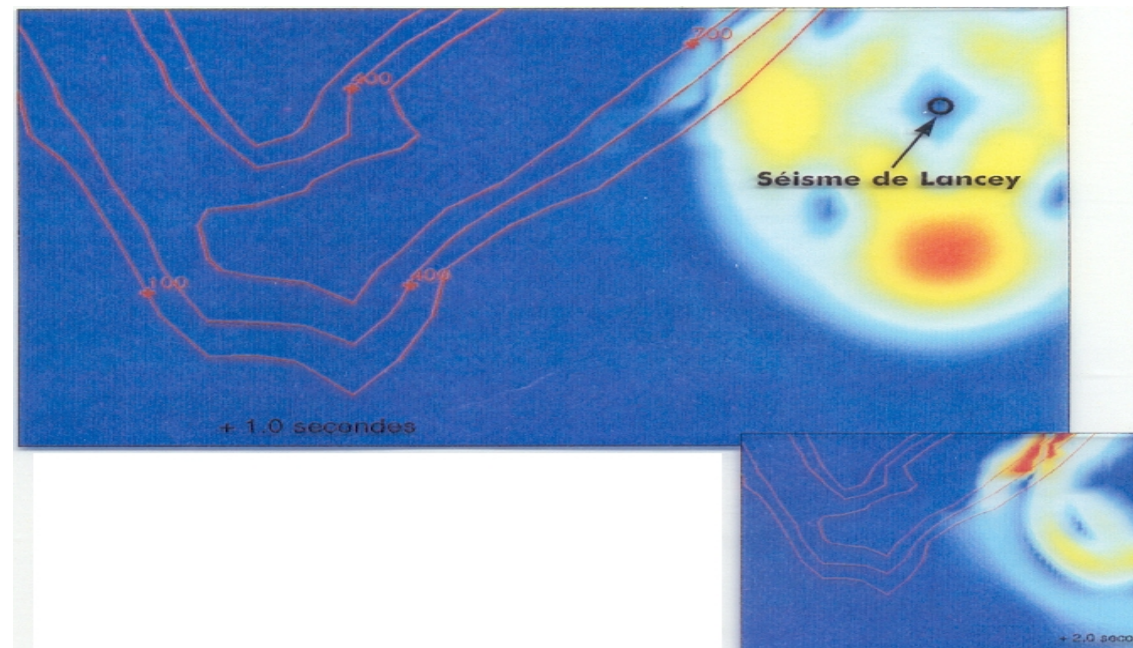
Laser control in Quantum mechanical and molecular systems to design coherent vibrational states.

In this case the fundamental equation is the Schrödinger one. Most of the theory we shall develop here applies in this case too. The Schrödinger equation may be viewed as a wave equation with infinite speed of propagation.



*P. Brumer and M. Shapiro, Laser Control of Chemical reactions, Scientific American, March, 1995, pp.34-39.*

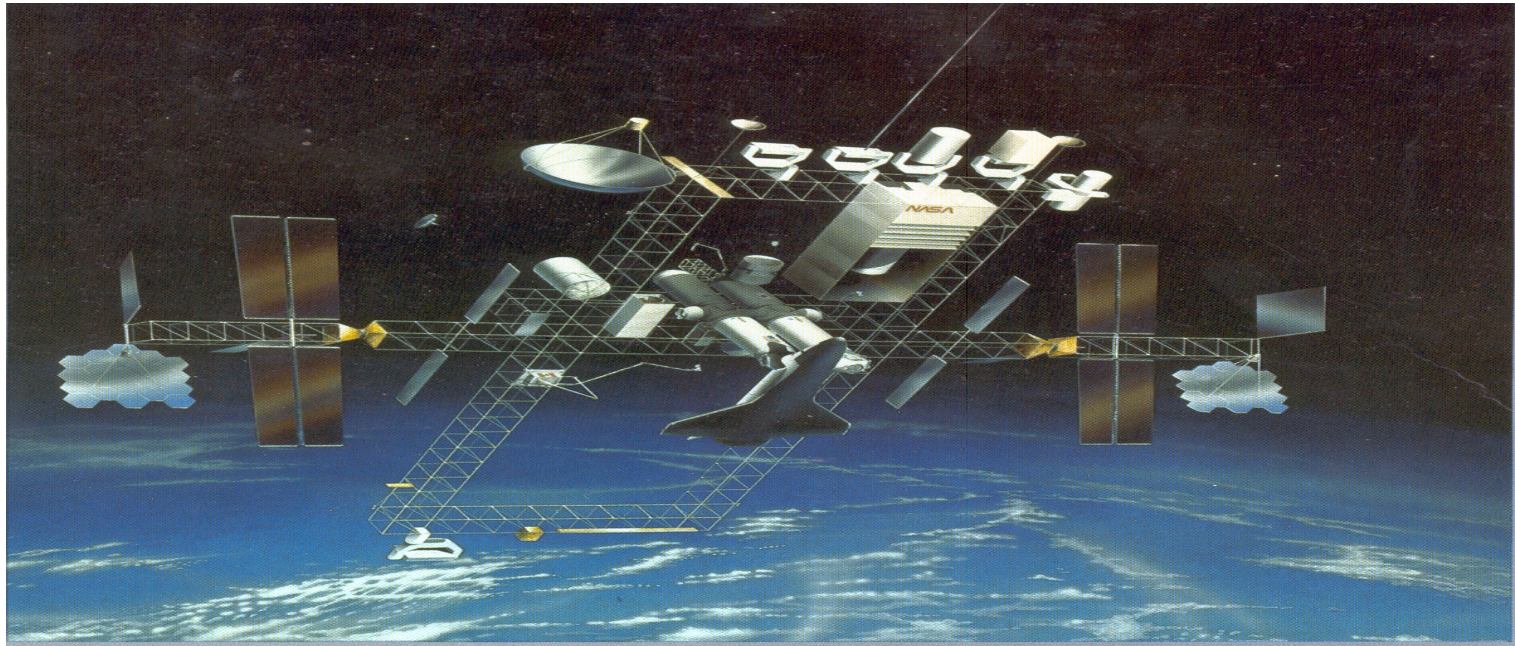
- Seismic waves, earthquakes.



*F. Cotton, P.-Y. Bard, C. Berge et D. Hatzfeld, Qu'est-ce qui fait vibrer Grenoble?, La Recherche, 320, Mai, 1999, 39-43.*

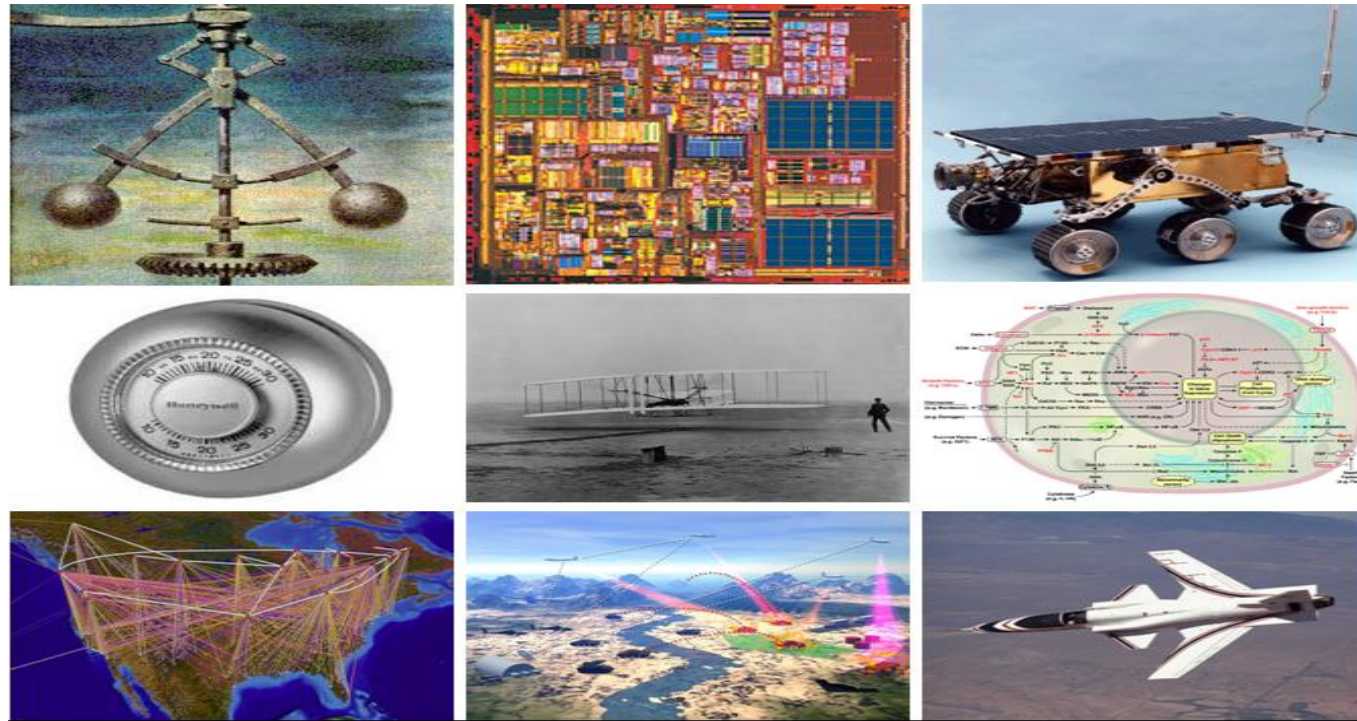


- Flexible structures.



*SIAM Report on “Future Directions in Control Theory. A Mathematical Perspective”, W. H. Fleming et al., 1988.*

- An many others...



*Control in an information rich World, SIAM, 2003.*

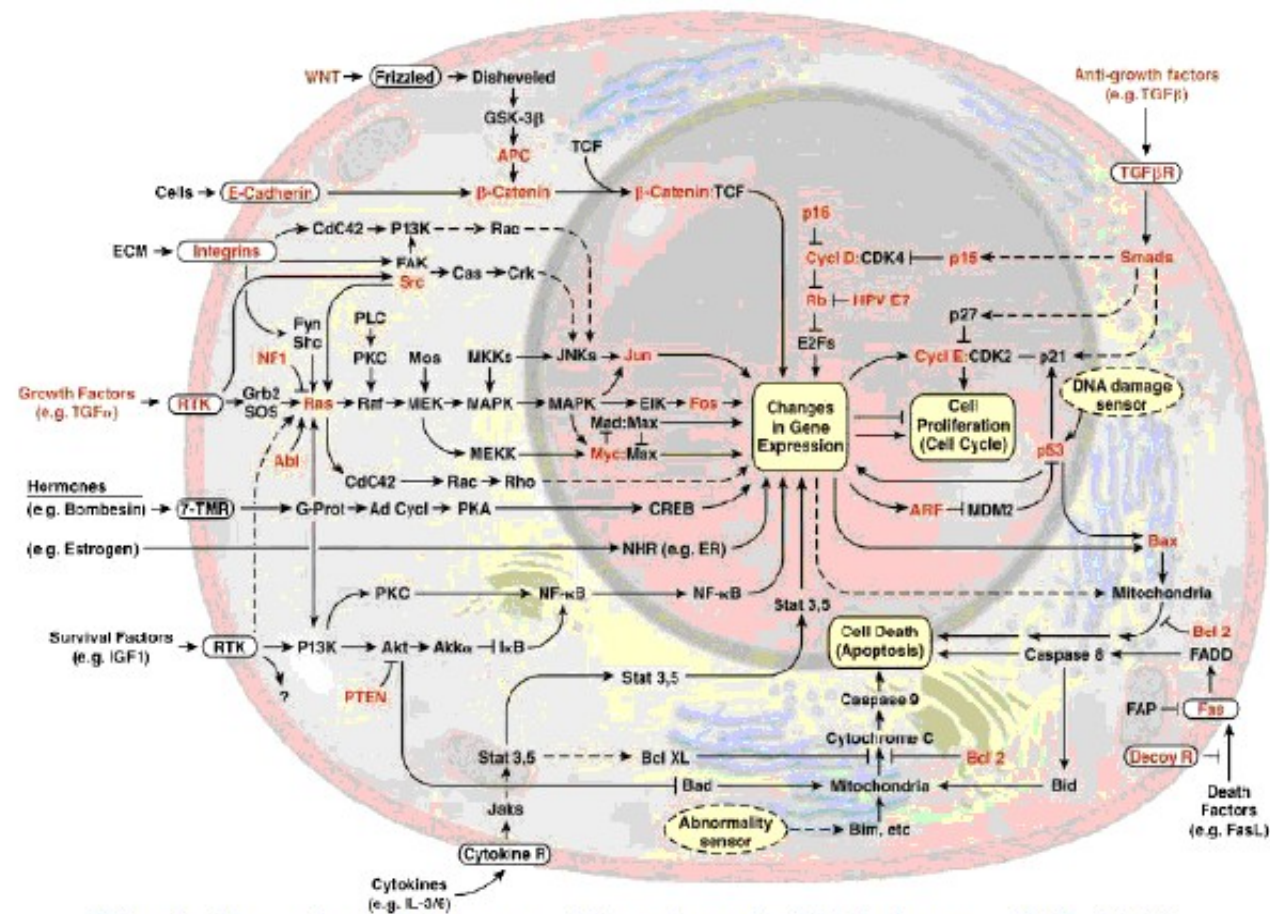
CONTROL THEORY is full of challenging, difficult and interesting mathematical problems.

Control is continuously enriched by the permanent **interaction with applications**.

This interaction works in both directions:

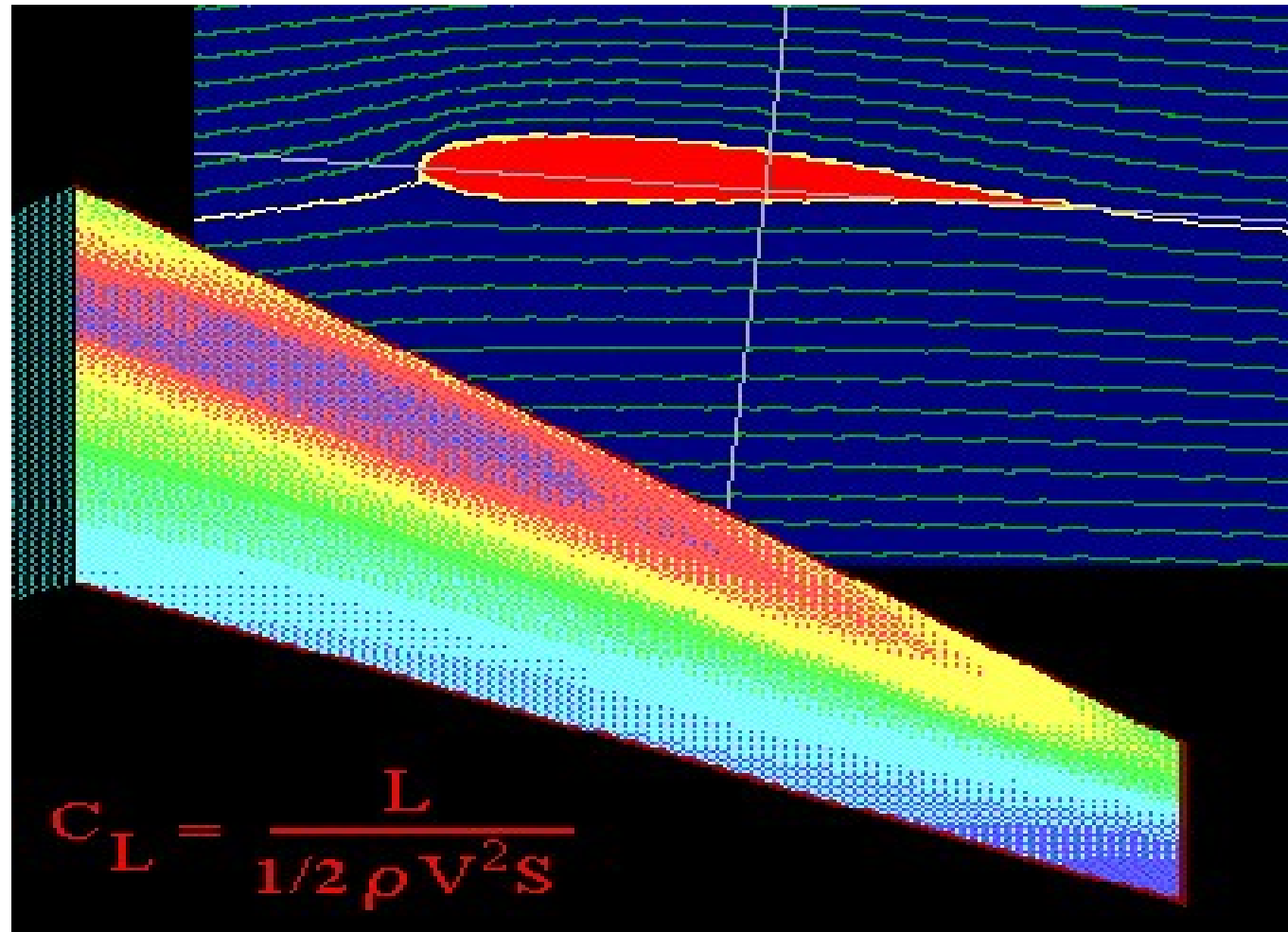
- \* **mathematical control theory provides the understanding** allowing to improve real-life control mechanisms;
- \* **Applications provide and bring new mathematical problems of increasing complexity.**





The hallmarks of cancer, Hanahan & Weinberg, Cell 2000













AN OPEN PROBLEM

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control one one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0, t) = 0; y(1, t) = v(t), & 0 < t < T \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & 0 < x < 1 \end{cases}$$

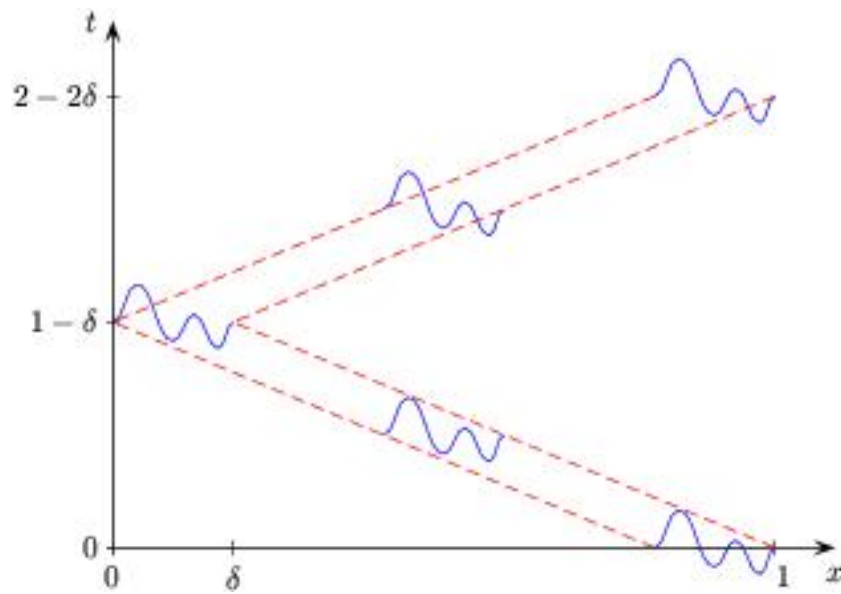
$y = y(x, t)$  is the state and  $v = v(t)$  is the control.

The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time  $T$ : Given initial data  $\{y^0(x), y^1(x)\}$  to find a control  $v = v(t)$  such that

$$y(x, T) = y_t(x, T) = 0, \quad 0 < x < 1.$$

We know that this property holds if and only if  $T \geq 2$ .

The problem being linear it can be reduced to an observability inequality (J. L. Lions' Hilbert Uniqueness method (HUM)).

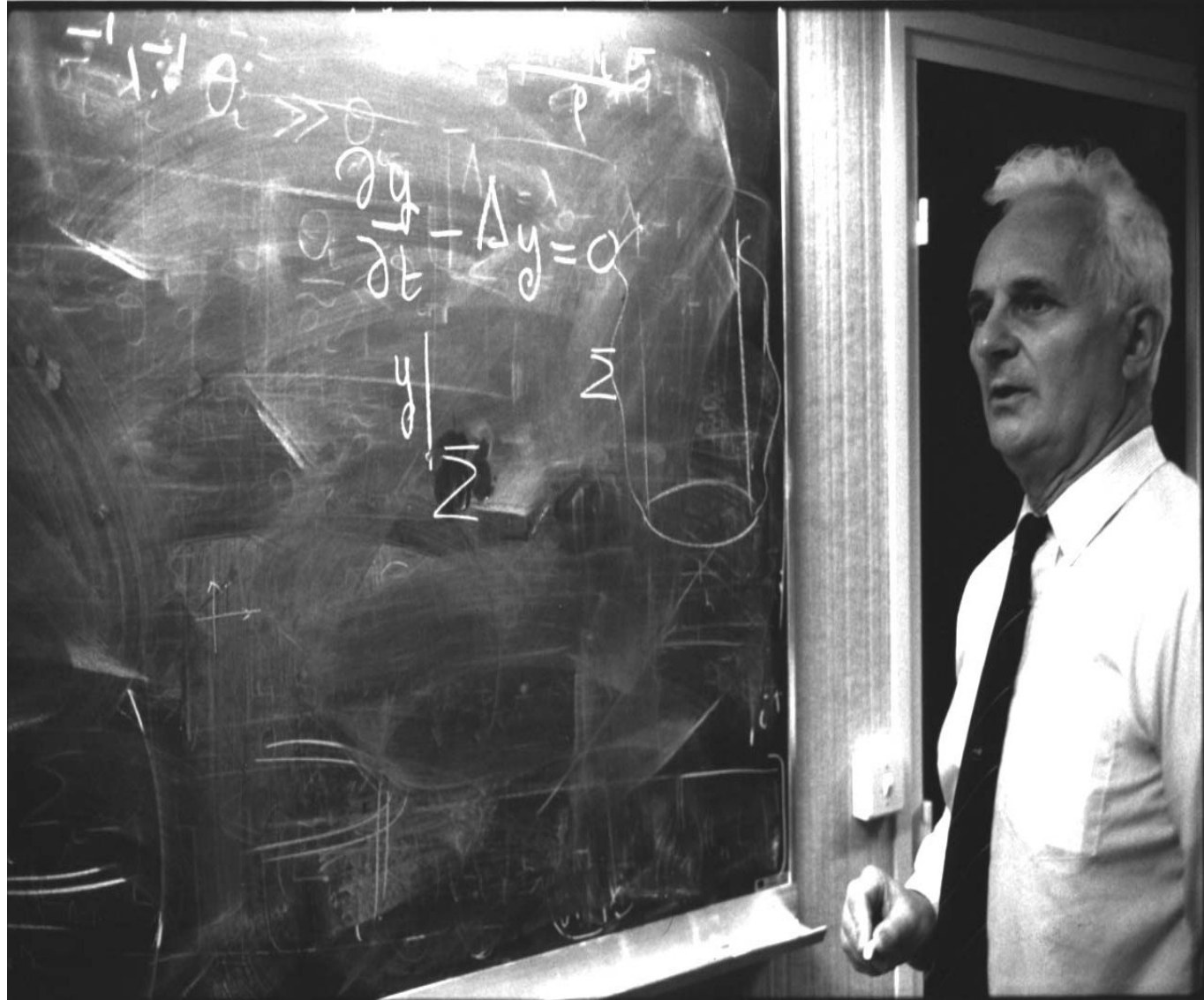


$$E(0) \leq C(T) \int_0^T |u_x(1, t)|^2 dt.$$

The open problem is: is the same true for the nonlinear wave equation?

$$\begin{cases} y_{tt} - y_{xx} + y^3 = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0, t) = 0; y(1, t) = v(t), & 0 < t < T \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x), & 0 < x < 1 \end{cases}$$

So far we know that for any data  $\{y^0(x), y^1(x)\}$ , controllability can be reached if  $T$  is LARGE ENOUGH. But it is unknown whether this can be done in a time which is independent of the data!





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